Algebras of Multilinear Forms on Groups

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Dedicated to Mischa Cotler on the occasion of his eightieth birthday

Abstract. For locally compact groups \(G_i, i = 1, 2, \cdots, n\), let \(CB(G_1, \cdots, G_n)\) denote the Banach space of completely bounded multilinear forms on \(C_0(G_1) \times \cdots \times C_0(G_n)\) in the completely bounded norm. \(CB(G_1, \cdots, G_n)\) has the structure of a Banach \(*\)-algebra under a multiplication and adjoint operation which agree with the convolution structure on the measure algebra \(M(G_1 \times \cdots \times G_n)\). If the \(G_i\) are all abelian, \(CB(G_1, \cdots, G_n)\) carries a naturally defined Fourier transform which generalizes the Fourier-Stieltjes transform on measure algebras. Various other aspects of \(CB(G_1, \cdots, G_n)\) are investigated.

1. Introduction

For locally compact spaces \(X_i, i = 1, 2, \cdots, n\), let \(MM(X_1, X_2, \cdots, X_n)\) denote the space of all bounded \(n\)-linear forms \(u : C_0(X_1) \times C_0(X_2) \times \cdots \times C_0(X_n) \to C\). Equipped with the usual norm on \(n\)-linear forms, \(MM(X_1, X_2, \cdots, X_n)\) is a Banach space. Its elements are called multimeasures on \(X_1 \times X_2 \times \cdots \times X_n\), bimeasures when \(n = 2\), and trimeasures if \(n = 3\).

In [8], [9], [10], [11] the second author and others showed that if \(G_1\) and \(G_2\) are locally compact groups, then \(MM(G_1, G_2)\) is a Banach \(*\)-algebra consistent with the convolution structure of the measure algebra \(M(G_1 \times G_2)\), and they studied the structure of \(MM(G_1, G_2)\) as a normed algebra. The arguments in the papers cited above, however, depend heavily on the classical inequality of Grothendieck, which is invalid when \(n \geq 3\). In fact, it has been shown that it is impossible to introduce an algebra structure on \(MM(G_1, G_2, \cdots, G_n)\) consistent with that of \(M(G_1 \times G_2 \times \cdots \times G_n)\) when \(n \geq 3\). (See [11, Theorem 6].)

Nevertheless, for locally compact groups \(G_1, G_2, \cdots, G_n\) there is an appropriate normed subspace of \(MM(G_1, G_2, \cdots, G_n)\) on which one can introduce a Banach-algebra structure, namely the Banach space \(CB(G_1, G_2, \cdots, G_n)\) of

1991 Mathematics Subject Classification. Primary 43A10; Secondary 46L05.

The final version of this work appears in Contemporary Math. 189 (1995), 497-511.
all completely bounded \(n\)-linear forms. In fact, it follows from Remark 2.3 below that this is the most appropriate setting for an extension of the results on \(MM(G_1, G_2)\). As we shall see below, there is a convolution and an adjoint operation on the space \(CB(G_1, G_2, \ldots, G_n)\) which generalize the convolution structure of the measure algebra of \(G_1 \times G_2 \times \cdots \times G_n\) and which will make \(CB(G_1, G_2, \ldots, G_n)\) into a Banach \(+\)-algebra. In the case of \(LCA\) (locally compact abelian) groups, we shall also define and study a Fourier transform for completely bounded \(n\)-linear forms which generalizes the Fourier-Stieltjes transform of measures.

Some of the material herein has been modified from the doctoral dissertation of the first author submitted to Wayne State University in July, 1990. The fact that \(CB(G_1, G_2, \ldots, G_n)\) is a Banach algebra (perhaps not the explicit formulations (10) and (11) below for the multiplication and adjoint, however) can, in fact, be extracted from more recent abstract results in the theory of operator spaces, for instance [2], [6], [7]. Nevertheless, given the widespread interest in Harmonic Analysis and convolutions, it seems appropriate to develop this context explicitly. We hope that this will stimulate further research into the harmonic analysis of multilinear forms on groups.

Throughout the paper the inner product on a Hilbert space will be denoted by \((\cdot | \cdot)\). For Hilbert spaces \(H\) and \(K\), \(L(H, K)\) denotes the space of all bounded linear operators from \(H\) to \(K\), and \(L(H) = L(H, H)\). Recall that the \(\sigma\)-weak (operator) topology on \(L(H)\) is the locally convex topology given by the family of seminorms

\[
p(T) = \sum_{n=1}^{\infty} (T \xi_n | \eta_n), \quad \sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty, \quad \sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty.
\]

\(X, Y, X_i\) will always denote locally compact Hausdorff spaces, \(A, B, A_i\) denote \(C^*\)-algebras, and \(G, H, G_i\) are locally compact groups. \(L^\infty(X)\) and \(C_0(X)\) are the spaces of bounded functions on \(X\) which are, respectively, Borel locally measurable and continuous with limit zero at infinity. As usual, \(C_0(X)^*\) is identified with \(M(X)\), the space of bounded regular Borel measures on \(X\).

If \(A\) is a \(C^*\)-algebra, \(M_k(A)\) denotes the algebra of all \(k \times k\) matrices with entries in \(A\). Recall that we may identify the bidual \(A^{**}\) of \(A\) with the enveloping von Neumann algebra \(N\) of \(A\), the double commutant \(A''\) when \(A\) is represented as an algebra of operators. Under this identification, the weak* topology on \(A^{**}\) becomes the \(\sigma\)-weak topology on \(N\). For the fundamental facts about \(C^*\)-algebras that we will need, we refer the reader to [12], [16].

For \(i = 1, 2, \ldots, n\), let \(A_i\) be a \(C^*\)-algebra and \(\theta_i: A_i \to L(H_i)\) be a representation. Then \(\theta_1 \otimes \cdots \otimes \theta_n\) denotes the unique representation of the (spatial) tensor product \(C^*\)-algebra \(A_1 \otimes \cdots \otimes A_n\) on the tensor-product Hilbert space \(H_1 \otimes \cdots \otimes H_n\) satisfying

\[
(\theta_1 \otimes \cdots \otimes \theta_n)(a_1 \otimes \cdots \otimes a_n) = \theta_1(a_1) \otimes \cdots \otimes \theta_n(a_n). \tag{1}
\]
For each $i$ there is a unique extension of $\theta_i$ to $\mathcal{A}_i''$ which is continuous with respect to the weak$^*$ topology on $\mathcal{A}_i'$ and the $\sigma$-weak topology on $L(H_i)$, namely $\theta_i''$. We shall denote these representations $\theta_i''$ by $\theta_i$ also. $\theta_1 \otimes \cdots \otimes \theta_n$ will thus be extended to $(\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n)''$. To clarify this extension, let us observe the following.

