Bimeasure Algebras on Locally Compact Groups

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For locally compact groups $G$ and $H$, let $BM(G, H)$ denote the Banach space of bounded bilinear forms on $C_0(G) \times C_0(H)$. Using a consequence of the fundamental inequality of A. Grothendieck, a multiplication and an adjoint operation are introduced on $BM(G, H)$ which generalize the convolution structure of $M(G \times H)$ and which make $BM(G, H)$ into a $K_G^*$-Banach *-algebra, where $K_G$ is Grothendieck's universal constant. Various topics relating to the ideal structure of $BM(G, H)$ and the lifting of unitary representations of $G \times H$ to *-representations of $BM(G, H)$ are investigated. © 1985 Academic Press, Inc.

1. INTRODUCTION

For locally compact spaces $X$ and $Y$, consider the space of all bounded, bilinear forms on $C_0(X) \times C_0(Y)$. In other words, if $V_0(X, Y) = C_0(X) \otimes C_0(Y)$ is the projective tensor product of the indicated spaces of functions, then we are considering elements of the dual $V_0^*(X, Y)$. $V_0(X, Y)$ is called the Varopoulos algebra on $X \times Y$, and such bilinear forms have been referred to traditionally as bimeasures. Following [17], for example, we denote the space of all bimeasures on $X \times Y$ by $BM(X, Y)$.

In [7] C. C. Graham and the third author showed that if $G$ and $H$ are LCA groups, then $BM(G, H)$ has the structure of a $K_G^*$-Banach algebra, where $K_G$ is Grothendieck's universal constant (see Theorem 1.1 below),

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and they studied the structure of $BM(G, H)$ as a normed algebra. In this paper we shall continue the study of the harmonic analysis of bimeasures. We shall extend the notion of a bimeasure algebra to the context of bimeasures on arbitrary locally compact groups and study these algebras (Sects. 2-4). In Section 3 we solve a problem left open in [7], in the present, more general context, by proving that the space $BM_+(G, H)$ of continuous bimeasures as defined in [7] is an ideal in $BM(G, H)$. In Section 5 we examine the lifting of unitary representations of $G \times H$ to $BM(G, H)$ and some consequences thereof. Finally, in Section 6 we return to the case of abelian groups $G$ and $H$ to demonstrate how the techniques employed earlier may be exploited to obtain information about the maximal ideal space of $BM(G, H)$.

A number of authors have studied topological tensor products in the context of Banach algebras, beginning with [16]. Recent work in this area has tended to depend heavily on the fundamental work on the metric theory of tensor products by A. Grothendieck, and ours is no exception to this phenomenon. Some of this work appears in [1-5, 12]. It should be noted, however, that in general $BM(G, H)$ does not arise as the completion of $M(G) \otimes M(H)$ with respect to any tensorial norm.

Throughout the paper $X$ and $Y$ will denote locally compact spaces and $G$ and $H$ are locally compact groups. As is customary $L^1(G)$ and $M(G)$ are the group and measure ($*$-)algebras of $G$. For any space $X$, $L^\infty(X)$, $C_b(X)$, $C_0(X)$ and $C_\infty(X)$ are the spaces of bounded functions on $X$ which are, respectively, Borel locally measurable, continuous, continuous with limit zero at infinity, and continuous with compact support. The norm in $C(X)$ is denoted by $\| \cdot \|_X$. Set $V_0(X, Y) = C_0(X) \otimes Y$ and $V(X, Y) = C(X) \otimes Y$. If $f \in C_b(G)$ and $x \in G$, then $f_x$ and $f^x$ denote the left and right translates of $f$ by $x$; viz., $f_x(y) = f(x^{-1}y)$ and $f^x(y) = f(xy)$.

If $E$ is a Banach space, $J: E \to E^{**}$ is the canonical embedding, while if $\mu$ is a probability measure on $X$, the identity embedding of $C_0(X)$ in $L^2(X, \mu)$ will be denoted by $I$. Thus, identifying $L^2(X, \mu)$ with its second dual, the operator $I^{**}: C_0(X)^{**} \to L^2(X, \mu)$ satisfies $I^{**}J = I$. Recall that one may identify the Hilbert-space tensor product $L^2(X, \mu) \otimes_h L^2(Y, \nu)$ with $L^2(X \times Y, \mu \times \nu)$. If $A$ is a commutative Banach algebra, $\mathcal{M}_A$ denotes its maximal ideal space.

The foundation of our arguments is the following theorem, which is a consequence of the fundamental inequality of A. Grothendieck. For proofs and applications of this result in a contemporary setting, we refer the reader to [11, 3-5]; in particular, see [4, Theorem 3.1].

**Theorem 1.1.** Let $u \in BM(X, Y)$. There exist regular Borel probability measures $\mu$ on $X$ and $\nu$ on $Y$ and a universal constant $K_G$ such that for $f \in C_0(X)$ and $g \in C_0(Y)$,
\[ |u(f, g)| \leq K_G \|u\| \left( \int_X |f|^2 \, d\mu \right)^{1/2} \left( \int_Y |g|^2 \, dv \right)^{1/2}. \]  

(1)

It follows from (1) that \( u \) can be extended to a bounded bilinear functional on \( L^2(X, \mu) \times L^2(Y, v) \) and hence that there is an operator \( T: L^2(X, \mu) \to L^2(Y, v) \) such that

\[ u(f, g) = \int_Y (Tf) \, g \, dv \]  

(2)

and \( \|T\| \leq K_G \|u\| \). Whenever a bimeasure \( u \) is represented as in (2) we shall refer to the probability measures \( \mu \) and \( v \) as Grothendieck measures for \( u \) and the operator \( T \) will be said to be associated with \( u \), \( \mu \) and \( v \).

We shall need the following extension of elements of \( BM(X, Y) \) to bilinear functionals on \( C_0(X)^{**} \times C_0(Y)^{**} \).

**Definition 1.2.** [4] Given \( u \in BM(X, Y) \), let \( S_u: C_0(X) \to C_0(Y)^* \) be the corresponding operator; namely,

\[ \langle g, S_u(f) \rangle = u(f, g), \quad f \in C_0(X), \quad g \in C_0(Y). \]

Thus \( S_u^{**}: C_0(X)^{**} \to C_0(Y)^{***} \). For \( \Phi \in C_0(X)^{**} \) and \( \Psi \in C_0(Y)^{**} \) set

\[ u^{**}(\Phi, \Psi) = \langle \Phi, S_u^{**}(\Psi) \rangle. \]

Then \( u^{**} \) is a bounded bilinear form on \( C_0(X)^{**} \times C_0(Y)^{**} \), and \( \|u^{**}\| = \|u\| \).

**Lemma 1.3.** Let \( \mu \) and \( v \) be Grothendieck measures for \( u \) and \( T \) be an associated operator. Then for \( \Phi \in C_0(X)^{**}, \Psi \in C_0(Y)^{**} \),

\[ u^{**}(\Phi, \Psi) = \int_Y T I^{**}(\Phi) \, I^{**}(\Psi) \, dv. \]  

(3)

In particular, \( u^{**} \) is separately weak-* continuous.

**Proof.** Consider the following commutative diagram, where the Grothendieck factorization is expressed as \( S_u = I^*_b T I_b \).

\[ \begin{array}{ccc}
C_0(X)^{**} & \xrightarrow{S_u^{**}} & C_0(Y)^{***} \\
\downarrow I_b^{**} & & \downarrow I_b^{**} \\
C_0(X) & \xrightarrow{S_u} & C_0(Y)^* \\
\downarrow I_b & & \downarrow I_b \\
L^2(X, \mu) & \xrightarrow{T} & L^2(Y, v)
\end{array} \]
Identifying \( T \) and \( T^{**} \), we obtain
\[
\langle \Phi, B_X \rangle = \langle \Phi, T^{**} X \rangle
\]
\[
= \langle \Phi, I^{**} T^{**} X \rangle
\]
\[
= \langle I^{**} \Phi, T^{**} X \rangle
\]
\[
= \int_Y T^{**} (\Phi) I^{**} (\Psi) \, d\gamma,
\]
which is (3). The continuity assertion follows easily from (3).

**Corollary 1.4.** Let \( f \in L^\infty(X) \) and \( g \in L^\infty(Y) \), and let \( \Phi_f \) and \( \Psi_g \) denote the functionals they induce via integration on \( M(X) \) and \( M(Y) \), respectively. Then
\[
\langle \Phi_f, \Psi_g \rangle = \int_Y (Tf) g \, d\gamma.
\]

Thus identifying \( \Phi_f \) and \( \Psi_g \) with \( f \) and \( g \), \( u^{**} \big| L^\infty(X) \times L^\infty(Y) \) is the unique extension of \( u \) to \( L^\infty(X) \times L^\infty(Y) \) such that (2) holds for all choices of Grothendieck measures \( \mu \) and \( \nu \) for \( u \) and associated operators \( T \).

