Contents

Row-convex representations for quantum gl_n.*

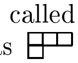
Brian D. Taylor
Wayne State University
Detroit, MI 48202, USA
bdt@math.wayne.edu

June 7, 2000

Abstract

We prove a basis theorem, branching rule, and excellent filtration for a quantized version of the Schur modules associated to row-convex shapes. Our primary technique, a quantum straightening law, specializes when $q = 1$ to a new straightening algorithm for row-convex Schur modules.

1 Introduction

This paper provides quantized versions of some results in what has been called "the calculus of shapes." As is well known, "partition shapes" such as  which are the Ferrer's diagram of a partition index, in characteristic 0, the irreducible representations of the general linear Lie algebra, the general linear group, and the symmetric group. A construction of these representations was given by Akin, Buchsbaum, and Weyman in [ABW82] that replaced the partition shapes with a generalized shape–any finite subset of the cells of an infinite matrix. These generalized shapes have been given significant recent attention, most notably in the work of Reiner and Shimozono [RS95, RS96p, RS97, RS98, S97], Magyar [M98], and Magyar and Lakshmibai [LM97].

One of the original motivations for the constructions in [ABW82] was Buchsbaum's program for the study of characteristic–free (projective) resolutions of the irreducible representations of GL_m; for some recent results in this program, see [B98, BR00]. The methods developed for this program in [AB85] required the use of shapes that were no longer skew. These shapes were studied in [W94] and generalizations to row-convex shapes, shapes with no gaps in their rows,were considered in [K96, K98]. A canonical basis for these row-convex representations was introduced in [T97a, T], together with the first "straightening" algorithm for expanding elements of the representation in terms of this basis; this algorithm generalized the known straightening law for skew and partition shapes.

*http://www.math.wayne.edu/~bdt/quantStraight.pdf

By comparison, the study of quantized Schur modules has mostly been restricted to the classical shapes. Analogues for most of the classical results on straightening were established by Taft and Towber in [TfTo91] and Huang and Zhang in [HZ94]. A good exposition, focusing on these modules as representations of the quantized universal enveloping algebra of gl_m can be found in Leclerc and Thibon's paper [LeTh96]. This last paper makes the combinatorial interest of quantized straightening laws apparent; the authors derive the Robinson-Schensted-Knuth bijection as the "crystal limit" of the straightening law at $q = 0$.

In this paper, we generalize a portion of the theory of general shapes to representations of quantum gl_m. We focus on quantum versions of the results developed in [T].

The construction of [ABW82] builds representations as submodules of the algebra of polynomial functions on the entries of a matrix. Thus, in Section 2, we review the algebra of quantized polynomial functions on the entries of a quantum matrix as well as the notion of a quantum determinant. We recapitulate some of the identities satisfied by quantum determinants and identify some immediate, but useful, variations.

With this as background, in Section 3, we present a quantized version of the Schur-modules constructed in [ABW82].

In Section 4, we prove a quantized straightening for quantized row-convex Schur modules. The basis into which this straightening law expands elements of the quantized Schur module is indexed by the same combinatorial objects, "straight tableau," introduced in [T] for ordinary Schur modules corresponding to row-convex shapes. Specializing this straightening law to the classical case (by setting $q = 1$) provides a significantly simpler approach to the straightening procedure of [T]. The drawback to this approach is that it fails to handle the Weyl modules and supersymmetric Schur modules discussed in [T].

Section 5 introduces the "flagging" results necessary for applying the straightening law and basis theorem of Section 4 to representations of $U_q(b_n)$, the quantized enveloping algebra of the Borel subalgebra of lower triangular matrices.

Finally, Section 6 proves a branching rule for restricting representations of $U_q(gl_m)$, the quantized enveloping algebra of gl_m, to representations of the subalgebra $U_q(gl_{m-1})$. Like all of the results in this paper, the branching rule is characteristic-free and is presented via an appropriate filtration. Further, this is an "excellent filtration" in the sense that, upon flagging, it provides a branching rule for restricting from $U_q(b_m)$ to $U_q(b_{m-1})$.

2 Quantum matrices and quantum determinants

Throughout this paper $\mathbf{Q}$ will be the rationals, $\mathbf{Z}$ the integers and $\mathbf{k}$ a commutative ring containing a distinguished invertible element q. Typically we take $\mathbf{k} = \mathbf{Z}[q, q^{-1}]$.

We begin by recalling some facts about quantum groups and quantized algebras, following [Ka, LeTh96, Ma88, Ma91].

Definition 2.1 Let R be a $\mathbf{k}$-algebra. An R-matrix $A = (a_{i,j})$ is called a *quantum matrix* if the $a_{i,j}$ satisfy the relations

$$a_{i,k}a_{i,l} = q^{-1}a_{i,l}a_{i,k}, \qquad a_{i,k}a_{j,k} = q^{-1}a_{j,k}a_{i,k}, \qquad a_{i,l}a_{j,k} = a_{j,k}a_{i,l},$$
$$a_{i,k}a_{j,l} - a_{j,l}a_{i,k} = (q^{-1} - q)a_{j,k}a_{i,l},$$

for $i < j$ and $k < l$. Such a quantum matrix is *generic* when the subalgebra of R generated by the $a_{i,j}$ satisfies no additional relations. When the matrix $t_{i,j}$ is generic, call this subalgebra $Mat_q^{\mathbf{k}}(t_{i,j})$. We abbreviate $Mat_q^{\mathbf{Z}[q,q^{-1}]}(t_{i,j})$ by $Mat_q(t_{i,j})$.

As pointed out in [Ma88], the above relations are imposed in order to make linear change of variables an algebra homomorphism of the q-deformed exterior (or symmetric) algebra. Specifically, define $\Lambda_q(\xi_1, \ldots, \xi_n)$, the q-deformation of the exterior algebra, to be the quotient of the free associative k-algebra generated by the ξ_i's by the relations $\xi_i^2 = 0$ and $-q^{-1}\xi_i\xi_j = \xi_j\xi_i$ for $i < j$. When $q + q^{-1}$ is not a zero divisor in R, a direct calculation shows that if $M = (a_{i,j})$ is an $n \times n$ R-matrix with entries in an associative algebra and those entries commute with the ξ_i then the two maps $\Phi_M(\xi_i) = \sum_j a_{i,j}\xi_j$ and $\Phi_{M^t}(\xi_i) = \sum_j a_{j,i}\xi_j$ both induce algebra homomorphisms iff M is a quantum matrix.

Definition 2.2 A monomial $\prod_k t_{i_k,j_k}$ is *ordered* if the sequence j_k weakly increases and if $i_k \geq i_{k+1}$ whenever $j_k = j_{k+1}$.

The relations in Definition 2.1 guarantee that the ordered monomials span the algebra $Mat_q^{\mathbf{k}}(t_{i,j})$. In fact they are linearly independent. This fact can be checked directly, albeit laboriously, by application of the "diamond lemma," Theorem 1.2 of [Be78]. For a proof when $\mathbf{k}$ is a field, see for example [Ma91]. This proof works with minimal modification for arbitrary $\mathbf{k}$.

Proposition 2.3 *The algebra $Mat_q^{\mathbf{k}}(t_{i,j})$ is a free $\mathbf{k}$-module with the ordered monomials as a basis.*

Since the statements of independence in the combinatorics literature often assume $\mathbf{k} = Q(q)$ with q transcendental over $\mathbf{Q}$, we sketch how to see the above independence result for arbitrary $\mathbf{k}$ directly from independence when $\mathbf{k} = \mathbf{Q}(q)$.

Independence in $Mat_q^{\mathbf{Q}(q)}(t_{i,j})$ shows independence in the $Z[q, q^{-1}]$-subalgebra of $Mat_q^{\mathbf{Q}(q)}(t_{i,j})$ generated by the $t_{i,j}$. This subalgebra is obviously a quotient of $Mat_q^{\mathbf{Z}[q,q^{-1}]}(t_{i,j})$ (in fact they're isomorphic) hence the ordered monomials are independent in $Mat_q^{\mathbf{Z}[q,q^{-1}]}(t_{i,j})$. Since the coefficients of the non-ordered monomials appearing in the relations of Definition 2.1 are units in $\mathbf{Z}[q, q^{-1}]$, independence of ordered monomials follows for all $Mat_q^{\mathbf{k}}(t_{i,j})$.

We will next recall the quantum determinants and several identities involving them; see [TfTo91, HZ94, LeTh96].

Definition 2.4 Let X be a quantum matrix. The q-determinant, $\det_q(X)$, of the minor of X indexed by rows $i_1, \ldots, i_k$ and columns $j_1 < \cdots < j_k$ is

$$\mathrm{Tab}_q(i_1,\ldots,i_k|j_1,\ldots,j_k) := \sum_{\sigma\in S_k} (-q)^{-l(\sigma)} t_{i_1,j_{\sigma(1)}} \cdots t_{i_k,j_{\sigma(k)}},$$

where the length of σ, $l(\sigma)$, is the number of inversions in $\sigma_1, \ldots, \sigma_k$.