**Lemma 1.1.** Let $\{T_\alpha\}$ be a net in $L(H)$ and $S \in L(K)$. If $\{T_\alpha\}$ converges $\sigma$-weakly to zero in $L(H)$, then $\{T_\alpha \otimes S\}$ converges $\sigma$-weakly to zero in $L(H \otimes K)$.

**Proof.** Choose two sequences $\{\xi_i\}$ and $\{\eta_i\}$ in the Hilbert-space tensor product $H \otimes K$ such that $\sum_1^\infty \|\xi_i\|^2 < \infty$ and $\sum_1^\infty \|\eta_i\|^2 < \infty$. If $\{e_{1\lambda}\}_{\lambda \in \Lambda}$ is an orthonormal basis of $H$ and $\{e_{2\omega}\}_{\omega \in \Omega}$ is an orthonormal basis of $K$, then $\{e_{1\lambda} \otimes e_{2\omega}\}_{(\lambda, \omega) \in \Lambda \times \Omega}$ is an orthonormal basis of $H \otimes K$. Thus we can easily write

$$\xi_i = \sum_{j=1}^\infty x_{ij} \otimes y_{ij} \quad \text{and} \quad \eta_i = \sum_{k=1}^\infty x'_{ik} \otimes y'_{ik}$$

with $x_{ij}, x'_{ij} \in H, y_{ik}, y'_{ik} \in K$, and $\{x_{ij}\}_{j=1}^\infty, \{y_{ij}\}_{j=1}^\infty$ being orthonormal sets in $H$ and $K$, respectively. Then $\|\xi_i\|^2 = \sum_{j=1}^\infty \|x_{ij}\|^2$ and $\|\eta_i\|^2 = \sum_{k=1}^\infty \|y'_{ik}\|^2$. Hence $\sum_{i=1}^\infty \sum_{j=1}^\infty \|x_{ij}\|^2 < \infty$ and $\sum_{i=1}^\infty \sum_{k=1}^\infty \|y'_{ik}\|^2 < \infty$. Note that

$$\sum_{i=1}^\infty (T_\alpha \otimes S)\xi_i |\eta_i| = \sum_{i=1}^\infty \left( \sum_{j=1}^\infty T_\alpha x_{ij} \otimes S y_{ij} \sum_{k=1}^\infty x'_{ik} \otimes y'_{ik} \right)$$

$$= \sum_{i=1}^\infty \sum_{j=1}^\infty \sum_{k=1}^\infty (T_\alpha x_{ij} |x'_{ik})(S y_{ij} |y'_{ik})$$

$$= \sum_{i=1}^\infty \sum_{j=1}^\infty \left( T_\alpha x_{ij} \sum_{k=1}^\infty (y'_{ik} |S y_{ij}) x'_{ik} \right).$$

Let $z_{ij} = \sum_{k=1}^\infty (y'_{ik} |S y_{ij}) x'_{ik}$, then

$$\sum_{i=1}^\infty \sum_{j=1}^\infty \|z_{ij}\|^2 = \sum_{i=1}^\infty \sum_{j=1}^\infty \sum_{k=1}^\infty \|(y'_{ik} |S y_{ij}) x'_{ik}\|^2$$

$$= \sum_{i=1}^\infty \sum_{j=1}^\infty \sum_{k=1}^\infty |(S^* y'_{ik} |y_{ij})|^2$$

$$\leq \sum_{i=1}^\infty \sum_{k=1}^\infty \|S^* y'_{ik}\|^2$$

$$\leq \|S\|^2 \sum_{i=1}^\infty \sum_{k=1}^\infty \|y'_{ik}\|^2$$

$$= \|S\|^2 \sum_{i=1}^\infty \sum_{k=1}^\infty \|\eta_k\|^2 < \infty.$$
By assumption,
\[ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (T_{n} x_{ij} | z_{ij}) \rightarrow 0. \]
It follows that
\[ \sum_{i=1}^{\infty} (T_{n} \otimes S \xi_{i} | \eta_{i}) \rightarrow 0. \]

**Corollary 1.2.** Let \( A_{i} \) be a C*-algebra and \( \theta_{i} : A_{i} \rightarrow L(H_{i}) \) be a representation, \( i = 1, 2, \ldots, n \). Then
\[ A_{1}'' \otimes \cdots \otimes A_{n}'' \subseteq (A_{1} \otimes \cdots \otimes A_{n})'' \]
and (1) holds for all \( a_{i} \in A_{i}'' \), \( i = 1, 2, \ldots, n \).

**Proof.** Clearly it suffices to consider the case \( n = 2 \). Let \( a_{1} \in A_{1}'' \) and \( a_{2} \in A_{2} \), and choose a net \( \{a_{n}\} \in A_{1} \) converging weak* to \( a_{1} \). Then by Lemma 1.1, \( a_{1}'' \otimes a_{2} \) converges weak* to \( a_{1} \otimes a_{2} \), so \( a_{1}'' \otimes a_{2} \in (A_{1}'' \otimes A_{2}'') \). Since \( \theta_{1} \otimes \theta_{2} \) is weak*-to-\( \sigma \)-weak continuous,
\[ (\theta_{1} \otimes \theta_{2})(a_{1}'' \otimes a_{2}) \rightarrow (\theta_{1} \otimes \theta_{2})(a_{1} \otimes a_{2}) \]
\( \sigma \)-weakly. But \( (\theta_{1} \otimes \theta_{2})(a_{1}'' \otimes a_{2}) = \theta_{1}(a_{1}'') \otimes \theta_{2}(a_{2}) \). By Lemma 1.1 again,
\[ \theta_{1}(a_{1}'') \otimes \theta_{2}(a_{2}) \rightarrow \theta_{1}(a_{1}) \otimes \theta_{2}(a_{2}) \]
\( \sigma \)-weakly. Therefore,
\[ (\theta_{1} \otimes \theta_{2})(a_{1} \otimes a_{2}) = \theta_{1}(a_{1}) \otimes \theta_{2}(a_{2}). \]
Now choose \( \{a_{2}''\} \in A_{2} \) converging weak* to \( a_{2} \) and argue as above to complete the proof. \( \square \)

Considering \( C_{0}(X) \) as a commutative C*-algebra and embedding \( L^{\infty}(X) \) in \( C_{0}(X)'' \) via integration, we may view \( L^{\infty}(X) \) as a subspace of \( C_{0}(X)'' \). In fact, as is well known, \( L^{\infty}(X) \) is a C*-subalgebra of \( C_{0}(X)'' \). By the weak* topology of \( L^{\infty}(X) \) we shall mean the topology inherited from the weak* topology on \( C_{0}(X)'' \). Recall that the C*-algebra \( C_{0}(X) \otimes C_{0}(Y) \) is canonically isomorphic to \( C_{0}(X \times Y) \). Corollary 1.2 implies that if \( \theta \) and \( \pi \) are representations of \( C_{0}(X) \) and \( C_{0}(Y) \) on \( H \) and \( K \), respectively, then
\[ (\theta \otimes \pi)(f \otimes g) = \theta(f) \otimes \pi(g) \]
for all \( f \in L^{\infty}(X) \) and \( g \in L^{\infty}(Y) \).