**Proof.** It is straightforward to check that \( I^{**} (\Phi_f) = f \, \mu \)-a.e. and \( I^{**} (\Psi_g) = g \, \nu \)-a.e. Thus
\[
\langle \Phi_f, \Psi_g \rangle = \int_Y T^{**} (\Phi_f) I^{**} (\Psi_g) \, d\gamma
\]
\[
= \int_Y (Tf) g \, d\gamma.
\]

For \( f \in L^\infty(X) \) and \( g \in L^\infty(Y) \), we shall write
\[
\langle \Phi_f, \Psi_g \rangle = u(f, g).
\]

Extending elements of \( BM(X, Y) \) to \( V(X, Y) \), we have for \( f \in V(X, Y) \),
\[
\| f \|_V = \sup \{ \| u(f) \| : u \in BM(X, Y), \| u \| \leq 1 \}.
\]

**2. Bimeasure Algebras**

There are two traditional, functional-analytic approaches to the definition of convolution on \( M(G) \). One definition involves product...
measures and converts a function in $C_0(G)$ to a function on $G \times G$, defining $\mu \ast v$ for $\mu, v \in M(G)$ by the formula

$$\int_G \int_{G\times G} f(x) \, d\mu \times v(x, y).$$

The second approach begins with the observation that if $f \in C_0(G)$ and $\mu, v \in M(G)$, then setting

$$\mu \ast f(x) = \int_G f(y^{-1} x) \, d\mu(y),$$

one defines $\mu \ast v$ by

$$\int_G f(x) \, d\mu \ast v = \int_G (\tilde{v} \ast f) \, d\mu.$$

Motivated by these classical lines, we shall introduce the algebra structure on $BM(G, H)$ by showing that Theorem 1.1 can be used to develop analogues of both of these definitions of convolution.

We begin by developing an abstract result which will be the key to our definition of the multiplication on $BM(G, H)$ when $G$ and $H$ are groups.

**Theorem 2.1.** Let $u \in BM(X_1, Y_1)$ and $v \in BM(X_2, Y_2)$. There exists a unique element $u \otimes v$ of $BM(X_1 \times X_2, Y_1 \times Y_2)$ such that

$$u \otimes v(f_1 \otimes f_2, g_1 \otimes g_2) = u(f_1, g_1) \, v(f_2, g_2)$$

for $f_1 \in C_0(X_i)$, $g_1 \in C_0(Y_i)$, $i = 1, 2$. If $\mu_1$ and $\nu_1$ are Grothendieck measures for $u$ with associated operator $T_1$ and $\mu_2$ and $\nu_2$ are Grothendieck measures for $v$ with associated operator $T_2$, then $\mu_1 \times \mu_2$ and $\nu_1 \times \nu_2$ are Grothendieck measures for $u \otimes v$ with associated operator $T_1 \otimes T_2$, so that for $f \in C_0(X_1 \times X_2)$ and $g \in C_0(Y_1 \times Y_2)$,

$$u \otimes v(f, g) = \int_{Y_1 \times Y_2} ((T_1 \otimes T_2) f) \, g \, d\nu_1 \times \nu_2.$$

Moreover,

$$\|u \otimes v\| \leq K_1^2 \|u\| \|v\|.$$

**Proof.** For $f \in C_0(X_1 \times X_2)$ and $g \in C_0(Y_1 \times Y_2)$, let $u \otimes v(f, g)$ be defined by (5), so that $u \otimes v \in BM(X_1 \times X_2, Y_1 \times Y_2)$. For $f_i$ and $g_i$ as in the theorem,
so (4) holds. Since $C_0(X_1) \otimes C_0(X_2)$ and $C_0(Y_1) \otimes C_0(Y_2)$ are uniformly dense in $C_0(X_1 \times X_2)$ and $C_0(Y_1 \times Y_2)$, respectively, we see that the definition of $u \otimes v$ is independent of the choices of the Grothendieck measures for $u$ and $v$ and the associated operators and that the uniqueness assertion in our theorem holds. Choosing our Grothendieck measures and operators so that $\|T_1\| \leq K_u \|u\|$ and $\|T_2\| \leq K_v \|v\|$, we have for all $f$ and $g$ as above,

$$
\|u \otimes v(f, g)\| = \int_{Y_1 \times Y_2} \left( (T_1 \otimes T_2)(f) \right) g \, dv_1 \times dv_2 \\
\leq \|T_1 \otimes T_2\| \left( \int_{X_1 \times X_2} |f|^2 \, d\mu_1 \times \mu_2 \right)^{1/2} \left( \int_{Y_1 \times Y_2} |g|^2 \, dv_1 \times dv_2 \right)^{1/2} \\
\leq \|T_1\| \|T_2\| \|f\|_{X_1 \times X_2} \|g\|_{Y_1 \times Y_2} \\
\leq K_u \|u\| \|v\| \|f\|_{X_1 \times X_2} \|g\|_{Y_1 \times Y_2}.
$$

Remark 2.2. Note that the use of the Grothendieck factorization was crucial to the proof of Theorem 2.1. Without it we could only proceed as far as defining $u \otimes v$ on $V_0(X_1, X_2) \times V_0(Y_1, Y_2)$.

Definition 2.3. If $G$ is a locally compact group and $\varphi$ is a function on $G$, set 

$$
M \varphi(x_1, x_2) = \varphi(x_1 x_2)
$$

for $x_1, x_2 \in G$. Recall that it is evident from the definition of convolution that if $\mu_1$ and $\mu_2$ are Borel probability measures on $G$ then $M$ defines an isometry of $L^2(G, \mu_1 * \mu_2)$ into $L^2(G \times G, \mu_1 \times \mu_2)$. We shall always denote such an operator by $M$. For $f \in C_b(G)$ let

$$
\tilde{f}(x) = f(x^{-1}) \quad \text{and} \quad f_*(x) = f(x^{-1}), \quad x \in G.
$$

Given $u, v \in BM(G, H)$, set

$$
\begin{align*}
\hat{u}(f, g) &= u \otimes v(Mf, Mg), \\
\tilde{u}(f, g) &= u(\tilde{f}, \tilde{g}).
\end{align*}
$$

(6)
and

$$u(f, g) - \bar{u}(f^*, g^*)$$

(7)

for $f \in C_0(G)$ and $g \in C_0(H)$. Note that we have implicitly extended $u \otimes v$ to $C_b(G \times G) \times C_b(H \times H)$ so that (6) is well defined.

Theorem 2.1 gives us information on Grothendieck measures and associated operators for $u \ast v$ as follows.

**Corollary 2.4.** Let $u, v \in BM(G, H)$. Suppose that $\mu_1$ and $v_1$ are Grothendieck measures for $u$ and $\mu_2$ and $v_2$ are Grothendieck measures for $v$ and that $T_1$ and $T_2$ are correspondingly associated operators for $u$ and $v$. Then $\mu_1 \ast \mu_2$ and $v_1 \ast v_2$ are Grothendieck measures for $u \ast v$, and with respect to these measures $u \ast v$ is given by

$$u \ast v(f, g) = \int_H (M^{-1} P(T_1 \otimes T_2) Mf) g \, dv_1 \ast v_2$$

for $f \in C_0(G)$ and $g \in C_0(H)$, where $P$ is the projection of $L^2(H \times H, v_1 \times v_2)$ onto $M(L^2(H, v_1 \ast v_2))$.

**Proof:**

$$u \ast v(f, g) = \int_{H \times H} ((T_1 \otimes T_2) Mf) Mg \, dv_1 \times v_2$$

$$= \int_H (M^{-1} P(T_1 \otimes T_2) Mf) g \, dv_1 \ast v_2.$$ 

**Definition 2.5.** If $g \in V(G \times G, H \times H)$ and $u \in BM(G, H)$, then to make our expressions explicit when necessary let $u_{(x', y')}(g(x', x, y', y))$ denote $u$ acting on the function $g(\cdot, x, \cdot, y)$, as a function of $(x, y)$. Now, by analogy with the definition of $\mu \ast f$ above, for $f \in V(G, H)$ set

$$u \ast f(x, y) = u_{(x', y')}(f((x', y')^{-1}(x, y))) = u(f_{(x, y)})$$

and

$$f \ast u(x, y) = u_{(x', y')}(f((x, y)(x', y')^{-1})) = u(f_{(x, y)}^{-1}).$$

Since right and left translation of a function in $V_0(G, H)$ is norm continuous and $\|f\|_{V_0} = \|f\|_{V_0}$, it is easy to see that $u \ast f, f \ast u \in C_b(G \times H)$ if $f \in V_0(G, H)$.