Proposition 2.5 *Suppose that $j_1 < \cdots < j_k$, then $\mathrm{Tab}_q(i_1,\ldots,i_k|j_1,\ldots,j_k) = 0$ whenever the $i_1, \ldots i_k$ are not distinct. If they are distinct, then*

$$Tab_q(i_{\rho(1)},\ldots,i_{\rho(k)}|j_1,\ldots,j_k) = (-q)^{-l(\rho)}\, Tab_q(i_1,\ldots,i_k|j_1,\ldots,j_k). \qquad \Box$$

We require some notation for indexing products of determinants.

Definition 2.6 Denote the quantum determinant $\mathrm{Tab}_q(i_1,\ldots,i_k|j_1,\ldots,j_k)$ by $[i_1,\ldots,i_k|j_1,\ldots,j_k]_q$. If $u_1,\ldots,u_k$, $\mu_1 < \cdots < \mu_k$, $v_1,\ldots,v_{k'}$, and $\nu_1 < \cdots < \nu_{k'}$ are sequences of indices, then let

$$\left[\begin{array}{c|c} u_1,\ldots,u_k & \mu_1,\ldots,\mu_{k'} \\ v_1,\ldots,v_{k'} & \nu_1,\ldots,\nu_{k'} \end{array}\right]_q := [u_1,\ldots,u_k|\mu_1,\ldots,\mu_{k'}]_q \cdot [v_1,\ldots,v_{k'}|\nu_1,\ldots,\nu_{k'}]_q\,.$$

We generalize this notation in the obvious fashion to arrays (bitableaux) with $m > 2$ rows indexing products of m quantum determinants.

The following proposition follows directly from the exchange lemma (Lemma 10 of [HZ94]) or from suitable manipulation of the quantum Laplace expansion found in [TfTo91].

Proposition 2.7 *Define $t = k + j + i - s$. If j exceeds the cardinality of $\{\mu_1,\ldots,\mu_s,\nu_1,\ldots,\nu_t\}$, then*

$$\sum_{\sigma\in S_j} (-q)^{-l(v_{\sigma(1)},\ldots,v_{\sigma(j)})} \left[\begin{array}{c|c} u_1\cdots u_i v_{\sigma(1)}\cdots v_{\sigma(s-i)} & \mu_1\cdots\mu_s \\ v_{\sigma(s-i+1)}\cdots v_{\sigma(j)} w_1\cdots w_k & \nu_1\cdots\nu_t \end{array}\right]_q = 0$$

where the sum ranges over σ such that $\sigma(1) < \cdots < \sigma(s-i)$, $\sigma(s-i+1) < \cdots < \sigma(j)$, and where $u_1 < \cdots < u_i$, $w_1 < \cdots < w_k$, $\mu_1 < \cdots < \mu_s$, $\nu_1 < \cdots < \nu_t$ and the entries $v_1, \cdots, v_j$ are distinct. $\Box$

We record the following generalization of the row commutation relation from [TfTo91].

Proposition 2.8 *Suppose that $\{l_1,\ldots,l_j\} \supseteq \{\nu_1,\ldots,\nu_t\} \supseteq \{i_1,\ldots,i_r\}$ that $s = j - r$, that $k = t - r$ and that each sequence, $\underline{v}, \underline{w}, \underline{i}, \underline{\nu}$, strictly increases. The exchange lemma of [HZ94] implies*

$$\sum_{\sigma\in S_j} (-q)^{-l(\sigma)} \left[\begin{array}{c|c} v_{\sigma(1)}\cdots\cdots\cdots\cdots v_{\sigma(s)} & l_1\cdots\hat{i_1}\cdots\hat{i_r}\cdots l_j \\ v_{\sigma(s+1)}\cdots v_{\sigma(j)} w_1\cdots w_k & \nu_1\cdots\cdots\cdots\cdots\nu_t \end{array}\right]_q$$

$$= (-q)^{-l(\tau)} \left[\begin{array}{c|c} v_1\cdots\cdots v_j & l_1\cdots\hat{i_1}\cdots\hat{i_r}\cdots l_j\, i_1\cdots i_r \\ w_1\cdots w_k & \nu_1\cdots\hat{i_1}\cdots\hat{i_r}\cdots\nu_t \end{array}\right]_q$$

$$= (-q)^{-l(\rho)-l(\tau)} \left[\begin{array}{c|c} v_1\cdots\cdots v_j & l_1\cdots\cdots\cdots\cdots\cdots\cdots l_j \\ w_1\cdots w_k & \nu_1\cdots\hat{i_1}\cdots\hat{i_r}\cdots\nu_t \end{array}\right]_q$$

where $\sigma(1) < \cdots < \sigma(s)$, $\sigma(s+1) < \cdots < \sigma(j)$, $\rho = l_1 \cdots \hat{i_1} \cdots \hat{i_r} \cdots l_j i_1 \cdot i_r$ *and* $\tau = i_1 \cdots i_r \nu_1 \cdots \hat{i_1} \cdots \hat{i_r} \cdots \nu_t$. □

In this paper, we shall require the special case in which $\nu_1, \ldots, \nu_t = l_1, \ldots, l_j = c_0, c_0+1, \ldots, c_3$ and $i_1, \ldots, i_r = c_0, c_0+1, \ldots, c_1 - 1, c_2+1, \ldots, c_3 - 1, c_3$ with $c_0 \leq c_1 \leq c_2 \leq c_3$:

$$\sum_{\sigma \in S_j} (-q)^{-l(\sigma)} \left[\begin{array}{l|l} v_{\sigma(1)} \cdots\cdots\cdots v_{\sigma(s)} & \;c_1 \;\cdots\; c_2 \\ v_{\sigma(s+1)} \cdots v_{\sigma(j)} w_1 \cdots w_k & c_0 \cdots\cdots\cdots\cdots c_3 \end{array} \right]_q$$

$$= (-q)^{-l(\rho)-l(\tau)} \left[\begin{array}{l|l} v_1 \cdots\cdots v_j & c_0 \cdots\cdots\cdots c_3 \\ w_1 \cdots w_k & \;c_1 \;\cdots\; c_2 \end{array} \right]_q$$

where $\sigma(c_0) < \cdots < \sigma(s)$, $\sigma(s+1) < \cdots < \sigma(c_3)$, $\rho = c_1, \ldots, c_2, c_0, \ldots, c_1 - 1, c_2 + 1, \ldots, c_3$ and $\tau = c_0, c_0+1, \ldots, c_1 - 1, c_2+1, \ldots, c_3 - 1, c_3, c_1, \ldots, c_2$ and $a_1, \ldots, \hat{a}_i, \ldots, a_h$ denotes the sequence $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_h$.

A similar computation derives the following.

Proposition 2.9 *Given* $c_0 < c_1 \leq c_2 \leq c_3$ *where* $s = c_3 - c_0 + 1$, $c_2 - c_1 + 1 = j - s + i$, *we have*

$$\sum_{\sigma \in S_j} (-q)^{-l(\sigma)} \left[\begin{array}{c|c} u_1 \;\ldots\; u_i \;\; v_{\sigma_1} \;\ldots\; v_{\sigma_{s-i}} & c_0 \;\ldots\ldots\ldots\; c_3 \\ v_{\sigma_{s-i+1}} \;\ldots\; v_{\sigma_j} & c_1 \;\ldots\; c_2 \end{array} \right]_q =$$

$$= (-q)^{N} \left[\begin{array}{c|c} u_1 \;\ldots\; u_i & c_1 \;\ldots\; c_2 \\ v_1 \;\ldots\ldots\; v_j & c_0 \;\ldots\ldots\ldots\; c_3 \end{array} \right]_q$$

where $N = -l(c_1, \ldots, c_2, 1, \ldots, s) - l(c_0, \ldots, c_1 - 1, c_2 + 1, \ldots, c_3, c_1, \ldots, c_2)$, $\sigma_1 < \cdots < \sigma_{s-i}$ *and* $\sigma_{s-i+1} < \cdots < \sigma_j$. □

The two preceding identities are minor variations on results of [TfTo91].

3 Quantum Schur modules

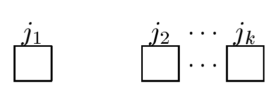
We define a quantized version of the Schur modules associated to a generalized shape in [ABW82]. The definition parallels the construction of ordinary Schur modules as the span of certain products of determinants.

Definition 3.1 A *shape* is a subset of cells in a matrix. A tableau of shape D on an index set $\mathcal{L}$ is a labelling of each cell in D by an element of $\mathcal{L}$.

We often draw a shape D by placing a box in the matrix cells which comprise D and write a tableau by writing the labels inside the matrix cells they label.

Definition 3.2 If D is the single-rowed shape

Column #:



with cells in columns $j_1 < \cdots < j_k$ and T is a tableaux

$$\boxed{i_1} \qquad \boxed{i_2} \cdots \boxed{i_k}$$

of shape D then define $[T]_q$ to be $\mathrm{Tab}_q(i_1, \ldots, i_k | j_1, \ldots, j_k)$.