The following lemma is obvious. Since it is needed in several places, we include it here for reference.

**Lemma 1.3.** If \( T : X \rightarrow Y \) is a continuous map, then the map \( f \rightarrow f \circ T \) from \( L^{\infty}(Y) \) to \( L^{\infty}(X) \) is weak* continuous.
2. Completely Bounded Multilinear Forms

Completely bounded multilinear operators and completely bounded norms were first introduced in [5].

**Definition 2.1.** Let \( A_1, A_2, \ldots, A_n \) and \( A \) be \( C^* \)-algebras, and let \( u : A_1 \times A_2 \times \cdots \times A_n \to A \) be an \( n \)-linear operator. For each \( k \geq 1 \), the \( n \)-linear operator \( u_k : M_k(A_1) \times M_k(A_2) \times \cdots \times M_k(A_n) \to M_k(A) \) is defined as follows. If \( u(a_1, \ldots, a_n) = v_1(a_1)v_2(a_2) \cdots v_n(a_n) \) for some linear operators \( v_i : A_i \to A \), then the entrywise extension of each \( v_i \) to \( M_k(A_i) \) induces an obvious extension \( u_k \) of \( u \) via multiplication in \( M_k(A) \). Motivated by this case, one defines \( u_k \) for any \( u \) by

\[
(2) \quad u_k(A_1, A_2, \ldots, A_n) = \left( \sum_{x, i, r, t} u(a_{1ir}, a_{2rt}, \ldots, a_{ntj}) \right)
\]

for all \( A_i = (a_{ij}) \in M_k(A_i) \) \((1 \leq l \leq n)\). The operator \( u \) is said to be **completely bounded** with completely bounded norm \( \|u\|_{cb} \) if

\[
(3) \quad \|u\|_{cb} = \sup \{ \|u_k\| : k \geq 1 \}
\]

is finite.

The following theorem is a special case of the Christensen-Sinclair representation theorem for completely bounded multilinear operators [5]. (See also [14],[1],[2].)

**Theorem 2.2.** A complex-valued \( n \)-linear form \( u \) on \( A_1 \times A_2 \times \cdots \times A_n \) is completely bounded if and only if there are Hilbert spaces \( H_1, H_2, \ldots, H_n \), operators \( U_i \in L(H_{i+1}, H_i) \) for \( i = 1, 2, \ldots, n-1 \), two vectors \( \xi \in H_n \) and \( \eta \in H_1 \), and representations \( \theta_i : A_i \to L(H_i) \) for \( i = 1, 2, \ldots, n \) such that

\[
(4) \quad u(a_1, a_2, \ldots, a_n) = (\theta_1(a_1)U_1\theta_2(a_2)U_2 \cdots \theta_n(a_n)\xi|\eta)
\]

for \( a_i \in A_i \), \( i = 1, 2, \ldots, n \). Moreover, we may choose \( \xi, \eta \) and \( U_i \) such that \( \|\xi\| = \|\eta\| = 1 \) and \( \|u\|_{cb} = \|U_1\|\|U_2\| \cdots \|U_n\| \).

Actually every completely bounded \( n \)-linear form has a representation of a simpler type [17], [4, Cor. 3.2], namely

\[
(5) \quad u(a_1, a_2, \ldots, a_n) = (\theta_1(a_1)\theta_2(a_2) \cdots \theta_n(a_n)\xi|\xi')
\]

where all the representations \( \theta_i \) act on the same Hilbert space \( K \), \( \xi, \xi' \in K \), and \( \|u\|_{cb} = \|\xi\|\|\xi'\| \). We will use (4) and (5) alternatively.

Whenever a completely bounded \( n \)-linear form \( u \) is represented as in (4) or (5), the representations \( \theta_i \) \((1 \leq i \leq n)\) are said to be associated with \( u \).

**Remark 2.3.** [5] For a bilinear form \( u \) on \( C_0(X) \times C_0(Y) \), the representation (4) is equivalent to the Grothendieck inequality, as is easily seen.
**Definition 2.4.** For $C^*$-algebras $A_1, A_2, \ldots, A_n$, let $CB(A_1, \ldots, A_n)$ be the space of all completely bounded $n$-linear forms on $A_1 \times \cdots \times A_n$, equipped with the completely bounded norm. It is known that $CB(A_1, \ldots, A_n)$ is the dual space of $A_1 \otimes_h A_2 \otimes_h \cdots \otimes_h A_n$, the Haagerup tensor product of the indicated algebras. For a proof, see [14, Theorem 3.1] and [13, Prop. 3.7]. In particular, $CB(A_1, \ldots, A_n)$ is a Banach space in the completely bounded norm. When $A_i = C_0(X_i)$, $i = 1, \ldots, n$, we shall denote the space $CB(A_1, A_2, \ldots, A_n)$ by $CB(X_1, X_2, \ldots, X_n)$.

For expository convenience, we shall just consider the case $n = 3$. The extension of our results to higher $n$ is just a matter of notation.

Each measure $\mu \in M(X_1 \times X_2 \times X_3)$ corresponds to a trilinear form $u_\mu$ on $C_0(X_1) \times C_0(X_2) \times C_0(X_3)$, namely

$$u_\mu(f_1, f_2, f_3) = \int_{X_1 \times X_2 \times X_3} f_1(x_1)f_2(x_2)f_3(x_3)d\mu(x_1, x_2, x_3).$$

The form $u_\mu$ is completely bounded, since

$$u_\mu(f_1, f_2, f_3) = (\theta_1(f_1)\theta_2(f_2)\theta_3(f_3)g)1,$$

where $\theta_1(f_1), \theta_2(f_2)$, and $\theta_3(f_3)$ are the multiplications by $f_1 \otimes 1 \otimes 1, 1 \otimes f_2 \otimes 1$, and $1 \otimes 1 \otimes f_3$, respectively, on $L^2(X_1 \times X_2 \times X_3, |\mu|)$, and $g \in L^\infty(X_1 \times X_2 \times X_3)$ is such that $d\mu = gd|\mu|$. The map $\mu \rightarrow u_\mu$ is obviously linear. It is injective since the span of the elements of the form $f_1(x_1)f_2(x_2)f_3(x_3)$ with $f_i \in C_0(X_i)$ is dense in $C_0(X_1 \times X_2 \times X_3)$. It is also continuous because $\|u_\mu\|_{cb} \leq \|g\|_2\|1\|_2 = \|\mu\|$

Identifying $\mu$ with $u_\mu$, we can view $M(X_1 \times X_2 \times X_3) \subseteq CB(X_1, X_2, X_3)$.