**Theorem 2.6.** Let $f \in V_0(G, H)$ and $u, v \in BM(G, H)$. Then $u \ast f, f \ast u \in V_0(G, H)$.
\[ \| u \ast f \|_{V_0} \leq K_\mathcal{V}^2 \| u \| \| f \|_{V_0}, \quad (8) \]
\[ \| f \ast u \|_{V_0} \leq K_\mathcal{V}^2 \| u \| \| f \|_{V_0}, \quad (8') \]
\[ u \ast v(f) = v(\tilde{u} \ast f) = v_{(x', y')}((u_{(x, y)}(f(xx', yy')))), \quad (9) \]

and
\[ u \ast v(f) = u(f \ast \tilde{v}) = u_{(x, y)}(v_{(x', y')}((f(xx', yy')))). \quad (9') \]

**Proof.** We shall prove only (8) and (9), the proofs of (8') and (9') being similar.

Let \( f^1, f^2 \in C_{\infty}(G) \) and \( g^1, g^2 \in C_{\infty}(H) \), and set \( f = f^1 \ast f^2 \) and \( g = g^1 \ast g^2 \). Assume that \( u \) has compact support. Then \( u \ast (f^1 \otimes g^1) \in C_{\infty}(G \times H) \), so
\[ u \ast (f^1 \otimes g^1)(x, y) = u_{(x', y')}((f(x' - x)x \otimes g^1(y' - y)y)) \]
\[ = u_{(x', y')}\left( \int_H \int_G f^1(x' - x) x \xi f^2(\xi^{-1}) y \eta g^1(y' - y) y \eta \, d\xi d\eta \right) \]
\[ = u_{(x', y')}\left( \int_H \int_G f^1(x' - x) x \xi f^2(\xi^{-1}) x \eta g^1(y' - y) y \eta \, d\xi d\eta \right) \]
\[ = \int_H \int_G u \ast (f^1 \otimes g^1)(\xi, \eta) f^2(\xi, \eta) g^2(\eta, \eta) \, d\xi d\eta. \]

That is, as a vector integral in \( V_0(G, H) \),
\[ u \ast (f \otimes g) = \int_H \int_G f^2 \otimes g^2 u \ast (f^1 \otimes g^1)(\xi, \eta) \, d\xi d\eta. \]

Let \( \varphi \in C_{\infty}(G) \) and \( \psi \in C_{\infty}(H) \) such that \( \varphi \otimes \psi = 1 \) on the support of \( u \). Since
\[ \int_G f_\xi^1(x') f_\xi^2(x) \, d\xi = \int_G f^1(x' - x) x \xi f^2(\xi^{-1}) x \eta \, d\xi \]
\[ = f(x' - x) = Mf(x' - x), \]
and similarly for \( g^1, g^2 \) and \( g \), we have
\[ \tilde{u} \ast v(f, g) = \tilde{u} \otimes v(Mf, Mg) \]
\[ = u \otimes v \left( \int_G f_\xi^1 \otimes f_\xi^2 \, d\xi, \int_H g_\eta^1 \otimes g_\eta^2 \, d\eta \right) \]
\[ = u \otimes v \left( (\varphi \otimes 1) \int_G f_\xi^1 \otimes f_\xi^2 \, d\xi, (\psi \otimes 1) \int_H g_\eta^1 \otimes g_\eta^2 \, d\eta \right) \]
\[ = u \otimes v \left( \int_H \int_G (\varphi f_\xi^1 \otimes f_\xi^2, \psi g_\eta^1 \otimes g_\eta^2) \, d\xi d\eta \right). \]
Now, $\phi j_{z}^{1} \otimes f_{z}^{2} = 0$ except on a compact set of $\zeta \in G$, and similarly for $\psi g_{n}^{1} \otimes g_{n}^{2}$. Hence we may consider the integral in the last expression above as a vector integral in $V(G \times G, H \times H)$. This last expression then equals

$$
\int_{H} \int_{G} (u \otimes v)(\phi j_{z}^{1} \otimes f_{z}^{2}, \psi g_{n}^{1} \otimes g_{n}^{2}) \, d\zeta \, d\eta
$$

$$
= \int_{H} \int_{G} u(\phi j_{z}^{1}, \psi g_{n}^{1}) \, v(f_{z}^{2}, g_{n}^{2}) \, d\zeta \, d\eta
$$

$$
= \int_{H} \int_{G} v(f_{z}^{2}, g_{n}^{2}) \, u(\phi j_{z}^{1}, \psi g_{n}^{1}) \, d\zeta \, d\eta
$$

$$
= v \left( \int_{H} \int_{G} f_{z}^{2} \otimes g_{n}^{2} \, u \ast (f^{1} \otimes g^{1})(\zeta, \eta) \, d\zeta \, d\eta \right)
$$

$$
= v(u \ast (f \otimes g)).
$$

Thus (9) follows with $f$ replaced by $f \otimes g$, and we have from Theorem 2.1 and (6) that

$$
\|u \ast (f \otimes g)\|_{V_0} = \sup_{\|v\| \leq 1} |v(u \ast (f \otimes g))|
$$

$$
= \sup_{\|v\| \leq 1} |\hat{u} \ast v(f \otimes g)|
$$

$$
\leq K_{G}^{2} \|u\| \|f\|_{G} \|g\|_{H}.
$$

Since sums of functions of the form $f \otimes g$ as above are dense in $V_0(G, H)$ and our last computation implies that (8) holds for such sums, and since bimeasures with compact support are dense in $BM(G, H)$ [7, Corollary 1.3], we can use (8) to pass to limits and obtain (8) and (9) for all $f \in V_0(G, H)$ and $u, v \in BM(G, H)$.

**Corollary 2.7.** Let $f$ and $g$ be bounded, uniformly continuous functions on $G$ and $H$, respectively, and let $u, v \in BM(G, H)$. Then the conclusions of Theorem 2.6 hold with $f$ and $V_0$ replaced by $f \otimes g$ and $V$.

**Proof.** Suppose that $f^2$ and $g^2$ in the proof of Theorem 2.6 are just assumed to be bounded and uniformly continuous. Then the argument presented there shows that $u \ast (f \otimes g) \in V(G, H)$. Proceeding as in the proof of Theorem 2.6, we conclude that (8)-(9') hold with $f$ and $V_0$ replaced by $f \otimes g$ and $V$ when $f = f^1 \ast f^2$ and $g = g^1 \ast g^2$ as above.

Recall that if $f$ is bounded and uniformly continuous on $G$, then $\|f^1 \ast f - f\|_{G} \to 0$ as $f^1$ ranges over an approximate identity, and similarly for $g$ bounded and uniformly continuous on $H$. Hence we can pass to limits as in the conclusion of the proof of Theorem 2.6 to obtain our corollary.
Theorem 2.8. The multiplication (6) and the adjoint operation (7) define a \( \ast \)-algebra structure on \( BM(G, H) \) which extends the \( \ast \)-algebra structure of \( M(G \times H) \). For \( u, v \in BM(G, H) \), \( \| \tilde{u} \| = \| u \| \) and
\[
\| u \ast v \| \leq K_G^2 \| u \| \| v \| .
\] (10)

Proof. If \( u, v \in BM(G, H) \) it follows from Theorem 2.1 that \( u \ast v \in BM(G, H) \) and (10) holds. It is easy to check that \( u \ast v, (u \ast v) \ast v = u \ast v \ast v, \) and \( \| u \ast v \| = \| u \| \ast v \| \). If \( \mu, \nu \in M(G \times H) \) and \( u \) and \( v \) are the corresponding elements of \( BM(G, H) \), then it is easy to see from (4) and the uniqueness assertion of Theorem 2.1 that for \( f \in C_0(G \times G) \) and \( g \in C_0(H \times H) \),
\[
u \otimes v(f, g) = \int_{G \times H} \int_{G \times H} f(x, x') g(y, y') \, d\mu(x, y) \, dv(x', y').
\]
Thus for \( f \in C_0(G) \) and \( g \in C_0(H) \), (6) gives
\[
u \ast v(f, g) = u \otimes v(Mf, Mg)
= \int_{(G \times H) \times (G \times H)} Mf \otimes Mg \, d\mu \times v
= \int_{(G \times H) \times (G \times H)} M(f \otimes g) \, d\mu \ast v
= \int_{G \times H} f \otimes g \, d\mu \ast v.
\]
Since (4) gives immediately that
\[
u_1 + \nu_2 \otimes v = \nu_1 \otimes v + \nu_2 \otimes v
\]
and
\[
u \otimes (\nu_1 + \nu_2) = \nu \otimes \nu_1 + \nu \otimes \nu_2,
\]
it follows that \( \ast \) is distributive. To see that the associative law holds in \( BM(G, H) \), let \( u, v, w \in BM(G, H) \). For \( f \in V_0(G, H) \), Theorem 2.6 gives
\[
u \ast v \ast w(f) = (u \ast v)(x, y)(w(x, y')(f(x', y')))
= u(x, y)(w(x, y')(f(x', y')))
= u(x, y)(v \ast w)(x, y')(f(x', y'))
= u \ast (v \ast w)(f).
\]
Finally, we must show that for $u, v \in BM(G, H)$,

$$(u * v)^* = \tilde{v} * \tilde{u}. $$

If $f \in V_0(G, H)$, we have

$$(u * v)^*(f) = u * v(f^*)$$

$$= u_{(x', y')}(v_{(x', y')}(f^*(xx', yy')))$$

$$= u_{(x', y')}(v_{(x', y')}(f'(x'^{-1}x, y'y^{-1})))$$

$$= u_{(x', y')}(v_{(x', y')}(f'(x, y)b))$$

$$= \tilde{u}_{(x', y')}(\tilde{b}_{(x', y')}(f(x, y)^*))$$

$$= \tilde{v} * \tilde{u}(f).$$

The proof of our theorem is now complete.

Remark 2.9. If $G$ and $H$ are abelian, then the multiplication introduced here on $BM(G, H)$ is the same as the one introduced in [7] via the Fourier transform. For if $\gamma$ and $\delta$ are characters on $G$ and $H$, respectively, and $^*$ denotes the Fourier transform on $BM(G, H)$ as introduced in [7, Definition 1.10], then

$$(u * v)^*(\gamma, \delta) = u * v(\tilde{\gamma}, \tilde{\delta})$$

$$= u \otimes v(M\tilde{\gamma}, M\tilde{\delta})$$

$$= u \otimes v(\tilde{\gamma} \otimes \tilde{\gamma}, \tilde{\delta} \otimes \tilde{\delta})$$

$$= u(\tilde{\gamma}, \tilde{\delta}) v(\tilde{\gamma}, \tilde{\delta})$$

$$= \tilde{u}(\gamma, \delta) \tilde{v}(\gamma, \delta).$$

Our assertion now follows from [7, Theorem 2.4, Definition 2.5].