In general suppose D is a shape with k rows and $b_{r,1} < \cdots < b_{r,l_r}$ are the column indices of the cells appearing in the rth row of D. If T is a tableau of shape D and $a_{i,j}$ is the contents of cell (i,j) in D, then define

$$[T]_q = \prod_{i=1}^{k} \mathrm{Tab}_q(a_{r,b_{r,1}}, \ldots, a_{r,b_{r,l_r}} | b_{r,1}, \ldots, b_{r,l_r}) .$$

Definition 3.3 Let $\mathcal{L} \subset \mathbf{Z}^+$ be an index set. Define $\mathcal{S}_q^D(\mathcal{L})$, the quantized or *q-Schur module* of shape D, to be the $\mathbf{Z}[q, q^{-1}]$-span of $[T]_q$ where T runs over all tableaux T of shape D, filled with elements of $\mathcal{L}$.

We first define a version of the diagonal term order from [Stu93] suitable for use with quantum determinants. See [T97a] for a treatment of noncommutative SAGBI bases that puts this term order in a broader context.

Definition 3.4 Let $\prec_{\mathrm{diag}}$ be the partial order defined on ordered monomials in $Mat_q(t_{i,j})$ by $\prod_l t_{i_l,j_l} \prec_{\mathrm{diag}} \prod_l t_{i'_l,j'_l}$ when $\underline{j} = \underline{j}'$ and $\underline{i} < \underline{i}'$ in lexicographic order.

For $p \in Mat_q(t_{i,j})$, define the *initial monomial*, $\mathrm{init}_{\prec_{\mathrm{diag}}}(p)$, of p to be the smallest monomial (under $\prec_{\mathrm{diag}}$) appearing in the expansion of p into the basis of ordered monomials. Likewise define the *leading term*, $\mathrm{LT}_{\prec_{\mathrm{diag}}}(p)$, of p to be $\mathrm{init}_{\prec_{\mathrm{diag}}}(p)$ multiplied by its coefficient in the expansion.

Definition 3.5 Define a biword $\substack{b_1,\ldots,b_l \\ a_1,\ldots,b_l}$ to be *sorted* when its upper sequence weakly increases and the lower sequence has a descent wherever the upper sequence has an equality. Define sort $\left(\substack{b_1,\ldots,b_l \\ a_1,\ldots,b_l}\right)$ to be the unique sorted biword obtainable by permuting the columns of $\substack{b_1,\ldots,b_l \\ a_1,\ldots,b_l}$.

When T is a tableau define c_T, the *column word* of T, to be the entries of T read column by column from left to right and from bottom to top within each column. Define the *column biword* $\mathbf{c}_T$ to be the biword whose bottom word is c_T and whose top word is $c_{Der^-(T)}$, where $Der^-(T)$ is the tableau having the same shape as T and whose ith column is filled entirely with i's.

Define the *modified column biword* $\mathbf{w}_T$ to be $\mathrm{sort}(\mathbf{c}_T)$ and define the *modified column word* w_T of T to be the bottom word of $\mathbf{w}_T$.

Definition 3.6 To any biword $\substack{j_1,\ldots,j_l \\ i_1,\ldots,i_l}$, we associate a monomial $\Psi\left(\substack{j_1,\ldots,j_l \\ i_1,\ldots,i_l}\right) = t_{i_1,j_1} \cdots t_{i_l,j_l}$.

The map Ψ is a bijection from sorted biwords to ordered monomials. Further, if M, M' are ordered monomials then $M < M'$ iff the lower word of $\Psi^{-1}(M)$ is lexicographically smaller than the lower word of $\Psi^{-1}(M)$ and their upper words are equal.

The initial monomial of a tableau T with strictly increasing rows (a *row-standard* tableau) can be read from $\mathbf{w}_T$.

Proposition 3.7 *If T is a row-standard tableau, then $\Psi^{-1}(\text{init}\,([T]_q)) = \mathbf{w}_T$.*

Before the proof we collect some useful lemmas.

Lemma 3.8 *The initial term of the q-determinant of any minor indexed by rows $i_1 < \cdots < i_k$ and columns $j_1 < \cdots < j_k$ is $\prod_{c=1}^{k} t_{i_c,j_c}$.* □

Lemma 3.9 *Suppose p, p' are in $Mat_q(t_{i,j})_{i=1\ldots m,\, j=1\ldots n}$ and suppose all monomials in p, p' are comparable under $\prec_{\text{diag}}$. If $\text{init}\,(p) = \prod_{j=1\ldots n} \prod_{i=m\ldots 1} t_{i,j}^{a_{i,j}}$ and $\text{init}\,(p') = \prod_{j,i} t_{i,j}^{b_{i,j}}$ then $\text{init}\,(pp') = \prod_{j,i} t_{i,j}^{a_{i,j}+b_{i,j}}$. Further, $\text{LT}\,(pp')$ is equal (up to mutliplication by a unit) to $\prod_{j,i} t_{i,j}^{a_{i,j}+b_{i,j}}$.*

Proof. We first prove the lemma when p, p' are ordered monomials, by the following observation: The biwords corresponding to the ordered monomials which appear in the expansion pp' are all produced from $\text{sort}(W)$, where W is the biword produced by concatenating $\Psi^{-1}(p)$ and $\Psi^{-1}(p')$, by successively creating inversions in the lower word $\text{sort}(W)$. (Each inversion created corresponds to an application of the commutation rule for $t_{i,k}$ and $t_{j,l}$ for $i < j$ and $k < l$.) Since the parts of the commutation rules that leave $\text{sort}(W)$ unchanged only change the monomial being reduced by multiplying by a unit, the leading *term* result also follows.

For general p, p' it suffices to observe that if $\frac{a}{\underline{v}}$, $\frac{b}{\underline{w}}$, $\frac{b}{\underline{w'}}$ are sorted biwords with $\underline{w} < \underline{w'}$ then the bottom word of $\text{sort}(\frac{a,b}{\underline{v},\underline{w}})$ is smaller than the bottom word of $\text{sort}(\frac{a,b}{\underline{v},\underline{w'}})$. □

The preceding lemma has the following corollary.

Lemma 3.10 *We have* $\text{init}\left(\Psi\left(\frac{j}{\underline{i}}\right)\Psi\left(\frac{j'}{\underline{i'}}\right)\right) = \text{sort}\left(\frac{\cdots j \cdots;\, \cdots j' \cdots}{\cdots \underline{i} \cdots;\, \cdots \underline{i'} \cdots}\right)$. □

Proof.(of Proposition 3.7) Lemma 3.8 proves the result for one-rowed tableaux. Applying the one-rowed case to each quantum minor in the product $[T]_q$ and combining the initial monomials using Lemma 3.10 proves the general version. □

Definition 3.11 When T is row-standard, define $\text{Tab}_q^{\prec}(T)$ to be the unique scalar multiple of $[T]_q$ such that $\text{init}\,([T]_q) = \text{LT}\left(\text{Tab}_q^{\prec}(T)\right)$.

Proposition 3.12 *If T is row-standard of shape D, and letting*

$$l(t) = \#\{(i,j),(i',j') \in D \times D \mid i > i', j < j', T_{i,j} = T_{i',j'}\} \\ - \#\{(i,c),(i,c') \in D \times D \mid i > i', T_{i',c} > T_{i,c}\}$$

then $\text{Tab}_q^{\prec}(T) = q^{l(t)}[T]_q$.

4 Straight bases for quantum Schur modules

We recall the definition of a straight tableau (for a Schur module) from [T].

A shape D is said to be *row-convex* if for all $i < j < k$ the cell (r, j) is in D whenever $(r, i), (r, k) \in D$. A row-convex shape is *sorted* (equivalently *northeast*) if when row r_1 ends in column c_1 and row r_2 in column c_2, we have $r_1 < r_2$ implies $c_1 \geq c_2$. Let T be a tableau of sorted, row-convex shape D and let $T_{r,c}$ be the entry in cell (r, c) of T. We define T to be *straight* when its rows strictly increase (i.e. T is *row-standard*) and when for $r_1 < r_2$ we have $T_{r_1,c} > T_{r_2,c}$ implies $(r_1, c-1) \in D$ and $T_{r_1,c-1} > T_{r_2,c}$.

A pair of cells $\{(r_1, c), (r_2, c)\}$ with $r_1 < r_2$ and $T_{r_1,c} > T_{r_2,c}$ is an *inversion* in T. An inversion $\{(r_1, c), (r_2, c)\}$ is *flippable* when it fails to satisfy the preceding condition, $(r_1, c-1) \in D$ and $T_{r_1,c-1} > T_{r_2,c}$.

Proposition 4.1 *Let D be a sorted, row-convex shape, and let n be the column index of the rightmost cell(s) in D. The elements $[T]_q$ of $Mat_q(t_{i,j})_{i=1,\dots,m;\, j=1\dots,n}$ indexed by all straight tableaux T of shape D filled with letters $\{1, \dots, m\}$ are linearly independent.*

The proposition remains true if $Mat_q(t_{i,j})$ is replaced by $Mat_q^{\mathbf{k}}(t_{i,j})$.

