OPEN CENTRALIZERS AND THE CONTINUITY OF GROUP REPRESENTATIONS

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Abstract. Let $G$ be a locally compact group, $\pi: G \rightarrow \mathcal{L}(L^2(G))$ the right regular representation of $G$, and $G^c = \{ x \in G : \text{the function } g \mapsto \pi(gxg^{-1}) \text{ is norm continuous} \}$. This note is devoted to the study of $G^c$. In particular, the compactly generated groups for which $G = G^c$ are characterized.

1. Let $G$ be a locally compact group and let $\pi: G \rightarrow \mathcal{L}(L^2(G))$ be the right regular representation of $G$ on $L^2(G)$ with respect to a right Haar measure. The function $\pi$ is continuous when $\mathcal{L}(L^2(G))$ is given the strong operator topology, but $\pi$ is not continuous with respect to the norm topology, except in trivial cases. Nevertheless, there is a middle ground, to which this note is devoted.

Following [5], [8], let $\mathcal{L}_G = \{ T \in \mathcal{L}(L^2(G)) : \text{the function } g \mapsto \pi(g)T\pi(g)^* \text{ is norm continuous} \}$. Then $\mathcal{L}_G$ is a $C^*$-algebra which contains the compact operators and has various pleasing properties (cf. [5, Theorem 2.2]). Let

$$G^c = \{ x \in G : \pi(x) \in \mathcal{L}_G \} = \{ x \in G : \text{the function } g \mapsto \pi(gxg^{-1}) \text{ is norm continuous} \}.$$

It is easy to see that $G^c$ is a subgroup of $G$. If $G$ is abelian or discrete, then $G^c = G$; in general it is much smaller. The relationship between $G^c$ and $G$ is the main subject of this note.

We shall denote the identity component and the center of $G$ by $G_0$ and $Z(G)$, respectively. For $x, y \in G$, $C_G(x)$ denotes the centralizer of $x$ in $G$ and $[x, y] = xyx^{-1}y^{-1}$.

Definition 1.1. Let $B$ be a Banach space of functions on $G$. Suppose that there exist constants $C, \delta > 0$ such that the following conditions are satisfied:

(i) If $\varphi \in B$ and $x \in G$, then $B\pi(x)\varphi \in B$, and $\| B\pi(x)\varphi \| \leq C\| \varphi \|$, where $B\pi(x)\varphi(t) = \varphi(x^{-1}t)$.

(ii) Given $\varphi \in B$, there exists $\lambda(\varphi) > 0$ such that $\| \varphi + \psi \| > \lambda(\varphi)$ for all $\psi \in B$ such that $\varphi \cdot \psi = 0$.

(iii) For every neighborhood $U$ of $e$ in $G$ there exists $0 \neq \varphi \in B$ such that $\varphi = 0$ off $U$ and $\lambda(\varphi) > \delta \| \varphi \|$.

Then $B$ will be called a homogeneous separating Banach space of functions on $G$.

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EXAMPLES 1.2. Most of the Banach spaces commonly encountered in harmonic analysis satisfy the conditions of Definition 1.1. The following is a sampling of such spaces.

(i) $L^p(G)$, $1 \leq p \leq \infty$,

(ii) $C_0(G)$,

(iii) the Fourier algebra $A(G)$ [1],

(iv) the Sobolev spaces $W^s_p(G)$ for $G$ a Lie group,

(v) the spaces $A^s_0(G) = L^p(G) \hat{\otimes} L^q(G)$ of Figà-Talamanca and Gaudry [2], [3] on $G \times G$.

DEFINITION 1.3. Let $B$ be a homogeneous separating Banach space of functions on $G$ and $\pi: G \to \mathcal{L}(B)$ be the regular representation as in (i) of Definition 1.1. Set

$$\pi^c = \{ x \in G : \text{the function } g \mapsto \pi(gx^{-1}) \text{ is norm continuous} \}.$$

The following observation provides the technical tool which is the key to studying $\pi^c$.

THEOREM 1.4. Let $B$, $\pi$ and $\pi^c$ be as in Definition 1.3, and let $x \in G$. Then $x \in \pi^c$ if and only if $C_G(x)$ is an open subgroup of $G$.

In view of Theorem 1.4 the subscript $B$ in $\pi^c$ is redundant and will be omitted following the proof of this theorem. Furthermore, the superscript in $G^c$ may be read as referring to (norm) continuity or to (open) centralizers.

PROOF. Suppose first that $C_G(x)$ is open in $G$. Then the function $g \mapsto \pi(gx^{-1})$ is constant when restricted to the open subgroup $C_G(x)$ of $G$, so in particular it is norm continuous at the identity. Since $\pi$ is a norm-bounded representation of $G$ (1.1(i)), it follows that the function in question is continuous on all of $G$.