Theorem 2.10. Let $H_i$ be a closed, normal subgroup of $G_i$ and $\varphi_i : G_i \to G_i/H_i$ be the quotient map, $i = 1, 2$. For $u \in BM(G_1, G_2)$, define $\sigma(u) \in BM(G_1/H_1, G_2/H_2)$ by

$$\sigma(u)(f, g) = u(f \circ \varphi_1, g \circ \varphi_2), \quad f \in C_0(G_1/H_1), \quad g \in C_0(G_2/H_2).$$

Then $\sigma$ is a surjective algebra homomorphism of norm one.
Proof. Since \( f \circ \varphi_1 \in C_b(G_1) \) and \( g \circ \varphi_2 \in C_b(G_2) \), \( \sigma \) is a well-defined linear map. And by Corollary 1.9,

\[
|\sigma(u)(f, g)| \leq \|u\| \|f \circ \varphi_1\|_{G_1} \|g \circ \varphi_2\|_{G_2} = \|u\| \|f\|_{G_1/H_1} \|g\|_{G_2/H_2}.
\]

If \( u, v \in BM(G_1, G_2) \), let \( \mu_i, v_i \) and \( T_i, i = 1, 2 \), be as in the statement of Corollary 2.4. Define \( \bar{\mu}_i \in M(G_1/H_1) \) and \( \bar{v}_i \in M(G_2/H_2) \) by

\[
\int_{G_1/H_1} f \, d\bar{\mu}_i = \int_{G_1} f \circ \varphi_1 \, d\mu_i, \quad f \in C_0(G_1/H_1)
\]

and

\[
\int_{G_2/H_2} g \, d\bar{v}_i = \int_{G_2} g \circ \varphi_2 \, dv_i, \quad g \in C_0(G_2/H_2),
\]

\( i = 1, 2 \). Then \( \bar{\mu}_1, \bar{v}_1 \) are Grothendieck measures for \( \sigma(u) \) and similarly for \( \bar{\mu}_2, \bar{v}_2 \) and \( \sigma(v) \). Let \( T_1 \) and \( T_2 \) be associated operators for \( \sigma(u) \) and \( \sigma(v) \), respectively, so that for \( i = 1, 2 \), \( f \in L^2(G_1/H_1, \bar{\mu}_i) \) and \( g \in L^2(G_2/H_2, \bar{v}_i) \),

\[
\int_{G_2/H_2} (T_1 f) \, g \, d\bar{v}_i = \int_{G_2} (T_i(f \circ \varphi_1)) \, g \circ \varphi_2 \, dv_i.
\]

Then since

\[
M(f \circ \varphi_1)(x, x') = Mf(\varphi_1(x), \varphi_1(x'))
\]

and similarly for \( g \) and \( \varphi_2 \), we have for \( f \in C_0(G_1/H_1) \) and \( g \in C_0(G_2/H_2) \),

\[
\sigma(u * v)(f, g) = u * v(f \circ \varphi_1, g \circ \varphi_2)
\]

\[
= u \otimes v(M(f \circ \varphi_1), M(g \circ \varphi_2))
\]

\[
= \int_{G_2 \times G_2} ((T_1 \otimes T_2) M(f \circ \varphi_1)) M(g \circ \varphi_2) \, dv_1 \times v_2
\]

\[
= \int_{(G_2/H_2) \times (G_2/H_2)} ((\bar{T}_1 \otimes \bar{T}_2) Mg) \, M \, d\bar{v}_1 \times \bar{v}_2
\]

\[
= \sigma(u) \otimes \sigma(v)(Mf, Mg)
\]

\[
= \sigma(u) * \sigma(v)(f, g).
\]

The fact that \( \sigma \) is surjective follows exactly as in the proof of [7, Theorem 3.1].
3. Continuous and Discrete Bimeasures

In [7] the spaces $BM_c(X, Y)$ of continuous bimeasures and $BM_d(X, Y)$ of discrete bimeasures were defined, and it was shown that $BM(X, Y)$ is the topological direct sum of the closed subspaces $BM_c(X, Y)$ and $BM_d(X, Y)$. Specifically, $u \in BM_d(X, Y)$ if there exist increasing sequences $\{E_n\}$ and $\{F_n\}$ of finite subsets of $X$ and of $Y$, respectively, such that $u = \lim_{n \to \infty} u|_{E_n \times F_n}$ in norm, where $u|_{E \times F}$ denotes the restriction of $u$ to $E \times F$. On the other hand, $u \in BM_c(X, Y)$ if $u|_{E \times F} = 0$ for all finite subsets $E$ of $X$ and $F$ of $Y$. Note that if for all $x \in X$ and $y \in Y$ we partially order the family of all pairs $(U, V)$ of neighborhoods $U$ of $x$ and $V$ of $y$ in the usual way, then the bimeasure $u$ is continuous if and only if for all $x \in X$ and $y \in Y$,

$$\lim \sup_{(U, V)} |u(f, g)| = 0,$$

the suprema being taken over all $f \in C_0(X)$ and $g \in C_0(Y)$ such that $\|f\| \leq 1$, $\|g\| \leq 1$, $f = 0$ outside $U$ and $g = 0$ outside $V$.

Remarks 3.1. Since $BM_d(X, Y)$ is closed it contains any bimeasure which is the limit of bimeasures (measures) with finite support. It now follows from Theorem 2.8 that for groups $G$ and $H$ $BM_d(G, H)$ is closed under multiplication. It is easy to see from Theorem 2.6 that if $u$ is the bimeasure defined by evaluation at a point of $G \times H$ and $v \in BM(G, H)$, then $u \ast v$ and $v \ast u$ are translates of $v$. Since the right or left translate of a continuous bimeasure is clearly continuous, we may again appeal to Theorem 2.8 to conclude that if $u \in BM_d(G, H)$ and $v \in BM_c(G, H)$, then $u \ast v$, $v \ast u \in BM_c(G, H)$. Our main purpose in this section is to show that if $u, v \in BM_c(G, H)$, then $u \ast v \in BM_c(G, H)$, so $BM_c(G, H)$ is an ideal in $BM(G, H)$. But our first goal is of a more general nature, namely to characterize the discrete and continuous bimeasures in terms of their associated operators.

Definition 3.2. Let $\mu$ and $\nu$ be Borel probability measures on $X$ and $Y$, respectively, and let $T \in \mathcal{B}(L^2(X, \mu), L^2(Y, \nu))$. Define the operators $T_d$ and $T_c$ as follows. Let $P_d$ be the projection on $L^2(X, \mu)$ given by

$$P_d f(x) = P_d^* f(x) = f(x), \quad \mu(\{x\}) > 0$$

$$-0, \quad \mu(\{x\}) = 0,$$

and similarly define $P_d$ on $L^2(\nu)$. Set $T_d = P_d TP_d$ and $T_c = T - T_d$. Denote $I - P_d$ by $P_c$. 
**Theorem 3.3.** Let $u \in BM(X, Y)$, and let $\mu$ and $\nu$ be Grothendieck measures for $u$ and $T$ be an associated operator.

(i) $u$ is continuous if and only if $T_d = 0$.

(ii) $u$ is discrete if and only if $T_c = 0$.

(iii) If $u_c$ and $u_d$ are the continuous and discrete components of $u$, respectively, then with respect to $\mu$ and $\nu$, $T_c$ and $T_d$ are corresponding associated operators.

The proof of Theorem 3.3 will be broken down into a sequence of steps. We begin with the following observation. Extending $u$ to a bilinear functional on $L^2(X, \mu) \times L^2(Y, \nu)$, for each $x \in X$ and $y \in Y$, let

$$t(x, y) = \int_Y (T\chi_{\{x\}}) \chi_{\{y\}} \, d\nu = u(\chi_{\{x\}}, \chi_{\{y\}}).$$

For each neighborhood $U$ of $x$ and neighborhood $V$ of $y$, choose $f_U \in C_0(X)$ and $g_V \in C_0(Y)$ such that $f_U(x) = 1$, $f_U = 0$ outside $U$, and $\|f_U\| = 1$ and similarly for $g_V$ and $y$. Then since $\chi_{\{x\}}$ can be approximated in $L^2(X, \mu)$ by functions of the form $\chi_U$ and $\chi_{\{x\}}$ can be approximated in $L^2(Y, \nu)$ by functions of the form $\chi_V$, we have

$$\lim_{(U, V)} u(f_U, g_V) = t(x, y).$$

Moreover, $t(x, y) = 0$ unless $\mu(\{x\}) \nu(\{y\}) > 0$. Thus $u$ is continuous if and only if $t(x, y) = 0$ for all $x \in X$ and $y \in Y$.

**Lemma 3.4.** For $f \in C_0(X)$ and $g \in C_0(Y)$, let $v(f, g) = \int_Y (T_c f) g \, d\nu$. Then $v \in BM_c(X, Y)$.

**Proof.** Recall that

$$T_c = T - T_d = T - P_d TP_c = P_c T + P_d TP_c.$$

Considering $v$ as a bounded bilinear form on $L^2(X, \mu) \times L^2(Y, \nu)$, we have

$$v(\chi_{\{x\}}, \chi_{\{y\}}) = \int_Y (P_c T\chi_{\{x\}}) \chi_{\{y\}} \, d\nu + \int_Y (P_d TP_c \chi_{\{x\}}) \chi_{\{y\}} \, d\nu$$

$$= \int_Y (T\chi_{\{x\}})(P_c \chi_{\{y\}}) \, d\nu = 0.$$

By the preceding observation, $v$ is continuous.