Proof. As shown in [T], if T, T' are distinct straight tableaux of shape D, then $w_T \neq w_{T'}$. Thus by Proposition 3.7 the initial terms of the $[T]_q$ corresponding to straight tableaux are distinct, hence they are linearly independent.

By Proposition 2.3 linear independence also holds over $\mathbf{k}$. $\square$

The straightening law of [T] relied on being able to straighten a pair of rows at a time. We define an analogous straightening relation.

Lemma 4.2 *Let m_1, m_2 and $c_1 \leq c_2 \leq \lambda_2 \leq \lambda_1$ be column indices with $\min(m_1, m_2) \leq c_1$. Further suppose that $m_1 \leq m_2$ or $m_2 = c_1$. Let $T = \begin{smallmatrix} u_{m_1} & \cdots & u_{\lambda_1} \\ v_{m_2} & \cdots & v_{\lambda_2} \end{smallmatrix}$ be a tableau of sorted, row-convex, two-rowed shape given, when $m_1 \leq m_2$, by*

$$T = \begin{array}{cccccccccccc} v'_{m_1} & \cdots & \cdots & v'_{c_1-1} & u_{c_1} & \cdots & u_{c_2-1} & v'''_{c_2} & \cdots & \cdots & \cdots & v'''_{\lambda_1} \\ & w_{m_2} & \cdots & w_{c_1-1} & v''_{c_1} & \cdots & & v''_{c_2} & w_{c_2+1} & \cdots & v_{\lambda_2} & \end{array} .$$

Let $m = \min(m_1, m_2)$. If $m_1 \leq m_2$, then

$$[T]_q = \sum_{\sigma \in S_d} (-q)^{-n_\sigma + n_e} [T_\sigma]_q$$

where σ is nontrivial, $\sigma_1 < \cdots < \sigma_{c_1-m}$, $\sigma_{c_1-m+1} < \cdots < \sigma_{c_2-m+1}$, $\sigma_{c_2-m+1} < \cdots < \sigma_{\lambda_1-m}$, and

$$T_\sigma = \begin{array}{cccccccccccc} v_{\sigma_1} & \cdots & \cdots & v_{\sigma_{a-1}} & u_{c_1} & \cdots & u_{c_2-1} & v_{\sigma_{b+1}} & \cdots & \cdots & \cdots & v_{\sigma_d} \\ & w'_{m_2} & \cdots & w'_{c_1-1} & v_{\sigma_a} & \cdots & & v_{\sigma_b} & w''_{c_2+1} & \cdots & w''_{\lambda_2} & \end{array}$$

with $a = c_1 - m + 1$, $b = c_2 - m + 1$, and $d = \lambda_1 - m + 2$ and where

$$n_\sigma = l(\underline{v}'_\sigma, \underline{u}) - l(\underline{u}, \underline{v}'_\sigma) + l(\underline{w}', v''_\sigma) - l(v''_\sigma, \underline{w}') + l(v'_\sigma, v'''_\sigma, v''_\sigma)$$

with $\underline{v} = \underline{v}', \underline{v}'', \underline{v}'''$; $\underline{v}'_\sigma = v_{\sigma_1}, \ldots, v_{\sigma_{a-1}}$; $\underline{v}''_\sigma = v_{\sigma_a}, \ldots, v_{\sigma_b}$; and $\underline{v}'''_\sigma = v_{\sigma_{b+1}}, \ldots, v_{\sigma_d}$.

If $m_1 > m_2 = c_1$, we maintain the notation used when $m_1 \le m_2$, only letting $v'_{m_1}, \ldots, v'_{c_1-1}$, $w_{m_2}, \ldots, w_{c_1-1}$ and $u_{c_1}, \ldots, u_{m_1-1}$ be empty sequences. Thus $\underline{u}$ reduces to $u_{m_1}, \ldots, u_{c_2-1}$. We have

$$[T]_q = \sum_\sigma (-q)^{-n_\sigma + n_e} [T_\sigma]_q$$

where n_σ specializes to $n_\sigma - l(v'''_\sigma, v''_\sigma)$.

Proof. We apply Proposition 2.5 to q-commute the letters into the format required by Lemma 2.7 and check that the size of $\underline{v}$ is larger than the number of columns in D. □

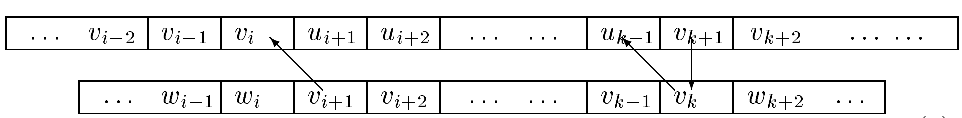
Informally, we describe the relation in the preceding lemma as a q-symmetrization of the entries $\underline{v}$ in T.

Proposition 4.3 *If D is a sorted two-rowed, row-convex shape then $\mathcal{S}_q^D(\mathcal{L}^-)$ has a basis consisting of all elements $[T]_q$ indexed by the straight tableaux of shape D filled with elements of $\mathcal{L}$.*

Proof. At first, let us assume that the top row contains the bottom row.

We start by briefly describing the two-rowed straightening algorithm. Suppose that T is a row-standard but non-straight tableaux. Let k be the leftmost column in which T has a flippable inversion. Let $i+1$ be the column of the leftmost cell in the bottom row of T such that the contents of that cell are bigger than the contents of the cell in column i of the top row.

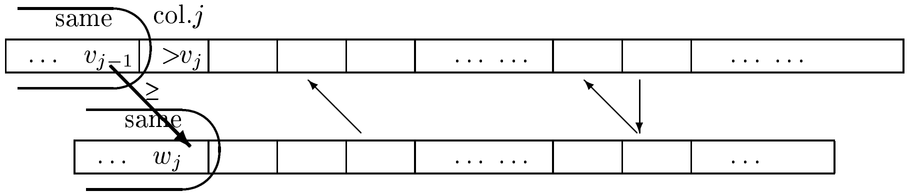
Now use Lemma 4.2 to replace $[T]_q$ with a linear combination of $[T_\sigma]_q$ by q-symmetrization of the entries in all cells of the top row, except those in columns $i+1$ through $k-1$, together with the contents of all cells in the bottom row that fall in columns $i+1$ through k. Suppose T is as pictured below,



(1)

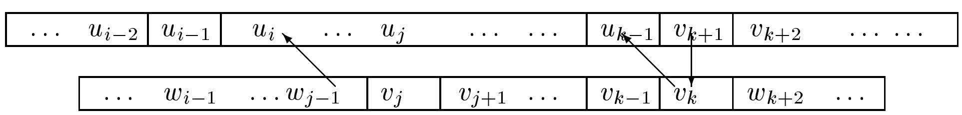
then the strictly increasing sequence $\ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_k, v_{k+1}, \ldots$ is being q-symmetrized. Now resort the rows using Proposition 2.5, picking up powers of $-q$ as needed. To finish the proof of Proposition 4.3, it suffices to prove that each such application of Lemma 4.2 produces only tableaux T_σ with strictly larger column word than the column word of T. Let $j = \min_{\sigma(v_c) \neq v_c} c$. The proof decomposes into three cases.

Case I. ($j \leq i$)
The tableau T_σ is partly described by,

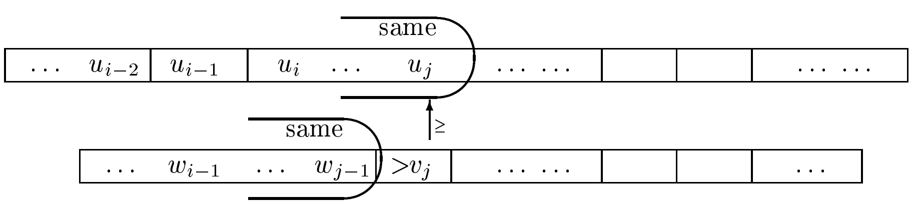


where *the arrows of equation 1 have been copied solely to aid comparison.* The circled entries in the top row remain the same since applying σ only brings elements larger than v_j to the top row. The circled entries in the bottom row remain the same because to change them, the permutation σ would have had to bring some element x smaller than w_j down from some position weakly right of column j in the top row. But since by construction $v_{j'-1} \geq w_{j'}$ for all $j' \leq i$, we have $x > v_{j-1} \geq w_j$. As the entry in cell j of the top row increased upon permuting and only further increased on resorting the rows, we find that the column word increases.

Case II. ($i < j < k$)
In this case T_σ is a tableau arising from permuting $v_j, v_{j+1}, \cdots$ in the following picture, where again the arrows indicate the relations that held in T:



Since all the permuted entries are strictly larger than w_{j-1}, row-sorting does not move the first $j-1$ entries in the bottom row of T. Further since k was the column of the leftmost flippable inversion, we must have had that the top cell in column j of T had a value less than or equal to the value of the bottom cell–otherwise, there exists a flippable inversion left of column k since there exists an inversion left of column k and the leftmost inversion in a pair of rows is always flippable. This information is recorded in the following picture of T_σ:



where the arrows indicate the relations that held in T. The inequality holding in column j guarantees that none of the v's that are moved up to the top row can be resorted into the first j cells of that row–thus we can guarantee that those cells are indeed unchanged. Since all entries in the bottom row right of w_{j-1} are strictly bigger than v_j, the bottom cell of column j increases and we conclude that the column word increases.