Conversely, suppose that $C_G(x)$ is not an open subgroup of $G$. Then any neighborhood $V$ of $e$ must contain some $v \in V \setminus C_G(x)$. We shall show that $\| \pi(gx^{-1}) - \pi(x) \|$ is bounded away from zero for all such $g$, hence $x \notin \pi^c$.

As $[g, x] \neq e$ there is a neighborhood $W$ of $e$ such that $W \cap W[g, x] = \emptyset$. Let $0 \neq \varphi \in B$ such that $\varphi = 0$ off $W$ and $\lambda(\varphi) > \delta \| \varphi \|$. Then $\pi([g, x])$ is supported on $W[g, x]$. Thus

$$\| \pi(gx^{-1}) - \pi(x) \| = \| \pi([g, x]) - I \pi(x) \| > \| \pi([g, x]) - I \| / C > \| \pi([g, x]) \varphi - \varphi \| / C \| \varphi \| > \lambda \varphi / C \| \varphi \| > \delta / C.$$

COROLLARY 1.5. Let $G$ be a connected group. Then $G^c = Z(G)$.

PROOF. For any group $G$ one has $Z(G) \subseteq G^c$ trivially. Conversely, suppose that $x \in G^c$. Then $C_G(x)$ is an open subgroup of $G$. But $G$ is connected, so $C_G(x) = G$. Thus $x \in Z(G)$. □

COROLLARY 1.6. Suppose that $G^c = G$. Then $G_0 \subseteq Z(G)$.

The remainder of this note is devoted to the study of $G^c$. In §2 we consider the case when $G = G^c$ and we obtain complete information when $G$ is compactly generated. §3 is concerned with other cases.
2. In this section we characterize the compactly generated groups for which \( G = G^c \) and we obtain various equivalent formulations of this property.

**Lemma 2.1.** Suppose that \( G = G^c \). Let \( A \) and \( B \) be compact sets of \( G \). Then

\[
[A, B] \equiv \{ [x, y] : x \in A, y \in B \}
\]

is a finite set.

**Proof.** Let \( w, x, y, z \in G \) with

\[
z \in C_G(x) \cap C_G(y)
\]  

(2.2)

and

\[
w \in C_G(x) \cap C_G(y) \cap C_G(z).
\]  

(2.3)

Then

\[
[z, w] = zwxwy^{-1}x^{-1}y^{-1}w^{-1}
\]

\[
= xwyx^{-1}w^{-1}x^{-1}y^{-1} \quad \text{by (2.2) and (2.3)}
\]

\[
= xy[z, w]x^{-1}y^{-1}
\]

\[
= [x, y] \quad \text{by (2.3)}.
\]

Choose \( x \in A \) and \( y \in B \). Then there is a neighborhood \( U_x \times V_y \) of \((x, y) \in A \times B \) such that \([u, v] = [x, y]\) for all \((u, v) \in U_x \times V_y \). Pick a finite subcover from the open cover \( \{U_x \times V_y\} \), and the lemma follows. \( \square \)

**Theorem 2.4.** Let \( G \) be a compactly generated group. Then \( G = G^c \) if and only if \( Z(G) \) is an open subgroup of \( G \).

**Proof.** As previously noted, \( Z(G) \subset C_G(x) \) for all \( x \in G \). If \( Z(G) \) is open, then every group \( C_G(x) \) is open, and hence \( G^c = G \).

Conversely, suppose that \( G = G^c \). Let \( K \) be a compact neighborhood of \( e \) which topologically generates \( G \). It suffices to prove that \( C_G(K) \) is open, since \( C_G(K) = Z(G) \). By Lemma 2.1, there is a neighborhood \( U \subset U \) of \( e \) with \( U \subset K \) and \([K, K] \cap U = \{e\}\). Let \( V \) be a symmetric neighborhood of \( e \) with \([V, V] \subset U \), so that \([V, V] = \{e\}\). Let \( H \) be the subgroup of \( G \) generated by \( V \). Then \( H \) is an abelian subgroup of \( G \), and \( H \) is open since \( V \) is open. Choose \( k_1, \ldots, k_n \in K \) such that \( K \subset U \cup \bigcup_{j=1}^{n} k_j H \). Let \( L \) be the open subgroup of \( G \) defined by

\[
L = H \cap \bigcap_{j=1}^{n} C_G(k_j).
\]

If \( k \in K \) then \( k = k_j y \) for some \( j \) and for some \( y \in L \). Thus for any \( x \in L \), one has \([k, x] = k_j y, x = e\), since \( L \subset C_G(k_j) \). This shows that \( L \subset C_G(K) \) and so \( C_G(K) \) is an open subgroup, completing the proof. \( \square \)

