1. (Fourier Series) Fill in the details below:

(a) For each $n \in \mathbb{N}$, set $D_n(\theta) = \sum_{k=-n}^{n} e^{ik\theta}$. For $f, g \in \mathcal{P}$, extend $f$ and $g$ to be periodic, continuous functions on $\mathbb{R}$ with period $2\pi$, and define

$$f * g(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta - \varphi)g(\varphi)d\varphi.$$ 

Show that for $f \in \mathcal{P}$, $S_n(\theta) = f * D_n(\theta)$, so in particular $S_n(0) = f * D_n(0)$.

(b) For each $n = 1, 2, \ldots$, and for $f \in \mathcal{P}$, set

$$F_n(f) = f * D_n(0).$$ 

Show that $F_n \in \mathcal{P}^*$ with $\|F_n\| \leq \|D_n\|$, for $n \in \mathbb{N}$

(c) Prove that $\|F_n\| = \|D_n\|$.

[Hints: First show that $D_n$ is real valued. Then set

$$g(\theta) = \begin{cases} 
1 & \text{if } D_n(\theta) \geq 0, \\
-1 & \text{if } D_n(\theta) < 0.
\end{cases}$$

Thus $gD_n = |D_n|$. Next show that there exists $\{f_j\}_{j=1}^{\infty} \subset \mathcal{P}$ such that $-1 \leq f_j(\theta) \leq 1$ and

$$f_j(\theta) \to g(\theta) \quad -\pi \leq \theta \leq \pi.$$ 

Then apply the Dominated Convergence Theorem.]

(d) Show that

$$\lim_{n \to \infty} \|F_n\| = \lim_{n \to \infty} \|D_n\| = \infty.$$ 

[Hints: Multiply $D_n$ first by $e^{\frac{1}{2}i\theta}$ and then by $e^{-\frac{1}{2}i\theta}$ and subtract one of the resulting functions from the other to get

$$D_n(\theta) = \frac{\sin(n + \frac{1}{2})\theta}{\sin(\frac{1}{2}\theta)}.$$ 

Since $|\sin x| \leq x$, for all $x \in \mathbb{R}$, you can justify the following:
\[
\|D_n\| \geq \frac{2}{n} \int_0^{\pi} |\sin(n + \frac{1}{2})\theta| \frac{d\theta}{\theta}
\]
\[
= \frac{2}{n} \int_0^{(n+\frac{1}{2})\pi} |\sin\theta| \frac{d\theta}{\theta}
\]
\[
> \frac{2}{n} \sum_{k=1}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin\theta| d\theta
\]
\[
= \frac{4}{n^2} \sum_{k=1}^{n} \frac{1}{k} \rightarrow \infty.
\]

(e) Use the Uniform boundness Principle to conclude that there exists \(f \in \mathcal{P}\) such that \(\{S_n(0)\}\) does not converge.

2. (The Gelfand Representation of a Commutative Banach Algebra with Identity) Let \(A\) be a (complex) commutative Banach algebra with identity.

(a) Let \(x \in A\) with \(\|x\| < 1\). Then \(e - x\) is invertible in \(A\).

[Hint: Analogize the fact that for \(|z| < 1\),
\[
\frac{1}{1-z} = 1 + z + z^2 + \cdots,
\]
and the series converges absolutely.]

(b) Let \(\Phi\) be the set of all algebra homomorphisms form \(A\) to \(\mathbb{C}\). Show that \(\Phi\) is included in \(A^*\) and \(\|\phi\| = 1\), for all \(\phi \in \Phi\).

[Hint: If \(\phi \in \Phi\), then \(\phi(e) = 1\), so
\[
\sup_{\|x\| \leq 1} |\phi(x)| \geq 1.
\]

Suppose \(\phi(x) > 1\) for some \(x\) with \(\|x\| \leq 1\). Then \(\phi(e - \frac{x}{\phi(x)}) = 0\), and by (a), \(e - \frac{x}{\phi(x)}\) is invertible.]

(c) Show that \(\Phi\) is weak* closed, and hence compact in the weak* topology.

(d) For \(x \in A\) and \(\phi \in \Phi\), define \(\hat{x}(\phi) = \phi(x)\). Show that the mapping \(x \rightarrow \hat{x}\) is an algebra homomorphism of \(A\) into \(C(\Phi)\), and \(\|\hat{x}\|_\Phi \leq \|x\|\) for all \(x \in A\).