**Lemma 3.5.** For $f \in C_0(X)$ and $g \in C_0(Y)$, let $w(f, g) = \int_Y (T_d f) g \, d\nu$. Then $w \in BM_d(X, Y)$.
Proof. If μ or ν is a continuous measure, the result is trivial. Otherwise write
\[ \{x \in X: \mu(\{x\}) > 0\} = \{x_1, x_2, \ldots\} \]
and
\[ \{y \in Y: \nu(\{y\}) > 0\} = \{y_1, y_2, \ldots\}, \]
and set \( t_{ij} = t(x_i, y_j), i, j = 1, 2, \ldots \). For each nonnegative integer \( n \) let \( P^n_d \) be the projection of \( L^2(X, \mu) \) onto the span of \( \{\chi_{\{x_i\}}\}_{i=1}^n \), and let it also denote the projection of \( L^2(Y, \nu) \) onto the span of \( \{\chi_{\{y_j\}}\}_{j=1}^n \). Define \( w_n \in BM(X, Y) \) by
\[
w_n(f, g) = \sum_{i,j=1}^{n} t_{ij} f(x_i) g(y_j) = \int_{Y} (T P^n_d f)(P^n_d g) \, d\nu.
\]
We claim that \( w_n \to w \). Given \( \varepsilon > 0 \) choose \( n \) so large that
\[
\sum_{i=n+1}^{\infty} \mu(\{x_i\}) < \varepsilon \quad \text{and} \quad \sum_{j=n+1}^{\infty} \nu(\{y_j\}) < \varepsilon.
\]
If \( f \in C_0(X) \) and \( g \in C_0(Y) \) such that \( \|f\| \leq 1 \) and \( \|g\| \leq 1 \), then recalling that \( Td = P_d TP_d \), we see that
\[
\left| (w - w_n)(f, g) \right| = \int_{Y} (T P_d f)(P_d g) \, d\nu - \int_{Y} (T P^n_d f)(P^n_d g) \, d\nu
\]
\[
\leq \int_{Y} (T(P_d - P^n_d)f)(P_d g) \, d\nu + \int_{Y} (T P^n_d f)((P_d - P^n_d)g) \, d\nu
\]
\[
\leq \|T\| \|P_d - P^n_d\| \|P_d g\| + \|T\| \|P^n_d f\| \|P_d - P^n_d\| \|g\|
\]
\[
> \|T\| \|P_d - P^n_d\| \|f\| + \|T\| \|P_d - P^n_d\| \|g\|
\]
\[
< 2 \|T\| \varepsilon^{1/2},
\]
since
\[
\|P_d - P^n_d\| \|f\|^2 = \sum_{i=n+1}^{\infty} |f(x_i)|^2 \mu(\{x_i\}) < \varepsilon
\]
and similarly for \( \|P_d - P^n_d\| \|g\|^2 \). Our claim is proven, and the lemma follows.

Proof of Theorem 3.3. (i) For all \( \phi \in L^2(X, \mu) \) and \( \psi \in L^2(Y, \nu) \) we have
\[
\int_{Y} (T_d \phi) \psi \, d\nu - \sum_{i,j=1}^{\infty} t_{ij} \phi(x_i) \psi(y_j).
\]
Hence by our initial observation, if \( u \) is continuous, then \( T_d = 0 \). On the other hand, if \( T_d = 0 \), so \( \ell(x_i, y_j) = 0 \) for all \( i \) and \( j \), then \( u \) is continuous, since \( \ell(x, y) = 0 \) for all other \( x \) and \( y \).

(ii) If \( v \) is as in Lemma 3.4 and we assume \( u \) is discrete, then by Lemma 3.5 \( v \) is discrete, since \( T_c = T - T_d \). But by Lemma 3.4 \( v \) is also continuous. Hence \( v = 0 \), so \( T_c = 0 \). If \( T_c = 0 \), so \( T = T_d \), then by Lemma 3.5 \( u \) is discrete.

The assertion (iii) follows immediately from (i) and (ii).

Let us now restrict our attention to the group situation and proceed to our main result.

**Lemma 3.6.** Let \( \sigma \) and \( \tau \) be probability measures on \( G \). Set \( P_d = P_{\sigma \times \tau}^a \) and \( Q_d = P_{\sigma \times \tau}^a \). Then \( MP_d = Q_d M \).

**Proof.** Let \( E = \{ x: \sigma(\{ x \}) > 0 \} \) and \( F = \{ y: \tau(\{ y \}) > 0 \} \). For \( f \in L^2(G, \sigma \times \tau) \), \( P_d f = f \chi_{EF} \), so

\[
MP_d f(x, y) = f(xy) \chi_{EF}(xy) = Mf(x, y) \chi_{C}(x, y) \quad \sigma \times \tau \text{-a.e.,}
\]

where \( C = \{ (x, y): xy \in EF \} \supset E \times F \). And since \( (\sigma \times \tau)_d = \sigma_d \times \tau_d \),

\[
Q_d Mf = (Mf) \chi_{E \times F}.
\]

Hence it suffices to show that \( \chi_{E \times F} = \chi_{C} \sigma \times \tau \text{-a.e.} \). Write

\[
\sigma \times \tau = \sigma_d \times \tau_d + \sigma_c \times \tau_d + \sigma_d \times \tau_c + \sigma_c \times \tau_c.
\]

Note that for any \( x, y \in G \), the slices

\[
C_x = \{ y: (x, y) \in C \} = \{ y: y \in x^{-1}EF \}
\]

and

\[
C^x = \{ x: (x, y) \in C \} = \{ x: x \in EFy^{-1} \}
\]

are translates of \( EF \) and hence countable. So by Fubini's Theorem,

\[
\sigma \times \tau(C) = \sigma_d \times \tau_d(C) = \sigma(E) \tau(F),
\]

and our lemma is proved.

We are now in a position to prove the main result of this section.

**Theorem 3.7.** Let \( u, v \in BM(G, H) \). Then

\[
(u * v)_d = u_d * v_d.
\]
Proof. Let $P$, $T_1$ and $T_2$ be as in Corollary 2.4, and let $P_d$ and $Q_d$ be as in Lemma 3.6 for either $\mu_1$ and $\mu_2$ or $v_1$ and $v_2$, where $\mu_1$ and $v_1$ are Grothendieck measures for $u$ and $\mu_2$ and $v_2$ are Grothendieck measures for $v$. Then for $f \in C_c(G)$ and $g \in C_c(H)$, Corollary 2.4 and Theorem 3.3(iii) imply that

\[
(u \ast v)_d(f, g) = \int_{H} ((M^{-1}P(T_1 \otimes T_2) M_d f) g \, dv_1 \ast v_2
\]

\[
= \int_{H} (M^{-1}P(T_1 \otimes T_2) MP_d f) (P_d g) \, dv_1 \ast v_2
\]

\[
= \int_{H} (M^{-1}P(T_1 \otimes T_2) Q_d Mf) (P_d g) \, dv_1 \ast v_2
\]

\[
= \int_{H \times H} ((T_1 \otimes T_2) Q_d Mf) (MP_d g) \, dv_1 \times v_2
\]

\[
= \int_{H \times H} ((T_1 \otimes T_2) Q_d Mf) (Q_d Mg) \, dv_1 \times v_2
\]

\[
= \int_{H \times H} ((Q_d(T_1 \otimes T_2) Q_d Mf) (Mg) \, dv_1 \times v_2
\]

\[
= \int_{H \times H} ((T_1 \otimes T_2) Q_d Mf) (Mg) \, dv_1 \times v_2
\]

But it is easy to see that

\[
(T_1 \otimes T_2)_d = T_{1,d} \otimes T_{2,d}.
\]

Hence again using Corollary 2.4 and Theorem 3.3(iii) we get

\[
(u \ast v)_d(f, g) = \int_{H \times H} ((T_{1,d} \otimes T_{2,d} Mf) (Mg) \, dv_1 \times v_2
\]

\[
= \int_{H} (M^{-1}P(T_{1,d} \otimes T_{2,d}) Mf) g \, dv_1 \ast v_2
\]

\[
= (u_d \ast v_d)(f, g).
\]

Combining Theorem 3.7 with the Remarks 3.1 we have the following.

Theorem 3.8. In $BM(G, H)$, $BM_d(G, H)$ is a closed subalgebra and $BM_\ast(G, H)$ is a closed ideal.
4. THE CLOSURE OF $L^1(G \times H)$

**Definition 4.1.** Let $BM_a(G, H)$ denote the closure of $L_1(G \times H)$ (considered as the space of absolutely continuous measures relative to a fixed left Haar measure on $G \times H$) in $BM(G, H)$. It follows immediately from the definitions that

$$BM_a(G, H) = L^1(G) \otimes \lambda L^1(H)$$

isometrically, where $\lambda$ denotes the least cross norm (injective tensor product). Elements of $BM_a(G, H)$ might be called "absolutely continuous bimeasures," for as we shall see below, $BM_a(G, H)$ plays a role in $BM(G, H)$ similar to that played by $L^1(G \times H)$ in $M(G \times H)$. Clearly $BM_a(G, H)$ is the closure of any of the spaces $C_{00}(G \times H)$, $C_{00}(G) \otimes C_{00}(H)$, or the "trigonometric polynomials" (span of the matrix coefficients of unitary representations) when $G$ and $H$ are compact. For $f \in L^1(G \times H)$, let $u_f$ denote the bimeasure determined by $f$.