It is well known that, in general, $M(X_1 \times X_2 \times X_3)$ is not dense in $MM(X_1, X_2, X_3)$. Our next theorem shows that this is still true if we replace $MM(X_1, X_2, X_3)$ by $CB(X_1, X_2, X_3)$.

**Theorem 2.5.** If any two of the spaces $X_1, X_2, X_3$ contain nonvoid perfect sets, then $M(X_1 \times X_2 \times X_3)$ is not dense in $CB(X_1, X_2, X_3)$.

**Proof.** Suppose $X_1$ and $X_2$ contain nonvoid perfect sets. We shall show that if $M(X_1 \times X_2 \times X_3)$ is dense in $CB(X_1, X_2, X_3)$, then $M(X_1 \times X_2)$ is dense in $MM(X_1, X_2)$. Indeed, if $\epsilon > 0$ and $u \in MM(X_1, X_2)$, then by Remark 2.3,

$$u(f_1, f_2) = (\theta_1(f_1)U_1\theta_2(f_2)\xi|\eta),$$

where $\theta_1, \theta_2$ are representations of $C_0(X_1)$ and $C_0(X_2)$ on some Hilbert spaces $H_1$ and $H_2$, respectively, $U_1 \in L(H_2, H_1)$, $\xi \in H_2$, and $\eta \in H_1$. Choose $x_{3,0}$ in $X_3$.

Let $\theta_3$ be the representation of $C_0(X_3)$ on $H_3 = H_2$ defined by $\theta_3(f_3) = f_3(x_{3,0})I$.

Set

$$u'(f_1, f_2, f_3) = (\theta_1(f_1)U_1\theta_2(f_2)I\theta_3(f_3)\xi|\eta);$$

then $u' \in CB(X_1, X_2, X_3)$. If $M(X_1 \times X_2 \times X_3)$ is dense in $CB(X_1, X_2, X_3)$, then there exists a measure $\mu' \in M(X_1 \times X_2 \times X_3)$ such that in the trimeasure
norm,
\[ \|\mu' - u\| \leq \|\mu' - u\|_{cb} < \epsilon. \]

Now choose \( f_{3,0} \in C_0(X_3) \) such that \( f_{3,0}(x_{3,0}) = 1 = \|f_{3,0}\| \). Let \( \mu \) be the measure on \( X_1 \times X_2 \) given by
\[
\int_{X_1 \times X_2} f(x_1, x_2)d\mu(x_1, x_2) = \int_{X_1 \times X_2} f(x_1, x_2)f_{3,0}(x_3)d\mu'(x_1, x_2, x_3)
\]
for \( f \in C_0(X_1 \times X_2) \). Since \( u'(f_1, f_2, f_{3,0}) = u(f_1, f_2) \), we have
\[
\|\mu - u\| = \sup\{\|\mu(f_1, f_2) - u(f_1, f_2)\| : \|f_1\|_\infty \leq 1, \|f_2\|_\infty \leq 1\}
\]
\[
= \sup\{\|\mu'(f_1, f_2, f_{3,0}) - u'(f_1, f_2, f_{3,0})\| : \|f_1\|_\infty \leq 1, \|f_2\|_\infty \leq 1\}
\]
\[
\leq \|\mu' - u'\| < \epsilon.
\]

Therefore \( M(X_1 \times X_2) \) is dense in \( MM(X_1, X_2) \). But [11, Theorem 3] says this is not the case, and the proof is complete. \( \square \)

We now extend the elements of \( CB(X_1, X_2, X_3) \) to completely bounded trilinear forms on \( C_0(X_1)^{**} \times C_0(X_2)^{**} \times C_0(X_3)^{**} \).

**Proposition 2.6.** Let \( u \in CB(A_1, A_2, A_3) \). There is a unique, separately weak*-continuous, completely bounded, trilinear form \( \tilde{u} \) on \( A_1^{**} \times A_2^{**} \times A_3^{**} \) which extends \( u \) and satisfies \( \|u\|_{cb} = \|\tilde{u}\|_{cb} \).

**Proof.** Let \( u \) be represented as in (4) (with \( n = 3 \)) so that \( \|u\|_{cb} = \|U_1\|\|U_2\| \). Extend each \( \theta_i \) to \( A_i^{**} \) as above. The right-hand side of (4) extends to \( A_1^{**} \times A_2^{**} \times A_3^{**} \), defining a trilinear form \( \tilde{u} \) on that space. By Theorem 2.2, \( \tilde{u} \) is completely bounded with \( \|u\|_{cb} = \|\tilde{u}\|_{cb} \). The separate weak* continuity of \( \tilde{u} \) follows from the weak*-to-\( \sigma \)-weak continuity of \( \theta_i \), and the uniqueness is clear, since \( A_i \) is weak* dense in \( A_i^{**} \), \( i = 1, 2, 3 \). \( \square \)

**Corollary 2.7.** Each \( u \in CB(X_1, X_2, X_3) \) extends isometrically to a (unique) separately weak*-continuous, completely bounded, trilinear form on \( \mathcal{L}^\infty(X_1) \times \mathcal{L}^\infty(X_2) \times \mathcal{L}^\infty(X_3) \) (which will also be denoted by \( u \)).

**Theorem 2.8.** If \( u \in CB(A_1, A_2, A_3) \) and \( v \in CB(B_1, B_2, B_3) \), then there exists a unique \( u \otimes v \in CB(A_1 \otimes B_1, A_2 \otimes B_2, A_3 \otimes B_3) \) satisfying
\[
(u \otimes v)(a_1 \otimes b_1, a_2 \otimes b_2, a_3 \otimes b_3) = u(a_1, a_2, a_3)v(b_1, b_2, b_3).
\]

Moreover, (6) holds for \( a_i \in A_i^{**} \) and \( b_i \in B_i^{**} \), \( i = 1, 2, 3 \), and
\[
\|u \otimes v\|_{cb} \leq \|u\|_{cb}\|v\|_{cb}.
\]

**Proof.** Let
\[
\begin{align*}
\theta_1(a_1)\theta_2(a_2)\theta_3(a_3)\xi' & = u(a_1, a_2, a_3) = (\theta_1(a_1)\theta_2(a_2)\theta_3(a_3)\xi') \\
\xi(\pi_1(b_1)\pi_2(b_2)\pi_3(b_3)\eta') & = v(b_1, b_2, b_3) = (\pi_1(b_1)\pi_2(b_2)\pi_3(b_3)\eta')
\end{align*}
\]
with \( \|u\|_{cb} = \|\xi\| \|\xi'\| \) and \( \|v\|_{cb} = \|\eta\| \|\eta'\| \), where for \( i = 1, 2, 3 \), \( \theta_i \) is a representation of \( \mathcal{A}_i \) on \( H \), \( \pi_i \) is a representation of \( \mathcal{B}_i \) on \( K \), \( \xi, \xi' \in H \), and \( \eta, \eta' \in K \).