Case III. ($j = k$)

The last case is particularly simple. We are guaranteed that T and T_σ agree in their first $k-1$ columns and the bottom rows of T and T_σ differ precisely by replacing v_k with some $v_l > v_k$. Since the bottom element of column k increases even before row-sorting, we are done.

If the bottom row ends at least as far left as the top row, i.e. shape D is skew, then we set $i+1$ to be the first cell of the bottom row. The above arguments follow identically in Cases II and III; Case I is now superfluous. □

Porism 4.4 *If T is a row-standard tableau of two-rowed, sorted, row-convex shape D and if $\sum_\iota \alpha_\iota [T^{(\iota)}]_q$ is the expansion of $[T]_q$ in terms of straight tableaux $T^{(\iota)}$, then $c_T < c_{T^{(\iota)}}$ and $w_T \leq w_{T^{(\iota)}}$ in lexicographic order.* □

If D is skew, we can duplicate the classical straightening procedure by applying the preceding syzygy to any non-straight tableau and iterating. However, unlike straightening for skew tableaux, the row-convex straightening law of [T] required that the straightening syzygies be applied when two not-necessarily adjacent rows form a non-straight tableau. The next several lemmas circumvent the problems caused by the fact that quantum minors do not necessarily quasi-commute. We choose the pair of rows forming a non-straight subdiagram to be sufficiently close together, and show how to pull out the intervening rows, apply the straightening syzygy, and then put the relevant rows back.

First observe that duplicating the proofs of Lemma 4.2 and Proposition 4.3 proves the following.

Proposition 4.5 *Let D be a sorted row-convex shape. Let $\bar{D}$ be the shape obtained by interchanging the rows of D. For any tableau T of shape D, let $\bar{T}$ be the shape $\bar{D}$ tableau obtained by interchanging the rows of T. $\mathcal{S}_q^{\bar{D}}(\mathcal{L})$ has a basis consisting of all elements $[\bar{T}]_q$ as T ranges over all straight tableaux of shape D filled from $\mathcal{L}$.* □

Lemma 4.6 *Suppose that $T = \begin{smallmatrix} \cdots T_1 \cdots \\ \cdots T_2 \cdots \end{smallmatrix}$ is a row-standard tableau of two-rowed, row-convex shape D. Further suppose that one row of D contains the other. Under these assumptions, and maintaining the notation of Proposition 4.5, $\mathrm{Tab}_q(T)$ can be written as $\mathrm{Tab}_q(\bar{T}) + \sum_\iota \alpha_\iota \mathrm{Tab}_q(\bar{S}^{(\iota)})$ where each tableau $\bar{S}^{(\iota)}$ is row-standard of shape $\bar{D}$ and has strictly larger modified column word than T has.*

Proof. Suppose that D is sorted. Because all straight tableaux have different modified column words, the basis elements $[T]_q$, or alternately $\mathrm{Tab}_q(T)$, found in Proposition 4.3 must have different initial terms. By the identity following Proposition 2.8 we can write $\mathrm{Tab}_q(T)$ as a linear combination of $\mathrm{Tab}_q(\bar{T}^{(\kappa)})$ where the tableaux $\bar{T}^{(\kappa)}$ have shape $\bar{D}$. Either the $\bar{T}^{(\kappa)}$ or (when D is sorted) the $T^{(\kappa)}$ may be taken to be straight. Taking initial terms it is clear that each $\bar{S}^{(\iota)}$ has lexicographically weakly larger modified column word than T. □

We call a diagram *weakly skew* when it is either a skew shape or a partition shape.

Lemma 4.7 *Suppose that D is a sorted, row-convex diagram in which two rows $r < s$ form a weakly skew subdiagram and suppose there exists no intermediate row $r < i < s$ such that those three rows form a weakly skew subdiagram of D. There exists a diagram D' obtained by permuting the rows of D such that $\mathcal{S}_q^D = \mathcal{S}_q^{D'}$ and such that under this permutation the rows r and s end up adjacent with r still preceding s.*

Proof. The rows lying between r and s fall come in two (not necessarily disjoint) types. Rows of type A are contained in row r and rows of type B contain row s. This implies that any row of type B appearing above a row of type A contains the type A row. Thus by the equation after Proposition 2.8, we can permute the rows of type A up to before row r; start with the northmost such row and move it up by successive interchanges, then move up the second northmost row of type A etc. Once this is done, carry out an analogous process (starting with the southernmost row of type B) to move the rows of type B south of row s. □

Lemma 4.8 *Suppose D is a sorted, row-convex diagram. Suppose $r < s$ are rows of D where row r contains row s. There exists a shape D' obtained by permuting the rows of D such that rows r and s end up adjacent in D' and such that $\mathcal{S}_q^D = \mathcal{S}_q^{D'}$.*

Proof. Again we construct the permutation by sequentially interchanging adjacent rows for which one row contains the other. The intervening rows are of type A–which are contained in row r and type B–which fail to be contained in row r. Any row of type B appearing above a row of type A contains that row and any row of type B certainly contains row s. We permute the rows of type A up and then permute the rows of type B down just as in the previous lemma. □

Porism 4.9 *Let diagrams D and D' be given as in either Lemma 4.7 or Lemma 4.8 and let π be the permutation taking D to D' which was constructed in the proof of the lemma. Any row-standard tableau T of shape D (respectively D') can be rewritten as a linear combination $Tab_q(T') + \sum_\iota \alpha_\iota Tab_q(T^{(\iota)})$ of tableaux of shape D' (respectively D) where each $w_{T^{(\iota)}}$ is strictly larger than w_T, and where T' is obtained by using π (respectively π^{-1}) to permute the rows of T.*

Proof. Apply Lemma 4.6. □

These lemmas allow enough quasi-commutation modulo an appropriate filtration to push through the straightening algorithm.

Theorem 4.10 *Suppose D is a sorted row-convex shape. The expressions $[T]_q$ indexed by all straight tableaux of shape D with entries in $\mathcal{L}$ form a basis for $\mathcal{S}_q^D(\mathcal{L})$.*

Proof. By Proposition 4.1 it suffices to prove spanning. The spanning algorithm is now easy. Present any element of $\mathcal{S}_q^D$ as a linear combination of $[T]_q$ (or $\mathrm{Tab}_q(T)$) where T ranges over some collection of row-standard tableaux of shape D. For any non-straight tableaux T pick a pair of rows $r < s$ that form a non-straight subtableau. If row r contains the lower row, then use Lemma 4.8 and Porism 4.9 to rewrite $\mathrm{Tab}_q(T)$ as a $\mathrm{Tab}_q(T') + \sum_\iota \alpha_\iota \mathrm{Tab}_q(T^{(\iota)})$ so that T' arises from permuting the rows of T in some fashion that leaves rows r immediately above row s. Note that the shape of T' need not be sorted. Since r, s are now adjacent, we can apply Proposition 4.3 to rewrite these rows in T' as a linear combination where these rows have larger column word and weakly larger modified column word. Reapplying Lemma 4.8 and Porism 4.9 to all of these tableaux gives a linear combination of tableaux where all but one have modified column words that are strictly larger than that of T. The tableau with the same modified column word has had its column word increased.

The same argument goes through, using Lemma 4.7 rather than Lemma 4.8, if non-straight rows $r < s$ form a strictly skew subtableau. We are guaranteed that we can meet the extra conditions of Lemma 4.7 since any three-rowed skew subtableaux of D in which the first and third rows are non-straight (and thus non-standard) must have a non-straight subtableaux formed by either the first and second rows or the second and third rows. □

Porism 4.11 *If T is a row-standard tableau of sorted, row-convex shape D and if $\sum_\iota \alpha_\iota [S^{(\iota)}]_q$ is the expansion of $[T]_q$ in terms of shape D straight tableaux, $S^{(\iota)}$, then $w_T \leq w_{S^{(\iota)}}$ and if $w_T = w_{S^{(\iota)}}$ then $c_T < c_{S^{(\iota)}}$ in lexicographic order.* □

Setting $q = 1$, the above straightening algorithm for Schur modules is simpler than the one presented in [T]. One consequence of this for homogeneous coordinate rings of a certain configuration varieties (see [LM97]) follows.

Corollary 4.12 *Let D be a row-convex shape, $\mathcal{L}$ an index set, and $R^D(\mathcal{L})$ the ring generated by $\mathcal{S}_{q=1}^D(\mathcal{L})$. Let I be the ideal of relations among the $[T]_{q=1}$ for all tableaux T of shape D. The ideal I has a Groebner basis consisting of the relations which say that $[T]$ vanishes when T has a row with repeated letters, permuting any two letters in one row of $[T]$ gives $-[T]$, skew symmetrizing k letters of T and T' within the product $[T][T']$ yields 0 whenever the k letters were chosen from a pair of rows occupying fewer than k columns.* □

The simplified straightening law holds only for Schur modules. For the Weyl and super-Schur modules treated in [T], the straightening law of this section sometimes fails to terminate.

