MULTIPLICATIVE PROPERTIES OF JENSEN MEASURES

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Abstract. It is shown that if $\psi$ is a linear functional on a function algebra with $\psi(1) = 1$, and if $\psi$ satisfies the Jensen inequality with respect to some finite nonnegative measure, then $\psi$ is multiplicative.

Let $A$ be a function algebra (closed, separating subalgebra of $C(X)$ containing the constants) on the compact space $X$, and let $\psi$ be a nontrivial linear functional on $A$. We shall say that a finite, nonnegative measure $\mu$ on $X$ is a Jensen measure for $\psi$ if the Jensen inequality

$$|\psi(f)| \leq \exp \left( \int_X \log |f| \, d\mu \right), \quad f \in A,$$

holds for $\psi$ and $\mu$. It is a well-known theorem of Bishop [1], [2, §§2–5] that if $\psi$ is multiplicative on $A$ then there exists a Jensen representing measure $\mu$ for $\psi$ (i.e., $\mu$ is a probability measure such that (1) holds and $\psi(f) = \int_X f \, d\mu, f \in A$). The purpose of this note is to point out the following converse to Bishop’s theorem, thereby filling a gap in the proof of Lemma 2.5.5 in [2].

Theorem. If $\mu$ is a Jensen measure for $\psi$, then $\mu$ represents a multiplicative linear functional $\phi$ on $A$ such that $\psi = \phi(1)\phi$.

Proof. (1) implies the continuity of $\psi$ with $\|\psi\| \leq 1$, so for $f \in A$ $\psi(ze^f)$ is an entire function of the complex variable $z$ and $\psi(ze^f) = \sum_0^n \psi(f^n) z^n / n!$. If $\int_X f \, d\mu = 0$, then

$$|\psi(ze^f)| \leq \exp \left( \int_X \text{Re}(zf) \, d\mu \right) = \exp \left( \text{Re} z \int_X f \, d\mu \right) = 1,$$

so by Liouville’s theorem we have $\psi(ze^f) = \psi(1)$ for all $z$. In general, given $f \in A$ let $\alpha = \int_X f \, d\mu$. Then

$$\psi(ze^f) = \psi(\exp(z(f - \alpha \|\mu\|^{-1}))) \exp(z\alpha \|\mu\|^{-1}) = \psi(1) \exp(z\alpha \|\mu\|^{-1}).$$

Thus $\sum_0^n \psi(f^n) z^n / n! = \psi(1) \sum_0^n (\alpha \|\mu\|^{-1})^n z^n / n!$, so

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(2) \[ \psi(f^n) = \psi(1)^n \left( \frac{1}{\mu} \int_X f \, d\mu \right)^n, \quad n = 0, 1, 2, \ldots, f \in A. \]

Clearly \( \psi(1) \neq 0 \) if \( \psi \) is nontrivial, and from (1) we have \( e^t |\psi(1)| \leq e^t |\mu| \) for all real \( t \). Hence \( \mu \) must be a probability measure, so (2) takes the form \( \psi(f^n) = \psi(1)^n \int_X f \, d\mu \). Set \( \phi(f) = \psi(1)^{-1} \psi(f), \quad f \in A. \) Then \( \phi(f) = \int_X f \, d\mu \) and

\[ \phi(f^n) = \psi(1)^{-n} \psi(f^n) = (\phi(f))^n, \quad n = 0, 1, 2, \ldots, f \in A. \]

It follows easily that \( \phi \) is multiplicative on \( A \).

Combining this result with Bishop’s theorem yields

**Corollary.** Let \( \psi \) be a linear functional on \( A \) such that \( \psi(1) = 1 \). Then \( \psi \) is multiplicative on \( A \) if and only if there exists a Jensen representing measure for \( \psi \).

**References**


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