For \( x_i \in \mathcal{A}_i \otimes \mathcal{B}_i \), \( i = 1, 2, 3 \), define

\[
(u \otimes v)(x_1, x_2, x_3) = ((\theta_1 \otimes \pi_1)(x_1)(\theta_2 \otimes \pi_2)(x_2)(\theta_3 \otimes \pi_3)(x_3)(\xi \otimes \eta))\xi' \otimes \eta'.
\]

Then \( u \otimes v \in CB(\mathcal{A}_1 \otimes \mathcal{B}_1, \mathcal{A}_2 \otimes \mathcal{B}_2, \mathcal{A}_3 \otimes \mathcal{B}_3) \), and \( \|u \otimes v\|_{cb} \leq \|\xi \otimes \eta\| \|\xi' \otimes \eta'\| = \|\xi\| \|\eta\| \|\xi'\| \|\eta'\| = \|u\|_{cb} \|v\|_{cb} \). Furthermore,

\[
(u \otimes v)(a_1 \otimes b_1, a_2 \otimes b_2, a_3 \otimes b_3) = ((\theta_1(a_1) \otimes \pi_1(b_1))(\theta_2(a_2) \otimes \pi_2(b_2))(\theta_3(a_3) \otimes \pi_3(b_3))(\xi \otimes \eta))\xi' \otimes \eta' = \theta_1(a_1)\theta_2(a_2)\theta_3(a_3)\xi \pi_1(b_1)\pi_2(b_2)\pi_3(b_3)\eta \xi' \otimes \eta' = u(a_1, a_2, a_3)v(b_1, b_2, b_3).
\]

The uniqueness is clear, and the extension to second duals follows from Corollary 1.2. \( \square \)

**Remark 2.9.** For \( \mu \in M(X_1 \times X_2 \times X_3) \) and \( \nu \in M(Y_1 \times Y_2 \times Y_3) \), we have \( u_\mu \otimes u_\nu = u_{\mu \times \nu} \).

### 3. Banach \( \ast \)-Algebras of Completely Bounded Multilinear Forms

Let us introduce a multiplication and an adjoint operation on \( CB(G_1, G_2, G_3) \). Recall that for \( \mu, \nu \in M(G) \), \( \mu \ast \nu \) is defined by the formula \( \mu \ast \nu(f) = \mu \times \nu(Mf) \), \( f \in C_0(G) \), where \( Mf(x, y) = f(xy) \). Of course, we have implicitly extended \( \mu \times \nu \) from \( C_0(G \times G) \) to \( \mathcal{L}^\infty(G \times G) \). We shall use the same approach to define a convolution in \( CB(G_1, G_2, G_3) \).

**Definition 3.1.** If \( G \) is a locally compact group and \( f \) is a function on \( G \), set

\[
Mf(x, y) = f(xy), \quad \hat{f}(x) = f(x^{-1}), \quad f^*(x) = \overline{f(x^{-1})}.
\]

For \( u, v \in CB(G_1, G_2, G_3) \), and \( f_i \in C_0(X_i) \), \( i = 1, 2, 3 \), define

\[
(u \ast v)(f_1, f_2, f_3) = (u \otimes v)(Mf_1, Mf_2, Mf_3),
\]

\[
u^*(f_1, f_2, f_3) = u(f_1^*, f_2^*, f_3^*).
\]

If \( \theta \) is a representation of \( C_0(G) \) on \( H \) we denote by \( \tilde{\theta} \) the representation defined by \( \tilde{\theta}(f) = [\theta(f^*)]^* \).

**Remarks 3.2.** (i) If \( u, v \) are represented as in (7) and (8), then according to (9) and (10), \( u \ast v \) has a representation

\[
(u \ast v)(f_1, f_2, f_3) = ((\theta_1 \otimes \pi_1)(Mf_1)(\theta_2 \otimes \pi_2)(Mf_2)(\theta_3 \otimes \pi_3)(Mf_3)(\xi \otimes \eta))\xi' \otimes \eta'.
\]
Corollary 3.4. Let $\theta, \pi$ be two representations of $C_0(G)$ on $H$ and $K$, respectively. Then $(\theta \otimes \pi)(Mf) = (\tilde{\pi} \otimes \tilde{\theta})(M\tilde{f})$ for $f \in C_0(G)$ (identifying $H \otimes K$ with $K \otimes H$).

Proof. Define $T : G \times G \rightarrow G \times G$ by $T(x, y) = (y^{-1}, x^{-1})$. For $f, g \in C_0(G)$,

$$(\theta \otimes \pi)(f \otimes g) = \theta(f) \otimes \pi(g) = \pi(g) \otimes \theta(f)$$

$$= (\tilde{\pi}(g) \otimes \tilde{\theta}(f)) = (\tilde{\pi} \otimes \tilde{\theta})(g \otimes \tilde{f})$$

$$= ((\pi \otimes \tilde{\theta})(f \otimes g) \circ T).$$

Since $\theta \otimes \pi$ and $\tilde{\pi} \otimes \tilde{\theta}$ are continuous with respect to the weak$^*$ topology on $C_0(G \times G)^*$ and the $\sigma$-weak topology on $L(H \otimes K)$, by Lemma 1.3, $(\theta \otimes \pi)(h) = (\tilde{\pi} \otimes \tilde{\theta})(h \circ T)$ for all $h \in L^\infty(X \times Y)$. Since $(Mf) \circ T = M\tilde{f}$, the lemma follows.

□

Corollary 3.4. For $u, v \in CB(G_1, G_2, G_3), (u * v)^* = v^* * u^*$.