**Lemma 4.2.** Let $\phi \in V_0(G, H)$ and $v \in BM(G, H)$ both have compact support. Then $u_\phi \ast v$, $v \ast u_\phi \in BM_a(G, H)$. In fact,

$$u_{v \ast \phi} = v \ast u_\phi.$$

**Proof.** Let $f \in C_{00}(G)$ and $g \in C_{00}(H)$. Then

$$\int_H \int_G f(x) g(y) \phi^{(x,y)}(x', y') \, dx \, dy = \int_H \int_G f(x) \left( \int_G g(y) \phi(x, y) \, dy \right) \, dx$$

Thus by Theorem 2.6,

$$u_{v \ast \phi}(f, g) = \int_H \int_G f(x) g(y) v \ast \phi(x, y) \, dx \, dy$$

$$= \int_H \int_G f(x) g(y) \tilde{v}(\phi^{(x,y)}) \, dx \, dy$$

$$= \tilde{v} \left( \int_H \int_G f(x) g(y) \phi^{(x,y)} \, dx \, dy \right)$$
\[ I = \int_G \int_H \left( \int_G \int_H (f \otimes g)(x, y) \varphi(x, y) \, dx \, dy \right) \, d\nu \]
\[ = \int_G \int_H \nu \ast (f \otimes g)(x, y) \varphi(x, y) \, dx \, dy \]
\[ = \nu \ast (f \otimes g)(x, y) \]
\[ = v \ast u_{\phi}(f, g). \]

A similar calculation shows that if \( u_{\phi} \) and \( u_{\phi \ast v} \) are defined relative to a right Haar measure on \( G \times H \), then \( u_{\phi} \ast v = u_{\phi \ast v} \).

Since the bimeasures with compact support are dense in \( BM(G, H) \) [7, Corollary 1.3], the following theorem follows immediately from Definition 4.1 and Lemma 4.2.

**Theorem 4.3.** \( BM_a(G, H) \) is a closed ideal in \( BM(G, H) \).

**Theorem 4.4.** For \( x \in G \) and \( y \in H \), let \( R_{(x,y)}^* \) denote the operator on \( BM(G, H) \) given by

\[ (R_{(x,y)}^* u)(f, g) = u(f^x, g^y) \]

for \( u \in BM(G, H) \), \( f \in C_0(G) \) and \( g \in C_0(H) \). Then \( u \in BM_a(G, H) \) if and only if the function \( (x, y) \mapsto R_{(x,y)}^* u \) is norm-continuous.

**Proof.** Since \( R_{(x,y)}^* \) is an isometry on \( BM(G, H) \) for all \( x \) and \( y \) and right translation is norm continuous on \( L^1(G \times H) \), and since the \( L^1 \)-norm dominates the bimeasure norm, it is clear that the function of right translation is continuous on elements of \( BM_a(G, H) \).

Let \( v \in BM(G, H) \) such that the function \( (x, y) \mapsto R_{(x,y)}^* v \) is norm continuous. Given \( \varepsilon > 0 \), choose a neighborhood \( U \) of the identity in \( G \times H \) such that \( \| v - R_{(x,y)}^* v \| < \varepsilon \) if \( (x, y) \in U \) and a function \( \varphi \in C_0(G) \otimes C_0(H) \) such that \( \varphi = 0 \) off \( U \) and \( \int_G \int_H \varphi(x, y) \, dx \, dy = 1 \). If \( f \in C_0(G) \), \( g \in C_0(H) \), \( \| f \|_a \leq 1 \), and \( \| g \|_H \leq 1 \), then a calculation similar to that appearing in the proof of Lemma 4.2 yields

\[ \| (v - v \ast u_{\varphi})(f, g) \| = \left| \int_H \int_G v(f, g) \varphi(x, y) \, dx \, dy \right| \]
\[ - v \left( \int_H \int_G (f^x \otimes g^y) \varphi(x, y) \, dx \, dy \right) \]
\[ = \left| \int_H \int_G (v(f, g) - v(f^x, g^y)) \varphi(x, y) \, dx \, dy \right| \]
Thus $\|v - v \ast u_\phi\| < \varepsilon$, and our theorem follows from Theorem 4.3.

**THEOREM 4.5.** Let $T : BM_{a}(G, H) \to BM_{a}(G, H)$ be a right multiplier (i.e., $T(u \ast v) = u \ast Tv$). Then there exists $v \in BM(G, H)$ such that $Tu = u \ast v$, $u \in BM_{a}(G, H)$.

*Proof.* Let $\{ f_\alpha \}$ be an approximate identity consisting of elements of norm at most one for $L^1(G \times H)$. Set $u_\alpha = u_{f_\alpha}$ and $v_\alpha = Tu_\alpha$. Then $\{ u_\alpha \}$ is an approximate identity for $BM_{a}(G, H)$ consisting of elements of norm at most one. Let $v$ be a weak-* limit point of $\{ v_\alpha \}$ in $BM(G, H)$. Then it follows immediately from Theorem 2.6 that $u \ast v_\alpha \rightharpoonup u \ast v$. Hence for $u \in BM_{a}(G, H)$ and $f \in V_{0}(G, H),$

$$ (Tu)(f) = \lim_{\alpha} T(u \ast u_\alpha)(f) $$

$$ = \lim_{\alpha} (u \ast v_\alpha)(f) $$

$$ = u \ast v(f), $$

and our theorem follows.

**Remark 4.6.** There is one significant way in which the position of $BM_{a}(G, H)$ in $BM(G, H)$ differs from that of $L^1(G \times H)$ in $M(G \times H)$. Namely, there need not exist a bounded projection from $BM(G, H)$ onto $BM_{a}(G, H)$, so there is no natural subspace of $BM(G, H)$ whose elements might be called "singular bimeasures." For instance it is proved in [7, Theorem 4.73] that if $G$ is abelian and nondiscrete then no such projection exists from $BM(G, G)$ onto $BM_{a}(G, G)$.

5. LIFTING REPRESENTATIONS

In this section we shall show that the classical lifting of unitary representations of $G \times H$ to *-representations of $M(G \times H)$ has an analogue for the algebra $BM(G, H)$. To accomplish this task we must again appeal to Theorem 1.1. We shall then apply this result to study the relationship between $BM(G, H)$ and the group von Neumann algebra $VN(G \times H)$ for certain groups $G$ and $H.$
Lemmas. Let $\pi$ and $\rho$ be unitary representations of $G$ and $H$ on the Hilbert spaces $H_\pi$ and $H_\rho$, respectively, and let $H_\pi \otimes_H H_\rho$ denote the Hilbert-space tensor product of $H_\pi$ and $H_\rho$. For $\zeta, \omega \in H_\pi \otimes_H H_\rho$, set

$$\chi_{\zeta,\omega}(x, y) = \langle \pi(x) \otimes \rho(y) \zeta, \omega \rangle, \quad x \in G, y \in H.$$ 

Then $\chi_{\zeta,\omega} \in V(G, H)$, and $\|\chi_{\zeta,\omega}\|_V \leq K_G \|\zeta\| \|\omega\|$.

Proof. For $\zeta, \eta \in H_\pi$ and $\xi', \eta' \in H_\rho$, set

$$\varphi_{\zeta,\eta}(x) = \langle \pi(x) \xi, \eta \rangle, \quad x \in G \tag{11}$$

and

$$\psi_{\xi',\eta'}(y) = \langle \rho(y) \xi', \eta' \rangle, \quad y \in H. \tag{12}$$

Let $e_1, \ldots, e_M$ be orthonormal elements of $H_\pi$ and $e'_1, \ldots, e'_N$ be orthonormal elements of $H_\rho$, and suppose we can write

$$\zeta = \sum_{m=1}^{M} \sum_{n=1}^{N} \alpha_{mn} e_m \otimes e'_n,$$

$$\omega = \sum_{j=1}^{M} \sum_{k=1}^{N} \beta_{jk} e_j \otimes e'_k,$$

$$\xi_n = \sum_{m=1}^{M} \alpha_{mn} e_m, \quad 1 \leq n \leq N,$$

and

$$\eta'_j = \sum_{k=1}^{N} \beta_{jk} e'_k, \quad 1 \leq j \leq M.$$

Thus

$$\|\zeta\|^2 = \sum_{n=1}^{N} \sum_{m=1}^{M} |\alpha_{mn}|^2 = \sum_{n=1}^{N} \|\xi_n\|^2,$$

and

$$\|\omega\|^2 = \sum_{j=1}^{M} \sum_{k=1}^{N} |\beta_{jk}|^2 = \sum_{j=1}^{M} \|\eta'_j\|^2.$$
Then
\[
\chi_{\xi,\omega}(x, y) = \sum_{m,j=1}^{M} \sum_{n,k=1}^{N} a_{mn} \overline{B}_{jk} \langle \pi(x) e_m, e_j \rangle \langle \rho(y) e_n', e_k' \rangle
\]
\[
= \sum_{j=1}^{M} \sum_{n=1}^{N} \langle \pi(x) \xi_n, e_j \rangle \langle \rho(y) e_n', \eta_j' \rangle
\]
\[
= \sum_{j=1}^{M} \sum_{n=1}^{N} \varphi_{\xi_n,\eta_j'}(x) \psi_{\xi_n,\eta_j'}(y).
\]