**Example 2.5.** The hypothesis that \( G \) be compactly generated in Theorem 2.4 seems essential. We shall exhibit a group \( G \) such that \( G^c = G \) but \( Z(G) \) is not open. Let \( H_0 \) be a finite abelian group, and set \( H = \prod_{n=1}^{\infty} H_n \), where each \( H_n \) is isomorphic to \( H_0 \). Let

\[
\Sigma = \bigoplus_{n=1}^{\infty} (Z(2))^n.
\]
and let $\alpha: \Sigma \to \text{Aut}(H)$ be the isomorphism such that the image of the generator of the $j$th summand of $\Sigma$ interchanges the $2j$th and the $(2j+1)$th coordinates of elements of $H$. Set $G = H \times_{\alpha} \Sigma$ (semidirect product), where $\Sigma$ is given the discrete topology. Then $G$ is a locally compact group with open subgroup $H$. If $(h, \sigma) \in C_\sigma(x)$ for some element $x$ of $G$, then a direct computation shows that there exists a finite set $J$ of positive integers depending on $\sigma$ such that if $h_k' = h_k$ for $k \in J$, then $(h', \sigma) \in C_\sigma(x)$. Thus $C_\sigma(x)$ is open in $G$. Hence $G^c = G$. On the other hand, $Z(G)$ consists only of elements whose $H$-coordinates are periodic with period two, so $Z(G)$ is not open in $G$.

**Theorem 2.6.** The following conditions are equivalent for a locally compact group $G$.

(a) $G = G^c$.
(b) $C_\sigma(x)$ is an open subgroup for all $x \in G$.
(c) $C_\sigma(K)$ is open for all compact subsets $K$ of $G$.
(d) $C_\sigma(H)$ is open for every compactly generated closed subgroup $H$ of $G$.
(e) $Z(H)$ is open in $H$ for every compactly generated closed subgroup $H$ of $G$.

**Proof.** The implications (a) $\iff$ (b) are the content of Theorem 1.4. The implication (b) $\implies$ (c) follows from the proof of Theorem 2.4. The facts that (b)$\iff$(c) $\implies$ (d) $\implies$ (e) are routine. To see that (e) $\implies$ (d), recall that if $H$ is a compactly generated closed subgroup of $G$, then there is an open compactly generated subgroup $H'$ of $G$ containing $H$. Applying (e) to $H'$, we conclude that $C_\sigma(H)$ is open in $G$. \qed

**Theorem 2.7.** Let $G$ be a compact group. Then the following conditions are equivalent.

(a) $G = G^c$.
(b) $G$ has finite conjugacy classes.
(c) $G$ is a central extension of an open abelian subgroup of finite index.

**Proof.** The implication (a) $\iff$ (b) is immediate from Lemma 2.1. If (a) holds, then $Z(G)$ is an open normal subgroup by Theorem 2.4. Since $G$ is compact, any open subgroup must be of finite index, so (a) $\implies$ (c). (Note that the subgroup may be taken to be $Z(G)$.) Conversely, suppose that $H$ is an abelian subgroup of finite index in $G$ with $H$ central. Then $Z(G)$ is open, so $C_\sigma(x)$ is open for all $x \in G$. \qed

Condition (c) in Theorem 2.7 is parallel to a condition appearing in an important theorem of C. C. Moore, which we recall. A group $G$ is said to be of bounded degree if the dimensions of the irreducible unitary representations of $G$ are bounded [4]. The theorem of Moore [6] is as follows. The locally compact group $G$ is of bounded degree if and only if $G$ is an extension of an open abelian subgroup of finite index.

We see immediately that any compact group $G$ with $G = G^c$ must be of bounded degree, but that the converse is false. The simplest example was pointed out to us by I. Kaplansky: Take $G$ to be the noncentral extension of the circle group $T$ by the group of order two. Then $G$ is of bounded degree (in fact the irreducible unitary representations of $G$ have dimension at most two), but for appropriate choice $x$, $C_\sigma(x)$ has four elements. So $G \neq G^c$; in fact $G^c = T$.\[\]
3. §3 is devoted to an exploration of how various conditions imposed upon \( G^c \) are reflected in the structure of \( G \).

**Theorem 3.1.** \( G^c \) is an open subgroup of \( G \) if and only if \( G \) has an open abelian subgroup.

**Proof.** If \( H \) is an open abelian subgroup of \( G \), then \( H \subset C_G(x) \) for every \( x \in H \). Hence \( H \subset G^c \), so \( G^c \) is open. Conversely, suppose that \( G^c \) is open. Then \( G^c \) contains an open compactly generated subgroup \( H \). Theorem 2.4 implies that \( Z(H) \) is open in \( H \) and hence in \( G \). Thus \( Z(H) \) is an open abelian subgroup of \( G \).