Proof. Let $u, v$ be represented as in (7) and (8), respectively. Then

$$(u^*(f_1, f_2, f_3) = \overline{u(f_1^*, f_2^*, f_3^*)} = \left(\tilde{\theta}_3(f_3)\tilde{\theta}_2(f_2)\tilde{\theta}_1(f_1)\xi|\xi\right),$$

$$v^*(f_1, f_2, f_3) = (\tilde{\pi}_3(f_3)\tilde{\pi}_2(f_2)\tilde{\pi}_1(f_1)\eta|\eta).$$
Note that these are not standard representations for \( u^* \) and \( v^* \) because of the order of the \(*\)-representations. Nevertheless,
\[
(v^* \otimes u^*)(\Phi_1, \Phi_2, \Phi_3) = \\
\left( (\hat{\pi}_3 \otimes \hat{\theta}_3)(\Phi_3)(\hat{\pi}_2 \otimes \hat{\theta}_2)(\Phi_2)(\hat{\pi}_1 \otimes \hat{\theta}_1)(\Phi_1)(\eta' \otimes \xi')|\eta \otimes \xi \right)
\]
for all \( \Phi_i \in C_0(G_i \times G_i)^{**} \) because both sides are separately weak* continuous and equal when \( \Phi_i = f_i \otimes g_i \) with \( f_i, g_i \in C_0(G_i), \ i = 1, 2, 3 \). So
\[
(v^* \ast u^*)(f_1, f_2, f_3) = \\
\left( (\hat{\pi}_3 \otimes \hat{\theta}_3)(Mf_3)(\hat{\pi}_2 \otimes \hat{\theta}_2)(Mf_2)(\hat{\pi}_1 \otimes \hat{\theta}_1)(Mf_1)(\eta' \otimes \xi')|\eta \otimes \xi \right).
\]

On the other hand,
\[
(u \ast v)^*(f_1, f_2, f_3) = (u \ast v)\left(f_1^*, f_2^*, f_3^*\right) = \\
\left( (\theta_3 \otimes \pi_3)(Mf_3)^* \theta_2 \otimes \pi_2)(Mf_2)^* \theta_1 \otimes \pi_1)(Mf_1)^* \xi' \otimes \eta'\right) \eta \otimes \xi
\]
\[
= \left( (\theta_3 \otimes \pi_3)(Mf_3)(\theta_2 \otimes \pi_2)(Mf_2)(\theta_1 \otimes \pi_1)(Mf_1)(\xi' \otimes \eta')\xi \otimes \eta \right).
\]
Now the equation \( (u \ast v)^* = v^* \ast u^* \) follows from lemma 3.3. \( \Box \)

**Lemma 3.5.** Let \( \theta, \pi, \delta \) be representations of \( C_0(G) \) on \( H, K, L \) respectively. Then for \( f \in C_0(G) \),
\[
\{ \theta \otimes [(\pi \otimes \delta)M] \} Mf = \{ [(\theta \otimes \pi)M] \otimes \delta \} Mf.
\]

**Proof.** Let \( T_1, T_2 : G \times G \to G \times G \) be defined by \( T_1(x, y, z) = (xy, z) \) and \( T_2(x, y, z) = (x, yz) \), respectively. Note that Corollary 1.2 gives
\[
\{ \theta \otimes [(\pi \otimes \delta)M] \}(f \otimes g) = \theta(f) \otimes (\pi \otimes \delta)(Mg) = [\theta \otimes (\pi \otimes \delta)](f \otimes Mg) = [\theta \otimes (\pi \otimes \delta)]((f \otimes g) \circ T_2).
\]
Thus, as in the proof of Lemma 3.3, for all \( h \in L^\infty(G \times G) \),
\[
\{ \theta \otimes [(\pi \otimes \delta)M] \}(h) = [\theta \otimes (\pi \otimes \delta)]((h \circ T_2).
\]
A similar argument shows that
\[
\{ [(\theta \otimes \pi)M] \otimes \delta \}(h) = [\theta \otimes (\pi \otimes \delta)]((h \circ T_1).
\]
Our lemma now follows from the identity \( (Mf) \circ T_1 = (Mf) \circ T_2 \). \( \Box \)

**Corollary 3.6.** The multiplication defined by (10) is associative.

**Proof.** For \( u, v, w \in CB(G_1, G_2, G_3) \), let \( u, v \) be represented as in (7) and (8), respectively, and let
\[
w(f_1, f_2, f_3) = (\delta_1(f_1)\delta_2(f_2)\delta_3(f_3))\zeta' \zeta,
\]
where \( \delta_i \) is a representation of \( C_0(G_i), i = 1, 2, 3 \), on some Hilbert space \( L \), and \( \zeta, \zeta' \in L \). By (12), the associated representations of \( u \ast v \) are \( (\theta_i \otimes \pi_i)M, i = 1, 2, 3 \), and those of \( v \ast w \) are \( (\pi_i \otimes \delta_i)M, i = 1, 2, 3 \). Thus for \( f_i \in C_0(G_i), i = 1, 2, 3 \),

\[
((u \ast v) \ast w)(f_1, f_2, f_3) = (u \otimes v) \ast w(Mf_1, Mf_2, Mf_3)
\]

where

\[
U = \{(\theta_1 \otimes (\pi_1 \otimes \delta_1))Mf_1\{(\theta_2 \otimes (\pi_2 \otimes \delta_2))Mf_2\{(\theta_3 \otimes (\pi_3 \otimes \delta_3))Mf_3\).
\]

On the other hand,

\[
(u \ast (v \ast w))(f_1, f_2, f_3) = (u \otimes (v \ast w)) (Mf_1, Mf_2, Mf_3)
\]

where

\[
V = \{(\theta_1 \otimes (\pi_1 \otimes \delta_1))Mf_1\{(\theta_2 \otimes (\pi_2 \otimes \delta_2))Mf_2\{(\theta_3 \otimes (\pi_3 \otimes \delta_3))Mf_3\).
\]

By Lemma 3.5, \( U = V \) and hence \( (u \ast v) \ast w = u \ast (v \ast w) \). □

**Theorem 3.7.** The multiplication (10) and the adjoint operation (11) define a unital Banach \(*\)-algebra structure with isometric involution on \( CB(G_1, G_2, G_3) \) which extends the \(*\)-algebra structure of \( M(G_1 \times G_2 \times G_3) \).