For any \( u \in BM(G, H) \), let \( \mu \) and \( \nu \) be suitable Grothendieck measures for \( u \). Then
\[
|u(\chi_{\xi,\omega})| \leq \sum_{j=1}^{M} \sum_{n=1}^{N} |u(\varphi_{\xi_n,\eta_j'})| |\psi_{\xi_n,\eta_j'}|
\]
\[
\leq K_G \|u\| \sum_{j=1}^{M} \sum_{n=1}^{N} \left( \int_G |\varphi_{\xi_n,\eta_j'}|^2 \, d\mu \right)^{1/2} \left( \int_H |\psi_{\xi_n,\eta_j'}|^2 \, d\nu \right)^{1/2}
\]
\[
\leq K_G \|u\| \left( \sum_{n=1}^{N} \int_G \left( \sum_{j=1}^{M} |\varphi_{\xi_n,\eta_j'}|^2 \right) \, d\mu \right)^{1/2} \left( \sum_{n=1}^{N} \int_H \left( \sum_{j=1}^{M} |\psi_{\xi_n,\eta_j'}|^2 \right) \, d\nu \right)^{1/2}
\]
\[
= K_G \|u\| \|\xi\| \|\omega\|.
\]

Hence \( \|\chi_{\xi,\omega}\| \nu \leq K_G \|\xi\| \|\omega\| \). Thus if \( \zeta, \omega \in H_\pi \otimes_H H_\rho \) then \( \chi_{\zeta,\omega} \in V(G, H) \) and \( \|\chi_{\zeta,\omega}\| \nu \leq K_G \|\zeta\| \|\omega\| \).

**Theorem 5.2.** Let \( \pi \) and \( \rho \) be unitary representations of \( G \) and \( H \), respectively. There is a unique \,*\,-representation \( T \) of \( BM(G, H) \) by operators on \( H_\pi \otimes_H H_\rho \) satisfying
\[
\langle T_u(\xi \otimes \xi'), \eta \otimes \eta' \rangle = u(\varphi_{\xi,\eta}, \psi_{\xi',\eta'})
\]
for all \( \xi, \eta \in H_\pi \) and \( \xi', \eta' \in H_\rho \), where \( \varphi_{\xi,\eta} \) is given by (11) and \( \psi_{\xi',\eta'} \) by (12). Moreover,
\[
\|T_u\| \leq K_G \|u\|, \quad u \in BM(G, H).
\]

**Proof.** It follows from Lemma 5.1 that for all \( u \in BM(G, H) \) there exists an operator \( T_u \) on \( H_\pi \otimes_H H_\rho \) such that
\[
\langle T_u \zeta, \omega \rangle = u(\chi_{\zeta,\omega}), \quad \zeta, \omega \in H_\pi \otimes_H H_\rho,
\]
that \( \|T_u\| \leq K_G \|u\| \), and that \( T_u \) is uniquely determined by considering all vectors of the form \( \zeta = \xi \otimes \xi' \) and \( \omega = \eta \otimes \eta' \).

To see that \( T \) is a *-representation, we need only check that it is multiplicative and adjoint-preserving. Let \( u, v \in BM(G, H) \), \( \xi, \eta \in H_\pi \) and \( \xi', \eta' \in H_\rho \). Then

\[
\varphi_{\xi, \eta}(xx') = \varphi_{\pi(x') \xi, \eta}(x)
\]

and

\[
\psi_{\xi', \eta'}(yy') = \psi_{\rho(y') \xi', \eta'}(y),
\]

so

\[
\tilde{u} * (\varphi_{\xi, \eta} \otimes \psi_{\xi', \eta'})(x', y') = u(\varphi_{\pi(x') \xi, \eta}, \psi_{\rho(y') \xi', \eta'})
\]

\[
= \langle T_u(\pi(x')) \xi \otimes \rho(y') \xi', \eta \otimes \eta' \rangle
\]

\[
= \chi_{\xi \otimes \xi'} T_u^*(\eta \otimes \eta')(x', y').
\]

Hence applying Corollary 2.7,

\[
\langle T_{u*v}(\xi \otimes \xi'), \eta \otimes \eta' \rangle = u * v(\varphi_{\xi, \eta}, \psi_{\xi', \eta'})
\]

\[
= v(\tilde{u} * (\varphi_{\xi, \eta} \otimes \psi_{\xi', \eta'}))
\]

\[
= v(\chi_{\xi \otimes \xi'} T_u^*(\eta \otimes \eta'))
\]

\[
= \langle T_v(\xi \otimes \xi'), T_u^*(\eta \otimes \eta') \rangle
\]

\[
= \langle T_{u*v}(\xi \otimes \xi'), \eta \otimes \eta' \rangle,
\]

so \( T_{u*v} = T_u T_v \). Finally,

\[
\langle T_{\tilde{u}}(\xi \otimes \xi'), \eta \otimes \eta' \rangle = \tilde{u} * (\varphi_{\xi, \eta} \otimes \psi_{\xi', \eta'})
\]

\[
= u(\varphi_{\xi, \eta}^*, \psi_{\xi', \eta'}^*)
\]

\[
= u(\varphi_{\xi, \xi'}, \psi_{\eta, \eta'})
\]

\[
= \langle T_\mu(\eta \otimes \eta'), \xi \otimes \xi' \rangle
\]

\[
= \langle T^*_\mu(\xi \otimes \xi'), \eta \otimes \eta' \rangle.
\]

Thus \( T_{\tilde{u}} = T^*_\mu \), and our proof is complete.

For any locally compact group \( G \) with "dual object" \( \hat{G} \) and \( \mu \in M(G) \), set

\[
\|\mu\|_* = \sup_{\pi \in \mathcal{G}} \|\pi(\mu)\|,
\]

where \( \pi(\mu) = \int_G \pi(x) \, d\mu(x) \). Recall that the group C*-algebra \( C^*(G) \) is the
completion of $L^1(G)$ with respect to the norm $\|f\|_*$ and the group von Neumann algebra $VN(G)$ is the second dual of $C^*(G)$. Moreover, there is a natural algebra embedding of norm one taking $M(G)$ into $VN(G)$ which is a consequence of the following observation. Set $\mathcal{H}_\pi^G = \bigoplus_{\pi \in \pi} H_\pi$ and $\pi_*^G = \bigoplus_{\pi \in \pi} \pi$. Then $\|\mu\|_* = \|\pi_*(\mu)\|$ and $VN(G)$ is the weak-operator closure of $C^*(G)$ as a subalgebra of $\mathcal{B}(\mathcal{H}_\pi^G)$.

Return now to the consideration of our pair of groups $G$ and $H$. If either of the groups is of Type I, then it is well known that

$$(G \times H)^* = \hat{G} \otimes \hat{H} = \{ \pi \otimes \rho : \pi \in \hat{G}, \rho \in \hat{H} \}.$$ 

Thus $\mathcal{H}^G \times H = \mathcal{H} \otimes_H \mathcal{H}^H$ and $\pi_*^{G \times H} = \pi_*^G \otimes \pi_*^H$. We are thus led to the following consequence of Theorem 5.2.

**Corollary 5.3.** Suppose that at least one of the groups $G$ and $H$ is of Type I. Each of the following inclusions represents a $*$-algebra embedding of norm at most $K_G$.

\[ L^1(G \times H) \subseteq BM_a(G, H) = L_1(G) \otimes_\lambda L_1(H) \subseteq C^*(G \times H) \]
\[ M(G \times H) \subseteq BM(G, H) \subseteq VN(G \times H). \]

**Proof.** By Theorem 5.2, $\pi_*^{G \times H}$ lifts to a $*$-representation of $BM(G, H)$ of norm at most $K_G$. In particular, setting $\pi = \pi_*^{G \times H}$ we have

$$\pi(BM_a(G, H)) \subseteq \overline{\pi(L^1(G \times H))} = C^*(G \times H).$$

To prove the second assertion, let $\zeta, \omega \in \mathcal{H}_\pi^G \times H$ and let $\{f_\alpha\}$ be a suitable approximate identity in $L^1(G \times H)$. Then setting $u_\alpha = u_{f_\alpha}$, we have for all $v \in BM(G, H)$,

$$\lim_{\alpha} \langle \pi(u_\alpha \ast v), \omega \rangle = \lim_{\alpha} \langle \pi(u_\alpha \pi(v)), \zeta, \omega \rangle = \lim_{\alpha} \int_H \int_G \langle (\pi^G(x) \otimes \pi^H(y)) \pi(v), \zeta, \omega \rangle f_\alpha(x, y) \, dx \, dy = \langle \pi(v), \zeta, \omega \rangle.$$ 

Hence $\pi(u_\alpha \ast v) \to \pi(v)$ in the weak operator topology, so $\pi(v) \in VN(G \times H)$. 


6. Maximal Ideal Spaces

Throughout this section, let $G$ and $H$ be locally compact abelian groups. Our objective in this section is to show that there exists a natural injection

$$\iota: \mathcal{M}(G) \times \mathcal{M}(H) \to \mathcal{M}(BM(G, H)),$$

If either $G$ or $H$ is discrete, this mapping is surjective. Note that if $\pi: \mathcal{M}(BM(G, H)) \to \mathcal{M}(G \times H)$ is the obvious restriction mapping, then $\pi \iota$ injects $\mathcal{M}(G) \times \mathcal{M}(H)$ in $\mathcal{M}(G \times H)$, a fact which to the best of our knowledge was not known heretofore. The construction of $\iota$ is based on the extension of elements of $BM(X, Y)$ to bilinear functionals on $C_0(X)^* \times C_0(Y)^*$ described in Definition 1.2.