5 Flagged straightening

Classically, (i.e. when $q = 1$) the straightening law descends to a flagged version. The classical flagged straightening law applies to representations of B_n, the group of lower triangular matrices. Further, when D is a rectangle, flagged straightening provided an Algebra with Straightening Law structure to the coordinate rings of Schubert and skew Schubert varieties in the Grassmannian (see [Sta76, Sta78].) In this section, we describe a flagged version of the straightening law presented above.

The quantum, flagged Schur modules will be constructed as subquotients of the algebra $Mat_q(t_{i,j})$. We first construct the appropriate quotient of $Mat_q(t_{i,j})$.

Definition 5.1 Let $\mathcal{L} = \{1, \ldots, n\}$ and let X be a generic $n \times n$ quantum matrix. Let $\underline{a}, \underline{b}$ be weakly increasing sequences of $1 \ldots n$ each having length n and such that $b_j \leq a_j$ for each j. Let $\mathcal{I}_{\underline{a},\underline{b}} = \{t_{i,j} | i < b_j \text{ or } i > a_j\}$. Define $\phi_{\underline{a},\underline{b}} : Mat_q(t_{i,j}) \rightarrow Mat_q(t_{i,j})/(\mathcal{I}_{\underline{a},\underline{b}})$ to be the natural projection.

Proposition 5.2 *Let $X = (t_{i,j})$ be a generic quantum matrix and let $X_{\underline{a},\underline{b}}$ be the $Mat_q(t_{i,j})/(\mathcal{I}_{\underline{a},\underline{b}})$-matrix arising as the image of X under componentwise application of $\phi_{\underline{a},\underline{b}}$. By construction, $X_{\underline{a},\underline{b}}$ is a quantum matrix; a basis for the algebra $Mat_q(t_{i,j})/(\mathcal{I}_{\underline{a},\underline{b}})$ is given by (the images of) all ordered monomials containing only variables $t_{i,j}$ outside of $\mathcal{I}$.*

Proof. It suffices to check that under application of the relations in Definition 2.1 and the relations $t_{i,j} = 0$ for $t_{i,j} \in \mathcal{I}$ any monomial in the free algebra generated by the $t_{i,j}$'s reduces a unique ordered monomial. Then by the Diamond Lemma (Theorem 1.2 of [Be78]) we know that the ordered monomials are linearly independent. Since we already know that reduction is unique without imposing the relations setting variables $\mathcal{I}$ to 0, it suffices to show that under this original reduction, any monomial divisible by some element of $\mathcal{I}$ reduces only to linear combinations of monomials each divisible by some element of $\mathcal{I}$. Only reduction using the last relation requires any work and the desired result is then immediate from the following: Given $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$ if either $X_{\underline{a},\underline{b}_{i,k}}$ or $X_{\underline{a},\underline{b}_{j,l}}$ is in $\mathcal{I}$ then at least one of $X_{\underline{a},\underline{b}_{i,l}}$ or $X_{\underline{a},\underline{b}_{j,k}}$ lies in $\mathcal{I}$. □

Definition 5.3 Let D be a sorted, row-convex shape. Define $(\mathcal{S}_q^D)_{\underline{a},\underline{b}}(\{1, \ldots, n\})$ to be the image of $\mathcal{S}_q^D(\{1, \ldots, n\})$ under the algebra homomorphism $\phi_{\underline{a},\underline{b}}$.

Define a tableau T to be $\underline{a}, \underline{b}$*-flagged* when any element in column j of T is weakly larger than b_j and weakly smaller than a_j.

Theorem 5.4 *Let D be a sorted, row-convex tableaux. The flagged, quantum Schur module $(\mathcal{S}_q^D)_{\underline{a},\underline{b}}(\{1, \ldots, n\})$ has a $\mathbf{Z}[q, q^{-1}]$-basis consisting of all $[T]_q$ where T ranges over all $\underline{a}, \underline{b}$-flagged straight tableaux of shape D filled with letters in $\{1, \ldots, n\}$.*

Proof. Since T being flagged implies that $\text{LT}\left(\phi_{\underline{a},\underline{b}}([T]_q)\right) = \phi_{\underline{a},\underline{b}}(\text{LT}\,([T]_q))$, Proposition 5.2 proves independence. The straightening algorithm of the preceding section shows that the collection of $[T]_q$ indexed by unflagged straight tableaux spans. To complete the proof it suffices to observe that if $\underline{i}$ and $\underline{j}$ are increasing and there exists $1 \leq c \leq k$ such that $i_c < b_c$ or $i_c > a_c$ then $\phi_{\underline{a},\underline{b}}(\text{Tab}_q(i_1, \ldots, i_k \mid j_1, \ldots, j_k))$ vanishes. □

6 A quantum branching rule

Throughout this section fix $\mathcal{L} = \{1, \ldots, n\}$ and fix $(t_{i,j})$ to be an $n \times n$ generic quantum matrix.

The constructions of Section 3 produce modules over $U_q(gl_n)$, the quantized universal enveloping algebra of gl_n. We describe the action of $U_q(gl_n)$ on $\mathcal{S}_q^D(\mathcal{L})$ explicitly and produce a branching rule for describing the action of $U_q(gl_{n-1})$ on $\mathcal{S}_q^D(\mathcal{L})$.

6.1 Quantum $U(gl_n)$

We start by recalling the definition of $U_q(gl_n)$; this definition, Proposition 6.1, and Proposition 6.2 all follow the exposition in [LeTh96].

The algebra $U_q(gl_n)$ is described, up to isomorphism, as the algebra generated by noncommuting variables e_i, f_i for $i = 1, \ldots, n-1$ and $q^{\epsilon_i}, q^{-\epsilon_i}$ for $i = 1, \ldots, n$ subject to the relations,

$$\begin{array}{lll} q^{\epsilon_i} q^{-\epsilon_i} = 1 & q^{\epsilon_i} e_i q^{-\epsilon_i} = q e_i & q^{\epsilon_i} f_i q^{-\epsilon_i} = q^{-1} f_i \\ q^{-\epsilon_i} q^{\epsilon_i} = 1 & q^{\epsilon_i} e_{i-1} q^{-\epsilon_i} = q^{-1} e_{i-1} & q^{\epsilon_i} f_{i-1} q^{-\epsilon_i} = q e_{i-1} \\ q^{\epsilon_i} q^{\epsilon_j} = q^{\epsilon_j} q^{\epsilon_i} & q^{\epsilon_i} e_j q^{-\epsilon_i} = e_j \text{ and } q^{\epsilon_i} f_j q^{-\epsilon_i} = e_j \text{ for } j \neq i, i-1. & \end{array}$$

Further, impose the relation

$$e_i f_j - f_j e_i = \delta_{i,j} \left(q^{\epsilon_i} q^{-\epsilon_{i+1}} - q^{-\epsilon_i} q^{\epsilon_{i+1}} \right) / (q - q^{-1})$$

where $\delta_{i,j}$ is the Kronecker delta. Finally, require that all e_i's with indices differing by at least two commute (similarly for the f_j's) and when $|i - j| = 1$ impose the relations

$$e_j e_i^2 - (q + q^{-1}) e_i e_j e_i + e_i^2 e_j \text{ and } f_j f_i^2 - (q + q^{-1}) f_i f_j f_i + f_i^2 f_j \,.$$

We require the following action of $U_q(gl_n)$ on $Mat_q^{\mathbf{Q}(q)}(t_{i,j})$

Proposition 6.1 *There is a representation of $U_q(gl_n)$ on $Mat_q^{\mathbf{Q}(q)}(t_{i,j})$ defined by the action of the generators on the quantum variables,*

$$q^{\epsilon_i} t_{k,l} = q^{\delta_{i,k}} t_{k,l}, \; q^{-\epsilon_i} t_{k,l} = q^{-\delta_{i,k}} t_{k,l}, \; e_i t_{k,l} = \delta_{i+1,k} t_{k-1,l}, \; f_i t_{k,l} = \delta_{i,k} t_{k+1,l},$$

where $\delta_{i,j}$ is the Kronecker delta and by the quantized Leibniz formulas,

$$\begin{array}{ll} q^{\epsilon_i}(gh) = (q^{\epsilon_i} g)(q^{\epsilon_i} h), & q^{-\epsilon_i}(gh) = (q^{-\epsilon_i} g)(q^{-\epsilon_i} h), \\ e_i(gh) = (e_i g) h + (q^{-\epsilon_i} q^{\epsilon_{i+1}} g)(e_i h), & f_i(gh) = (f_i g)(q^{\epsilon_i} q^{-\epsilon_{i+1}} h) + g(f_i h). \end{array}$$