It would be of interest to characterize those groups \( G \) with \( G^c = \{ e \} \). One motivation for our interest in this question is expressed by the following theorem (3.2). Of course, if \( G \) is connected, then \( G^c = \{ e \} \) just means that \( G \) has trivial center. A class of totally disconnected groups with \( G^c = \{ e \} \) is described in Example 3.3.

**Theorem 3.2.** Let \( \pi : G \to \mathcal{L}(L^2(G)) \) be the right regular representation, and for \( x \in G \) let \( \Phi_x \) be the automorphism of \( \mathcal{L}_G \) given by

\[
\Phi_x(T) = \pi(x)T\pi(x)^*.
\]

If \( x \in G \setminus G^c \), then \( \Phi_x \) is outer. In particular, if \( G^c = \{ e \} \), then

\[
G \to \text{Aut}(\mathcal{L}_G)/\text{Inn}(\mathcal{L}_G)
\]

is injective.

**Proof.** For \( x \in G \setminus G^c \) restrict \( \Phi_x \) to the compact operators \( \mathcal{K} \). Then \( \Phi_x \) is implemented by conjugation by \( \pi(x) \), and \( \pi(x) \notin \mathcal{L}_G \). In fact, \( \pi(x) \) is the only unitary operator implementing \( \Phi_x \). For if \( U \) is a unitary operator implementing \( \Phi_x \), then \( \pi(x)U^{-1} \) would centralize \( \mathcal{K} \); hence \( \pi(x) = U \). Thus there is no unitary element of \( \mathcal{L}_G \) which implements \( \Phi_x \), so \( \Phi_x \) is outer.

**Example 3.3.** Consider the group \( \text{SL}(2, Q_p) \), where \( Q_p \) denotes the \( p \)-adic number field. Suppose that \( x \) is an element of this group with an open centralizer. Then \( x \) must commute with all matrices of the form \( \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \) for \( \alpha, \beta \) sufficiently close to zero with \( \alpha\beta = 0 \), since every neighborhood of \( I \) contains such matrices. An easy computation shows that \( x = \pm I \), so \( \text{SL}(2, Q_p)^c = Z(2) \). A similar computation yields \( \text{PSL}(2, Q_p)^c = \{ e \} \).

More generally, let \( G \) be a Zariski-connected semisimple affine algebraic group defined over a local field \( k \) (of arbitrary characteristic), and assume that \( G \) is almost simple and isotropic over \( k \). Let \( G(k) \) denote the group of \( k \)-rational points of \( G \). Then \( G(k) \) has a natural locally compact topology. Let \( H \) be a closed cocompact subgroup of \( G(k) \). Then every neighborhood of the identity in \( H \) is dense in \( G \) with respect to the Zariski topology [7, Lemma 2.1]. If \( C_H(x) \) is open for some \( x \in H \), then \( x \in Z(G) \). Thus \( H^c = Z(H) \). In particular, if \( G \) has trivial center (e.g., \( G \) is of adjoint type), then \( H^c = \{ e \} \).

In all of the examples considered so far in this note, \( G^c \) is a closed subgroup of \( G \). In Example 3.4 we show that this is not always the case. Although we do not
know how to characterize those groups $G$ for which $G^c$ is closed, Theorem 3.5 below provides a step in that direction.

**Example 3.4.** Let $H$ be a finite group with trivial center, and let $G = \prod_{n=1}^{\infty} H_n$, $H_n$ being isomorphic to $H$. Then it is easy to see that

$$G^c = \{ x = (x_n) \in G : x_n = e \text{ for all but finitely many } n \} = \bigoplus_{1}^{\infty} H_n.$$ 

In particular, $G^c$ is not closed in $G$.

**Theorem 3.5.** Let $Y(G) = \cap_{x \in G^c} C_G(x)$.

1. If $Y(G)$ is open in $G$, then $G^c$ is closed.
2. If $G$ is metrizable and $G^c$ is compact, then $Y(G)$ is open.

**Proof.** The first assertion is clear. To prove the second assertion, let $\{ H_n \}$ be a decreasing sequence of open subgroups of $G$ such that $G_0 = \cap_{n=1}^{\infty} H_n$ and $H_n / G_0$ is compact for all $n$. Then every open subgroup of $G$ contains some $H_n$. For each $n$, let $G_n = \{ x \in G^c : C_G(x) \supset H_n \}$. Then $\{ G_n \}$ is an increasing sequence of closed subgroups of $G$ whose union is $G^c$. If $G^c$ is compact, then the Baire Category Theorem implies that $G_k$ is open in $G^c$ for some $k$, and hence $G_m = G^c$ for some $m > k$. Thus $H_m \subset Y(G)$. \[ \Box \]

**Corollary 3.6.** If $G_0$ is open in $G$, then $G^c$ is closed.

**References**


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