1. Let $G$ be a compact abelian group. Then $\hat{G}$ is a complete orthonormal set in $L^2(G)$.

2. Let $G$ be a compact abelian group. Prove that the following are equivalent.
   
   (1) $G$ is metrizable.
   
   (2) $\hat{G}$ is countable.
   
   (3) $G$ has a countable base of neighborhoods of $e$.
   
   (4) $G$ is a closed subgroup of $T^\omega$, the countably infinite Cartesian product of circle groups.

3. (a) Let $G$ be a locally compact group, and let $f, g \in C_00(G)$. Then $f * g \in C_00(G)$.
   
   (b) Let $p, q \geq 1$ such that $1/p + 1/q = 1$. For $f \in L^p(G)$ and $g \in L^q(G)$, define
       
       $$ f * g(x) = \int_G f(y)g(y^{-1}x) \, d\lambda(y). $$

       Then $f * g$ exists and is an element of $C_0(G)$.
   
   (c) Let $A$ be a measurable subset of $G$ with $0 < \lambda(A) < \infty$. Then $A \cdot A$ has nonempty interior. (Hint: Consider $\chi_A$.)
   
   (d) Let $H$ be as closed subgroup of $G$. If $H$ is not (locally) $\lambda$-null, then $H$ is open.

4. Let $G$ be a group and $H$ be a subgroup. Prove that the characteristic function $\chi_H$ is of positive type.

5. Let $G$ be a locally compact group and $F$ be a positive functional on $L^1(G)$. Then
   
   $$ F(f) = \int_G f \phi d\lambda, \quad f \in L^1(G), $$

   for some continuous function $\phi$ of positive type on $G$. More specifically, let $\pi$ be the unitary representation of $G$ constructed in the proof of Theorem (3.22) and $\xi$ be the cyclic vector chosen there. Then (1) is satisfied with $\phi(x) = \langle \pi(x)\xi, \xi \rangle$. (Hints: Let $\{u_\alpha\}$ be an approximate identity in $L^1(G)$. Then show that $\{u_\alpha^*\}$ is also an approximate identity. Thus
       
       $$ F(f) = \lim_{\beta} \lim_{\alpha} F(u_\beta^* f * u_\alpha). $$

   Next show that
       
       $$ F(u_\beta^* f * u_\alpha) = \int_G f(y) [\pi(y)\tilde{u}_\alpha, \tilde{u}_\beta] \, d\lambda(y). $$

   Passing to the limit on $\beta$, moving $\pi(y)$ to the right as $\pi(y^{-1})$ in the inner product, and then passing to the limit on $\alpha$ leads to the desired result.)
6. Let \( G \) be a locally compact group. A bounded, continuous function \( f \) on \( G \) is called \textit{almost periodic} if the set of translates \( \{ L_a f : a \in G \} \) of \( f \) has compact closure in the uniform norm on the space \( C_b(G) \) of bounded continuous functions on \( G \). Let \( AP(G) \) denote the set of almost periodic functions on \( G \).

(a) The set \( AP(G) \) is a closed subalgebra of \( C_b(G) \).

(b) If \( G = \mathbb{R}^n \) and \( f \) is a periodic continuous function on \( G \), then \( f \) is almost periodic.

(c) If \( G \) is abelian, then every character of \( G \) is almost periodic. Thus any linear combination of characters (called a “trigonometric polynomial”) is almost periodic.

Remark: If \( G \) is abelian, it can be shown that every almost periodic function on \( G \) is the uniform limit of trigonometric polynomials.

7. Let \( G \) be abelian. Give the group \( \hat{G} \) the discrete topology, creating the discrete group \( \hat{G}_d \), and let \( bG = (\hat{G}_d) \), a compact abelian group. Define \( \beta : G \to bG \) by

\[
(\gamma, \beta(x)) = (x, \gamma), \quad x \in G, \gamma \in \hat{G}.
\]

The group \( bG \), along with the map \( \beta \), is called the \textit{Bohr compactification} of \( G \).

(a) Prove that \( \beta \) is a continuous isomorphism of \( G \) onto a dense subgroup of \( bG \).

(Hints: Use the Pontryagin Duality Theorem or the Gelfand-Raikov Theorem to say that \( \hat{G} \) separates points of \( G \). To show \( \beta(G) \) is dense in \( bG \), let \( H \) be the closure of \( \beta(G) \). If \( H \neq bG \), then \( bG/H \) has a nontrivial character, which leads to a contradiction.)

(b) A function \( f \) on \( G \) is almost periodic if and only if it has the form \( f = g \circ \beta \) for some \( g \in C(bG) \).