Proposition 6.2 *Choose integers $1 \leq i_1 < \cdots < i_k \leq n$ and $1 \leq j_1 < \cdots < j_k \leq n$. The generators of $U_q(n)$ act on quantum minors as follows,*

$$q^{\pm\epsilon_i} \mathrm{Tab}_q(i_1, \ldots, i_k | j_1, \ldots, j_k) = \begin{cases} q^{\pm 1} \mathrm{Tab}_q(i_1, \ldots, i_k | j_1, \ldots, j_k) & \text{if } i \in \underline{i} \\ \mathrm{Tab}_q(i_1, \ldots, i_k | j_1, \ldots, j_k) & \text{otherwise} \end{cases}$$

$$e_i \mathrm{Tab}_q(i_1, \ldots, i_k | j_1, \ldots, j_k) = \begin{cases} \mathrm{Tab}_q(i_1, \ldots, i_{l-1}, i, i_{l+1}, \ldots | j_1, \ldots, j_k) & \text{if } i+1 = i_l \\ 0 & \text{if } i+1 \notin \underline{i} \end{cases}$$

$$f_i \mathrm{Tab}_q(i_1, \ldots, i_k | j_1, \ldots, j_k) = \begin{cases} \mathrm{Tab}_q(i_1, \ldots, i_{l-1}, i+1, i_{l+1}, \ldots | j_1, \ldots, j_k) & \text{if } i = i_l \\ 0 & \text{if } i \notin \underline{i} \end{cases} .$$

Since the straightening law in Section 4 only produces coefficients in $\mathbf{Z}[q,q^{-1}]$, the representation-theoretic results of this paper will hold for an appropriately chosen $\mathbf{Z}$-form of $U_q(gl_n)$. Accordingly, and loosely following [Jn96], we define $U_q(n) = U_{\mathbf{Z}[q,q^{-1}]}(gl_n)$ to be the $\mathbf{Z}[q,q^{-1}]$-subalgebra of $U_q(gl_n)$ generated by $e_j^k/[k]_q^!$, $f_j^k/[k]_q^!$, $q^{\pm\epsilon_i}$, and $\left[\begin{smallmatrix}\epsilon_i+a\\ k\end{smallmatrix}\right]_q$ for all $1 \le j \le n-1$, all $1 \le i \le n$, and all integers $k, a \ge 0$. Here $[k]_q^! = (1-q^k)/(1-q)$ is the usual q-factorial and $\left[\begin{smallmatrix}n\\ k\end{smallmatrix}\right]_q = [n]_q^! / ([k]_q^! [n-k]_q^!)$ is the usual q-binomial coefficient.

The following is an immediate consequence of Proposition 6.2 and the straight basis theorem.

Proposition 6.3 *Let D be a sorted, row-convex shape. The quantum Schur module, $\mathcal{S}_q^D(\mathcal{L})$, is a module over $U_q(n)$ with action descending from the action of $U_q(n)$ on $Mat_q(t_{i,j})$. The rank of $\mathcal{S}_q^D(\mathcal{L})$ over $\mathbf{Z}[q,q^{-1}]$ equals the rank of $\mathcal{S}_1^D(\mathcal{L})$ over $\mathbf{Z}$.* □

6.2 A filtration

The branching rule and filtration of [T] carry over to quantum Schur modules.

Definition 6.4 Let D be a sorted, row-convex shape with cells in some subset of columns $\{1,\dots,n\}$. A *vertical strip* V in D is any subset of the cells of D such that one obtains a straight tableau on letters $\{0,1,2,\dots,\}$ by putting 0 in the cells of V and putting into any other cell (r,c) that cell's column index, c.

For a vertical strip, V, define a sequence $\underline{p}(V)$ by letting $p_i(V)$ be the number of cells of V in column i.

For any diagram, E, let $|E|$ be the number of cells in E. For two vertical strips V, V' define $V \le V'$ (read V is weakly left of V') when $|V| = |V'|$ and $\sum_{i=1}^{j} p_i(V) \le \sum_{i=1}^{j} p_i(V')$ for all j, i.e. when $\underline{p}(V) \le \underline{p}(V')$ as compositions of $|V|$ under the dominance order.

As observed in [T], $\underline{p}(V)$ uniquely determines V.

Definition 6.5 Let D be a sorted, row-convex shape with cells in some subset of columns $\{1,\dots,n\}$ and let V be a vertical strip in D. Define $\mathcal{S}_q^{D>V}(\{0,\dots,m\})$ (respectively $\mathcal{S}_q^{D\ge V}(\{0,\dots,m\})$) to be the $\mathbf{Z}[q,q^{-1}]$-span of $[T]_q$ as T ranges over all vertical strips V' strictly (respectively weakly) right of V and over all straight tableaux of shape D on letters from $\{0,\dots,m\}$ such that 0 appears in precisely the cells of V'.

Proposition 6.6 *The $\mathbf{Z}[q,q^{-1}]$-modules $\mathcal{S}_q^{D>V}(\{0,\dots,m\})$ and $\mathcal{S}_q^{D\ge V}(\{0,\dots,m\})$ are $U_q(gl_m)$-representations.*

Proof. Let T be a straight tableau. By Proposition 6.2, the $U_q(gl_m)$-action expands $[T]_q$ into a linear combination of $[T^{(\iota)}]_q$ where the cells occupied by 0 in T and in the $T^{(\iota)}$ are identical. These $T^{(\iota)}$ are not necessarily straight, but they may be taken to be row-standard. Since application of the straightening law to a nonzero row-standard tableaux cannot move a 0 to a column further left than the one in which it started, the proposition follows. □

We can now state the branching rule.

Theorem 6.7 *Let D be a sorted, row-convex shape. Let the list $V_1, V_2, \ldots, V_h$ be a linear extension of the partial order on vertical strips of D. Let S_i be the span of the modules $\mathcal{S}_q^{D\geq V_1}(\{0,\ldots,m\}),\ldots,\mathcal{S}_q^{D\geq V_i}(\{0,\ldots,m\})$. The ith quotient S_i/S_{i-1} of the filtration $0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_h = \mathcal{S}_q^D(\{0,\ldots,m\})$ is isomorphic over $U_q(gl_m)$ to $\mathcal{S}_q^{D/V_i}(\{1,\ldots,m\})$. Here D/V is the shape formed by removing the cells of V from D.*

The proof amounts to the following result.

Proposition 6.8 *Let D be a sorted, row-convex shape and V a vertical strip contained in D. Over $U_q(gl_m)$, we have*

$$\mathcal{S}_q^{D\geq \underline{V}}(\{0,\ldots,m\})/\mathcal{S}_q^{D>\underline{V}}(\{0,\ldots,m\}) \simeq \mathcal{S}_q^{D/V}(\{1,\ldots,m\})$$

where D/V is the shape formed by removing V from D. The isomorphism, θ, is given by $\theta([T]_q) = [\tilde{T}]_q$ where $\tilde{T}$ is the tableau of shape D/V formed by removing all 0's from T.

Proof. We will show that θ is a $U_q(gl_m)$-map by constructing it as the restriction of a $U_q(gl_m)$-map θ' on a submodule of a multigraded piece of $Mat_q(t_{i,j})$.

We first define the appropriate submodule of $Mat_q(t_{i,j})$. We say that a monomial $t_{i_1,j_l}\cdots t_{i_k,j_k}$ *contains* the monomial $t_{i'_1,j'_1}\cdots t_{i'_{k'},j'_{k'}}$ if the multiset $\{\{t_{i_1,j_l},\ldots,t_{i_k,j_k}\}\}$ contains the multiset $\{\{t_{i'_1,j'_1},\ldots,t_{i'_{k'},j'_{k'}}\}\}$. Define $Mat_q^{D,\geq V}(t_{i,j})$ to be the submodule of $Mat_q(t_{i,j})$ spanned by all degree $|D|$ monomials containing $\prod_l t_{0,l}^{c_l}$ for some $\underline{c}$ such that $\underline{c} \geq \underline{p}(V)$ and not containing any monomial $\prod_l t_{0,l}^{c'_l}$ where $\sum_l c'_l > |V|$.

Let $M_V = \prod_l t_{0,l}^{p_l(V)}$. We define

$$\theta' : Mat_q^{D,\geq V}(t_{i,j})_{i=0,\ldots,m;\, j=1,2,\ldots} \to Mat_q(t_{i,j})_{i=1,\ldots,m;\, j=1,2,\ldots}$$

by requiring $\theta'(M\cdot M_V) = M$ when M is a degree $|D/V|$ (ordered) monomial in $Mat_q(t_{i,j})_{i=1,\ldots,m;\, j=1,2,\ldots}$. This can be verified by checking that the preceding property is preserved under applyication of the fourth commutation rule in Definition 2.1. Similarly, define $\theta'(M\cdot M_{V'}) = 0$ when V' is a vertical strip located strictly right of V. The kernel of this map is precisely the module spanned by all monomials appearing in the definition of $Mat_q^{D,\geq V}(t_{i,j})_{i=0,\ldots,m;\, j=1,2,\ldots}$ which contain $\prod_l t_{i_l,j_l}\cdot\prod_l t_{0,l}^{c_l}$ where for some l, $c_l > p_l(V)$. For the obvious reasons, we call this kernel $Mat_q^{D,>V}(t_{i,j})_{i=0,\ldots,m;\, j=1,2,\ldots}$.