Recall that if $\Phi \in \mathcal{M}(G)$ and $\mu \in M(G)$ with $\mu \geq 0$, then there exists a bounded, measurable function $\Phi_\mu$ on $G$ such that

$$\langle v, \Phi \rangle = \int_G \Phi_\mu \, dv$$

for all $v \in M(G)$ such that $v \leq \mu$ and

$$\Phi_\mu(x + y) = \Phi_\mu(x) \Phi_\mu(y) \quad \mu \times \mu\text{-a.e.},$$

that is,

$$M\Phi_\mu = \Phi_\mu \otimes \Phi_\mu \quad \mu \times \mu\text{-a.e.}$$

The functions $\Phi_\mu$ are called generalized characters [15, Sects. 1, 2], [6, Sect. 5.1]. Let $\hat{G}$ and $\hat{H}$ denote the character groups of $G$ and $H$, respectively, and recall that $\hat{u}$ denotes the Fourier transform of a bimeasure $u$ on $G \times H$ (cf. Remark 2.9).

**Theorem 6.1.** For $\Phi \in \mathcal{M}(G)$ and $\Psi \in \mathcal{M}(H)$, define $\iota(\Phi, \Psi)$ in $BM(G, H)^*$ by

$$\langle u, \iota(\Phi, \Psi) \rangle = u^{**}(\Phi, \Psi).$$

Then $\iota$ is a separately continuous injection of $\mathcal{M}(G) \times \mathcal{M}(H)$ into $\mathcal{M}(BM(G, H))$ satisfying

$$\langle u, \iota(\Phi_\gamma, \Psi_\delta) \rangle = \hat{u}(\gamma, \delta), \quad \gamma \in \hat{G}, \delta \in \hat{H}.$$
$v \in BM(G, H)$ and let $\mu_1, \mu_2, \nu_1, \nu_2, T_1$ and $T_2$ be as in the statement of Corollary 2.4. Set

$$\varphi = \Phi_{\mu_1 + \mu_2 + \mu_1 \ast \mu_2}, \quad \psi = \Psi_{\nu_1 + \nu_2 + \nu_1 \ast \nu_2}. $$

Then relative to $\mu_1, \mu_2$ or $\mu_1 \ast \mu_2$, $I^{**}(\Phi) = \varphi$, while relative to $\nu_1, \nu_2$ or $\nu_1 \ast \nu_2$, $I^{**}(\Psi) = \psi$. Hence by Corollary 2.4, (13) and (14),

$$\langle u \ast v, \iota(\Phi, \Psi) \rangle = (u \ast v)^{**}(\Phi, \Psi)$$

$$= \int_H M^{-1}P(T_1 \otimes T_2) M1^{**}(\Phi) I^{**}(\Psi) \, dv_1 \ast v_2$$

$$= \int_H (M^{-1}P(T_1 \otimes T_2) M\varphi) \psi \, dv_1 \ast v_2$$

$$= \int_{H \times H} \left( (T_1 \otimes T_2) M\varphi (M\psi) \right) \, dv_1 \times v_2$$

$$= \int_{H \times H} \left( (T_1 \otimes T_2)(\varphi \otimes \varphi) \right) (\psi \otimes \psi) \, dv_1 \times v_2$$

$$= \int_H (T_1 \varphi) \psi \, dv_1 \int_H (T_2 \varphi) \psi \, dv_2$$

$$= \int_H T_1 I^{**}(\Phi) I^{**}(\Psi) \, dv_1 \int_H T_2 I^{**}(\Phi) I^{**}(\Psi) \, dv_2$$

$$= u^{**}(\Phi, \Psi) v^{**}(\Phi, \Psi)$$

$$= \langle u, \iota(\Phi, \Psi) \rangle \langle v, \iota(\Phi, \Psi) \rangle.$$

That $\iota$ is separately continuous was observed in Lemma 1.3. To see that $\iota$ is an injection, let us identify $M(G)$ and $M(H)$ with the subspaces of $BM(G, H)$ of bimeasures supported on $G \times \{0\}$ and on $\{0\} \times H$, respectively. If $\mu \in M(G)$, let $\sigma = |\mu|/\|\mu\|$. Then the bimeasure associated with $\mu$ is given by

$$u(f, g) = g(0) \int_G f \mu,$$

and it is easy to see that $\sigma$ and $\delta_0$ are Grothendieck measures for $u$, an associated operator being the operator of rank one

$$Tf = \int_G f \mu, \quad f \in L^2(G, \mu).$$
Thus if $\Phi \in \mathcal{M}_G$ and $\Psi \in \mathcal{M}_H$, then

$$
\langle u, i(\Phi, \Psi) \rangle = \left( \int_G I^*(\Phi) \, d\mu \right) \Psi(\delta_0)
$$

$$
= \left( \int_G I^*(\Phi) \frac{d\mu}{d\sigma} \, d\sigma \right) \Psi(\delta_0)
$$

$$
= \Phi(\mu).
$$

Similarly, if $v \in M(H)$ and $u$ is the bimeasure associated with $v$ then

$$
\langle v, i(\Phi, \Psi) \rangle = \Psi(v).
$$

It follows that $i$ is injective. Our last assertion is a consequence of Corollary 1.4.

It is now natural to ask whether the mapping $i$ is also surjective and hence whether $\mathcal{M}_{BM(G,H)}$ can be identified with $\mathcal{M}_G \times \mathcal{M}_H$. We shall conclude by showing that sometimes $i$ is a surjection and sometimes it is not.

**Theorem 6.2.** Let $i: \mathcal{M}_G \times \mathcal{M}_H \rightarrow \mathcal{M}_{BM(G,H)}$ be the map defined in Theorem 6.1.

(i) If either $G$ or $H$ is discrete, then $i$ is a surjection.

(ii) If $G = H$ and $G$ is nondiscrete, then $i$ is not a surjection.

**Proof:** (i) Since $M(G \times H)$ is dense in $BM(G, H)$ in this case [7, Theorem 5.12], it suffices to prove that if $\Theta \in \mathcal{M}_{BM(G,H)}$ there exist $\Phi \in \mathcal{M}_G$ and $\Psi \in \mathcal{M}_H$ such that if we identify measures with the bimeasures they induce, then

$$
\Theta(\mu) = i(\Phi, \Psi)(\mu), \quad \mu \in M(G \times H).
$$

Again let us identify $M(G)$ and $M(H)$ with the corresponding subalgebras of $BM(G, H)$, as in the proof of Theorem 6.1. If $\mu \in M(G)$ and $v \in M(H)$, then $\mu \times v = \mu * v$ (convolution in $M(G \times H)$). If either $G$ or $H$ is discrete, the algebra $M(G) \otimes M(H)$ generated by $M(G)$ and $M(H)$ is dense in $M(G \times H)$, hence in $BM(G, H)$. Set

$$
\Theta|_{M(G)} = \Phi, \quad \Theta|_{M(H)} = \Psi.
$$

Arguing as in the last paragraph of the proof of Theorem 6.1, we see that

$$
\langle \mu, i(\Phi, \Psi) \rangle = \Phi(\mu)
$$

and

$$
\langle v, i(\Phi, \Psi) \rangle = \Psi(v).
$$
Hence
\[
\Theta(\mu \times v) = \Phi(\mu) \Psi(v) = \langle \mu \times v, \iota(\Phi, \Psi) \rangle.
\]
Hence \(\Theta\) and \(\iota(\Phi, \Psi)\) agree on \(M(G) \otimes M(H)\), and assertion (i) follows.

(ii) Assume now that \(G\) is nondiscrete, and let \(A\) denote the diagonal in \(\hat{G} \times \hat{G}\). \(A\) is a closed subgroup of \(\hat{G} \times \hat{G}\), so the Fourier–Stieltjes algebra \(B(A)\) satisfies
\[
B(A) = \{ \phi|_A : \phi \in B(\hat{G} \times \hat{G}) \}.
\]
Let \(A\) denote the uniform closure of \(B(A)\) and \(C_u(A)\) be the algebra of bounded, uniformly continuous functions on \(A\). Then
\[
C_u(A) = \{ u|_A : u \in BM(G, H) \}
\]
[7, Theorem 5.8]. Each \(\omega \in \mathcal{M}_{C_u(A)}\) determines a multiplicative linear functional on \(BM(G, G)\), namely
\[
\Theta_\omega(u) = \omega(u|_A), \quad u \in BM(G, G).
\]
Since both \(A\) and \(C_u(A)\) are closed, conjugate-closed algebras, Stone’s Theorem says that the maximal ideal space of \(A\) is given by sets of constancy of elements of \(A\) in \(\mathcal{M}_{C_u(A)}\). That is, since \(A \subsetneq C_u(A)\), there is a quotient map \(\rho : \mathcal{M}_{C_u(A)} \to \mathcal{M}_A\), and \(\rho\) is not injective. But two elements of the image of \(\iota\) which agree on \(M(G \times H)\) must agree on \(BM(G, H)\). Thus if \(\omega_1, \omega_2 \in \mathcal{M}_{C_u(A)}\) such that \(\rho(\omega_1) = \rho(\omega_2)\), then at least one of \(\Theta_{\rho(\omega_1)}\) and \(\Theta_{\rho(\omega_2)}\) cannot be in the image of \(\iota\).

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