**Proof.** If \( u,v \in CB(G_1, G_2, G_3) \), we have shown in Remarks 3.2 that \( u \ast v,u^* \in CB(G_1, G_2, G_3) \), \( \|u^*\|_{cb} = \|u\|_{cb} \), and \( \|u \ast v\|_{cb} \leq \|u\|_{cb}\|v\|_{cb} \). It is easy to check, given Corollary 3.4, that \( u \to u^* \) is an adjoint operation in \( CB(G_1, G_2, G_3) \). If \( \mu \in M(G_1 \times G_2 \times G_3) \), then

\[
u^*_\mu(f_1, f_2, f_3) = \frac{u_\mu(f_1^*, f_2^*, f_3^*)}{\int_{G_1 \times G_2 \times G_3} f_1^*(x_1)f_2^*(x_2)f_3^*(x_3)d\mu(x_1, x_2, x_3)}
\]

\[
= \int_{G_1 \times G_2 \times G_3} f_1(x_1)f_2(x_2)f_3(x_3)d\mu^*(x_1, x_2, x_3)
\]

\[
= \mu^*(f_1, f_2, f_3).
\]

Since Theorem 2.8 gives immediately that

\[
(u_1 + u_2) \otimes v = u_1 \otimes v + u_2 \otimes v,
\]

\[
u \otimes (v_1 + v_2) = u \otimes v_1 + u \otimes v_2,
\]

it follows that the multiplication is distributive. This, along with Corollary 3.6, shows that (10) defines a multiplication on \( CB(G_1, G_2, G_3) \).
Let $e \in G$ be the identity and $\theta_e(f) = f(e), f \in C_0(G)$. Define $T : G \to G \times G$ by $T(x) = (x, e)$. If $\theta$ is a representation of $C_0(G)$ on $H$, then for $f, g \in C_0(G)$,

$$
(\theta \otimes \theta_e)(f \otimes g) = \theta(f) \otimes g(e) = g(e)\theta(f) = \theta(g(e)f) = \theta((f \otimes g) \circ T).
$$

Thus by Lemma 1.3, $(\theta \otimes \theta_e)(h) = \theta(h \circ T), h \in L^\infty(G \times G)$. In particular,

$$
(\theta \otimes \theta_e)(Mf) = \theta(Mf \circ T), \quad f, \ g \in C_0(G).
$$

Let $e = (e_1, e_2, e_3)$, where $e_i$ is the identity of $G_i$, and let $u_e$ be the trilinear form induced by the point mass $\delta_e$, that is,

$$
u_e(f_1, f_2, f_3) = f_1(e_1)f_2(e_2)f_3(e_3).
$$

Let us show that $u_e$ is an identity of $CB(G_1, G_2, G_3)$. We may write the above formula as

$$
u_e(f_1, f_2, f_3) = (\theta_{e_1}(f_1)\theta_{e_2}(f_2)\theta_{e_3}(f_3))11.
$$

Applying (13),

$$
(u \ast u_e)(f_1, f_2, f_3) = (u \otimes u_e)(Mf_1, Mf_2, Mf_3) = ((\theta_1 \otimes \theta_{e_1})(Mf_1)(\theta_2 \otimes \theta_{e_2})(Mf_2)(\theta_3 \otimes \theta_{e_3})(Mf_3)(\xi \otimes 1)(\xi' \otimes 1) = (\theta_1(f_1)\theta_2(f_2)\theta_3(f_3)(\xi\xi')) = u(f_1, f_2, f_3).
$$

Similarly, $u_e \ast u = u$.

Finally,

$$
u_{\mu \nu}f_1, f_2, f_3 = \mu\times \nu = \int_{G_1 \times G_2 \times G_3} f_1(x_1)f_2(x_2)f_3(x_3) d\mu \times \nu(x_1, x_2, x_3)
$$

$$= \int_{G_1 \times G_2 \times G_3} f_1(x_1y_1)f_2(x_2y_2)f_3(x_3y_3) d(\mu \times \nu)(x_1, \cdots, y_3)
$$

$$= u_{\mu \times \nu}(Mf_1, Mf_2, Mf_3)
$$

$$= (u_{\mu} \otimes u_{\nu})(Mf_1, Mf_2, Mf_3)
$$

$$= (u_{\mu} \ast u_{\nu})(f_1, f_2, f_3),
$$

and the proof is complete. □

**Theorem 3.8.** $CB(G_1, G_2, G_3)$ is commutative if and only if $G_1, G_2,$ and $G_3$ are all abelian.

**Proof.** Let the $G_i$ all be abelian. To see $u \ast v = v \ast u$, it follows from (12) that we need only show that for $G$ abelian and $\theta$ and $\pi$ representations of $C_0(G)$ (and identifying, as usual, $H \otimes K$ with $K \otimes H$), we have

$$
(\theta \otimes \pi)(Mf) = (\pi \otimes \theta)(Mf).
$$
But this follows easily by observing that \((\theta \otimes \pi)(f \otimes g) = (\pi \otimes \theta)(g \otimes f)\) for \(f, g \in C_0(G)\), and that in our case the map \(T(x, y) = (y, x)\) satisfies \(Mf \circ T = Mf\) and arguing as in Lemma 3.3.

Conversely, if \(CB(G_1, G_2, G_3)\) is commutative, then \(M(G_1 \times G_2 \times G_3)\), as a subalgebra of \(CB(G_1, G_2, G_3)\), is commutative. Hence \(G_1 \times G_2 \times G_3\) is abelian. □

**Theorem 3.9.** Let \(H_i\) be a closed, normal subgroup of \(G_i\) and \(\phi_i : G_i \to G_i/H_i\) be the quotient map, \(i = 1, 2, 3\). For \(u \in CB(G_1, G_2, G_3)\), define \(\sigma(u) \in CB(G_1/H_1, G_2/H_2, G_3/H_3)\) by

\[
\sigma(u)(f_1, f_2, f_3) = u(f_1 \circ \phi_1, f_2 \circ \phi_2, f_3 \circ \phi_3), \quad f_i \in C_0(G_i/H_i).
\]

Then \(\sigma\) is a \(\ast\)-algebra homomorphism with \(\|\sigma(u)\|_{cb} \leq \|u\|_{cb}\).