Since $\mathcal{S}_q^{D\geq V}(\{0,\ldots,m\}) \subseteq Mat_q^{D,\geq V}(t_{i,j})_{i=0,\ldots,m;\, j=1,2,\ldots}$, we can construct a map $\theta : \mathcal{S}_q^{D\geq V}(\{0,\ldots,m\}) \to Mat_q(t_{i,j})_{i=1,\ldots,m;\, j=1,2,\ldots}$ as the restriction of θ'.

Now a basis of $\mathcal{S}_q^{D\geq V}(\{0,\ldots,m\})$ is given by all $[T]_q$ where T is straight and the vertical strip V' formed by the 0's in T is weakly right of V. If V' is strictly right of V, then all monomials in $[T]_q$ lie in $Mat_q^{D,>V}(t_{i,j})_{i=0,\ldots,m;\, j=1,2,\ldots}$, hence $\mathcal{S}_q^{D>V}(\{0,\ldots,m\}) \subseteq Mat_q^{D,>V}(t_{i,j})_{i=0,\ldots,m;\, j=1,2,\ldots} = \ker\theta'$.

A direct check, as above, of the commutation relations 2.1 for the $t_{i,j}$ shows that if T is straight of shape D and its 0's occupy the vertical strip V then $\theta([T]_q) = \theta'([T]_q) = [\tilde{T}]_q$ where $\tilde{T}$ is the tableau of shape D/V formed by removing all 0's from T. Since these $[\tilde{T}]_q$ are independent, we have that $\ker\theta = \mathcal{S}_q^{D>,V}(\{0,\ldots,m\})$.

□

Perusal of the preceding proofs show that the same branching rule holds under flagging.

Porism 6.9 *Maintain the notation of Theorem 6.7 and fix a weakly increasing sequence, $\underline{a}$, of $0,1,\ldots,m$. Define $S_{i_{\underline{a}}} = \phi_{\underline{a},\underline{0}}$ where $\phi_{\underline{a},\underline{0}}(S_i)$ is the flagging function of Section 5. The modules $S_{i_{\underline{a}}}$ are modules over the subalgebras $U_q(b_{n+1})$ and $U_q(b_n)$ of $U_q(gl_{n+1})$ generated over $\mathbf{Q}[q,q^{-1}]$ by variables $q^{\epsilon_i}, q^{-\epsilon_i}, f_i$ for $i=0,\ldots,n$ and $i=1,\ldots,n$ respectively. (Over $\mathbf{Z}[q,q^{-1}]$ one can substitute the appropriate $\mathbf{Z}$-form as on page 16.) The sequence $0 \subseteq S_{1_{\underline{a}}} \subseteq S_{2_{\underline{a}}} \subseteq \cdots \subseteq S_{h_{\underline{a}}} = \mathcal{S}_q^D(\{0,\ldots,m\})$ forms a filtration of $(\mathcal{S}_q^D)_{\underline{a},\underline{0}}(\{0,\ldots,m\})$ and the ith quotient S_i/S_{i-1} is isomorphic over $U_q(b_m)$ to $(\mathcal{S}_q^{D/V_i})_{\underline{a},\underline{0}}(\{1,\ldots,m\})$.* □

7 Acknowledgments

A version of the quantum straightening law presented here appeared as part of the author's thesis [T97a]. I am grateful to Gian-Carlo Rota for insisting that the results of [T] be generalized to quantum Schur modules.

References

[AB85] Kaan Akin and David A. Buchsbaum, Characteristic-Free Representation Theory of the General Linear Group, *Advances in Mathematics* vol. 58, (1985), 149-200.

[ABW82] Kaan Akin, David A. Buchsbaum, and Jerzy Weyman, Schur functors and Schur complexes, *Advances in Mathematics* vol. 44, (1982), 207-278.

[Be78] George M. Bergman , The diamond lemma for ring theory, *Advances in Mathematics* vol. 29, (1978), 178–217.

[B98] David A. Buchsbaum, Letter-place methods and homotopy, in *Proceedings of the Rotafest and Umbral Calculus Workshop* Birkhauser, Boston, 1998.

[BR00] David A. Buchsbaum and Gian-Carlo Rota, Approaches to Resolution of Weyl Modules, preprint, 2000.

[B98] David A. Buchsbaum, Letter-place methods and homotopy, in *Proceedings of the Rotafest and Umbral Calculus Workshop* Birkhauser, Boston, 1998.

[DRS76] Peter Doubilet, Gian-Carlo Rota, and Joel Stein, On the foundations of combinatorial theory IX: Combinatorial methods in invariant theory, *Studies in Applied Mathematics* vol. 53, (1976) 185-216.

[HZ94] Rosa Q. Huang, James J. Zhang, Standard basis theorem for quantum linear groups, *Adv. Math.* vol. 102, (1993) 202-229.

[Jn96] Jens Carsten Jantzen, *Lectures on Quantum Groups*, Amer. Math. Soc., Providence, 1996.

[Ji86] M. Jimbo, A q-analogue of $U(gl(N+1))$, Hecke algebra, and the Yang-Baxter equation, *Lett. Math. Phys.* **11** (1986), 247-252.

[Ka] Christian Kassel, *Quantum Groups.* Springer-Verlag, Berlin, 1995.

[K96] Mike Klucznik, *Thesis*, Brandeis University, 1996.

[K98] Michael Klucznik, Presentations of Schur complexes attached to row convex shapes. *Adv. Math.* **135** (1998), no. 1, 41–75.

[LM97] V. Lakshmibai and P. Magyar, Standard monomial theory for Bott-Samelson varieties. C. R. Acad. Sci. Paris Sér. I Math. vol. 324 (1997), 1211–1215.

[LeTh96] Bernard Leclerc and Jean-Yves Thibon, The Robinson-Schensted correspondence, crystal bases, and the quantum straightening at $q = 0$. *The Foata Festschrift, Electron. J. Combin.* **3**, No. 2, Research Paper 11, (1996).

[M98] P. Magyar, Borel-Weil theorem for configuration varieties and Schur modules. Adv. Math. vol. 134 (1998), 328–366.

[Ma88] Yuri Manin, *Quantum Groups and Noncommutative Geometry.* CRM Lecture Notes, Université de Montréal, Montréal, 1988.

[Ma91] Yuri Manin, *Topics in Noncommutative Geometry.* Princeton University Press, Princeton, NJ, 1991.

[RS95] Victor Reiner and Mark Shimozono, Specht Series for Column-Convex Diagrams, *Journal of Algebra* vol. 174, (1995), 489-522.

[RS97] V. Reiner and M. Shimozono, Straightening for standard monomials on Schubert varieties. J. Algebra vol. 195 (1997), 130–140.

[RS98] V. Reiner and M. Shimozono, Percentage-avoiding, northwest shapes and peelable tableaux. J. Combin. Theory Ser. A vol. 82 (1998), 1–73.

[RS96p] V. Reiner and M. Shimozono, Flagged Weyl modules for two-column shapes, preprint 1996.

[ReTaFa89] N. Yu. Reshetikhin, L. A. Takhtadjian, and L. D. Faddeev, Quantization of Lie groups and Lie algebras. *Algebra i Analiz* **1** (1989), 178-206. English translation: *Leningrad Math. J.* **1** (1990), 193-225.

[S97] M. Shimozono, Specht modules for column-convex diagrams: characteristic-free results for Weyl modules. J. Algebra vol. 192 (1997), 810–822.

[Sta76] Richard P. Stanley, Some combinatorial aspects of the Schubert calculus, in *Springer LNM 579: Combinatoire et représentation du groupe symmetrique, Strasbourg 1976.* Springer-Verlag, Berlin 1977.

[Sta78] Richard P. Stanley, Hilbert functions of graded algebras, *Advances in Math.* **28** (1978), 57-83.

[Stu93] Bernd Sturmfels, *Algorithms in Invariant Theory.* Springer-Verlag, Berlin, 1993.

[TfTo91] Earl Taft and Jacob Towber, Quantum deformation of flag schemes and Grassmann schemes I-A q-deformation of the shape-algebra for $GL(n)$, *Journal of Algebra* **142**, (1991), 1-36 .

[T97a] Brian D. Taylor, Generalized straightening laws for products of determinants, *Thesis*, MIT 1997.

[T97b] Brian D. Taylor, Straightening laws for row-convex tableaux, extended abstract in *Conference Proceedings: Formal Power Series and Algebraic Combinatorics '97*, Vienna 1997.

[T00] B. D. Taylor, Row-convex tableaux and the combinatorics of initial terms, *Discrete Mathematics* , (2000) -.

[T] Brian D. Taylor, A straightening algorithm for row-convex tableaux, submitted to *Journal of Algebra.*

[W94] David J. Woodcock, A vanishing theorem for Schur modules, *Journal of Algebra* **165**, (1994), 483-506.