**Proof.** The map \(\delta_i : C_0(G_i/H_i) \to C(G_i)\) defined by \(\delta_i(f_i) = f_i \circ \phi_i\) is obviously a \(\ast\)-homomorphism. For \(f_i \in C_0(G_i/H_i), i = 1, 2, 3\), Proposition 2.6 allows us to write

\[
\sigma(u)(f_1, f_2, f_3) = u(f_1 \circ \phi_1, f_2 \circ \phi_2, f_3 \circ \phi_3) = (\theta_1(f_1 \circ \phi_1)\theta_2(f_2 \circ \phi_2)\theta_3(f_3 \circ \phi_3)\xi') = ((\theta_1\delta_1)(f_1)(\theta_2\delta_2)(f_2)(\theta_3\delta_3)(f_3)\xi').
\]

Thus \(\sigma(u) \in CB(G_1/H_1, G_2/H_2, G_3/H_3)\), and

\[
\|\sigma(u)\|_{cb} \leq \|u\|_{cb}.
\]

By Remark 3.2 (iii), we have that

\[
\sigma(u^\ast)(f_1, f_2, f_3) = u^\ast(f_1 \circ \phi_1, f_2 \circ \phi_2, f_3 \circ \phi_3) = u((f_1 \circ \phi_1)^\ast, (f_2 \circ \phi_2)^\ast, (f_3 \circ \phi_3)^\ast) = u(f_1^\ast \circ \phi_1, f_2^\ast \circ \phi_2, f_3^\ast \circ \phi_3) = \sigma(u)(f_1^\ast, f_2^\ast, f_3^\ast) = \sigma(u)^\ast(f_1, f_2, f_3).
\]

Since \(\sigma\) is obviously linear, to complete the proof we just need to show that

\[
\sigma(u \ast v) = \sigma(u) \ast \sigma(v), \quad u, v \in CB(G_1, G_2, G_3).
\]

Now by Remark 3.2 (iii),

\[
\sigma(u \ast v)(f_1, f_2, f_3) = (u \ast v)(f_1 \circ \phi_1, f_2 \circ \phi_2, f_3 \circ \phi_3) = (u \otimes v)(M(f_1 \circ \phi_1), M(f_2 \circ \phi_2), M(f_3 \circ \phi_3)) = (U(\xi \otimes \eta)\xi' \otimes \eta'),
\]

where \(U = (\theta_1 \otimes \pi_1)(M(f_1 \circ \phi_1)(\theta_2 \otimes \pi_2)(M(f_2 \circ \phi_2)(\theta_3 \otimes \pi_3)(M(f_3 \circ \phi_3)), \) and

\[
(\sigma(u) \ast \sigma(v))(f_1, f_2, f_3) = (\sigma(u) \otimes \sigma(v))(Mf_1, Mf_2, Mf_3) = (V(\xi \otimes \eta)\xi' \otimes \eta'),
\]
where \( V = (\theta_1 \delta_1 \otimes \pi_1 \delta_1)(Mf_1)(\theta_2 \delta_2 \otimes \pi_2 \delta_2)(Mf_2)(\theta_3 \delta_3 \otimes \pi_3 \delta_3)(Mf_3). \)

For \( g_i, h_i \in C_0(G_i/H_i), \)

\[
(\theta_i \delta_i \otimes \pi_i \delta_i)(g_i \otimes h_i) = \theta_i \delta_i(g_i) \otimes \pi_i \delta_i(h_i)
= \theta_i(g_i \circ \phi_i) \otimes \pi_i(h_i \circ \phi_i)
= (\theta_i \otimes \pi_i)((g_i \circ \phi_i) \otimes (h_i \circ \phi_i)).
\]

Hence, as before, \((\theta_i \delta_i \otimes \pi_i \delta_i)(Mf_i) = (\theta_i \otimes \pi_i)[M(f_i \circ \phi_i)], i = 1, 2, 3. \) So \( U = V, \) and hence \( \sigma(u * v) = \sigma(u) * \sigma(v). \) □

4. Fourier Transforms of Completely Bounded Multilinear Forms

**Definition 4.1.** Let \( G_i \) be an LCA group with character group \( \Gamma_i, i = 1, 2, 3, \) and let \( u \in CB(G_1, G_2, G_3). \) We define the Fourier transform \( \hat{u} \) of \( u \) by the formula

\[
\hat{u}(\gamma_1, \gamma_2, \gamma_3) = u(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3), \quad \gamma_i \in \Gamma_i, i = 1, 2, 3.
\]

It is obvious from Proposition 2.6 that \( \|\hat{u}\|_{\infty} \leq \|u\|_{cb}, u \in CB(G_1, G_2, G_3). \) It is also clear that \( \hat{u} \) determines \( u \) uniquely, because the trigonometric polynomials are weak* dense in \( C_0(G_i)^{**} \) and \( u \) is separately weak* continuous.

If \( \mu \in M(G_1 \times G_2 \times G_3), \) then clearly \( \hat{u}_\mu(\gamma_1, \gamma_2, \gamma_3) = \hat{\mu}(\gamma_1, \gamma_2, \gamma_3). \)

**Theorem 4.2.** If \( G_i, i = 1, 2, 3 \) are LCA groups and \( u, v \in CB(G_1, G_2, G_3), \) then

\[
\hat{u} * \hat{v} = \hat{u} \hat{v} \quad \text{and} \quad \hat{u}^* = \hat{u}.
\]

**Proof.** For \( \gamma_i \in \Gamma_i, i = 1, 2, 3, \)

\[
\hat{u} * \hat{v}(\gamma_1, \gamma_2, \gamma_3) = (u * v)(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)
= (u \otimes v)(M\bar{\gamma}_1, M\bar{\gamma}_2, M\bar{\gamma}_3)
= (u \otimes v)(\bar{\gamma}_1 \otimes \bar{\gamma}_1, \bar{\gamma}_2 \otimes \bar{\gamma}_2, \bar{\gamma}_3 \otimes \bar{\gamma}_3)
= u(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)v(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)
= \hat{u}(\gamma_1, \gamma_2, \gamma_3)\hat{v}(\gamma_1, \gamma_2, \gamma_3),
\]

and

\[
\hat{u}^*(\gamma_1, \gamma_2, \gamma_3) = u^*(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)
= u(\bar{\gamma}_1^*, \bar{\gamma}_2^*, \bar{\gamma}_3^*)
= u(\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3)
= \hat{u}(\gamma_1, \gamma_2, \gamma_3). \quad \square
\]

We conclude with a remark which compares our work with Ylinen’s work in [18].
Remark 4.3. For locally compact groups $G_1, G_2, \cdots, G_n$, Ylinen [18] studied completely bounded $n$-linear forms $\Phi : C^*(G_1) \times \cdots \times C^*(G_n) \to C$ and their Fourier transforms, where $C^*(G_i)$ is the group $C^*$-algebra of $G_i$. A convolution was introduced on the space of all such completely bounded $n$-linear forms via the Fourier transform. It turned out that the space equipped with the completely bounded norm is always a commutative unital Banach algebra with respect to this convolution [18, Sec. 6]. It is obvious that the space we study in this paper is quite different from the one studied in [18]. If the $G_i$ are all LCA groups, however, then the two spaces do coincide, only with the roles of $G_i$ and $\Gamma_i$ interchanged, since $C^*(G_i) = C_0(\Gamma_i)$ in this case. Moreover, since Fourier transforms for abelian $G_i$ in [18] were also defined via the unique separately weak* continuous extensions to $C^{**} \times \cdots \times C^{**}$, it is easy to see that the two definitions of Fourier transform are the same.

References


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