Primitive Monodromy Groups of Genus at most Two

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Introduction

Let X be a compact, connected Riemann surface of genus g, and let $\rho : X \to P^1(C)$ be a covering map of degree N. Then the monodromy group $\mathrm{Mon}(X, \rho)$ acts transitively on the fibre of a generic point. Such a group has genus g. We are concerned with the following question. Given an abstract finite group G and a non-negative integer g, does G arise as a monodromy group of genus g? The focus in the present paper is with primitive groups of genus at most two, that is, groups which have primitive permutation representations as monodromy groups of genus two or less.

For each non-negative integer g, there a finite set $\mathcal{E}_g$ of simple groups such that if G is a group of genus g and S is a nonabelian composition factor of G then either S is an alternating groups or $S \in \mathcal{E}_g$. See [FM01]. Our goal is to obtain not only the explicit list of elements of $\mathcal{E}_g$, $g = 0, 1, 2$, but also the monodromy groups G in which these sections appear.

Let $G = \mathrm{Mon}(X, \rho)$, as above. Then G has a distinguished generating tuple $\underline{x} = (x_1, \ldots, x_r)$ corresponding to the branch points S of ρ. (G is a homomorphic image of the fundamental group $\pi_1(P^1(C) \setminus S)$, and $x_1, \ldots, x_r$ are the images of natural generators of this group.) For present purposes, say that G is an *exceptional group of genus* g if, furthermore, G has a composition factor in $\mathcal{E}_g$. More precisely, we speak of the exceptional triple $(G, M, \underline{x})$ of genus g where M is a point stabilizer and $\underline{x}$ is as above. Our goal is to determine, up to natural equivalance, the complete list of exceptional triples $(G, M, \underline{x})$ of genus g for g at most 2.

For small values of g, most of the exceptional triples occur in *point actions*, that is, where G is an almost simple classical group and M is the stabilizer of a point in the action of G on the 1-spaces of its natural module. For that reason, our main results here concern such actions. Our analysis uses properties of the natual module, often regarded as a vector space over the prime field. Working toward explicit descriptions of $\mathcal{E}_0$, $\mathcal{E}_1$, and $\mathcal{E}_2$, we show here that when g is at most 2, the classical point actions associated to monodromy groups of genus g are of degree less than 10^5 and we provide an explcit list for all possible such actions that do not have degree less than 10000.

1 Statement of Results

Definition *$\underline{x} = (x_1, x_2, \ldots, x_r)$ is a* special generating r-tuple *for G provided*

1. *$G = \langle x_1, x_2, \ldots, x_r \rangle$*
2. *$x_1 x_2 \ldots x_r = 1$*
3. *$x_i \neq 1$, $i = 1, 2, \ldots, r$*

The Riemann Existence Theorem [GT90], guarantees that given a special generating tuple $\underline{x}$ for a permutation group G there is a covering $\rho : X \mapsto P^1C$ such that $G \cong \mathrm{Mon}(X, \rho)$

Definition *The* genus, *$g(\underline{x}, \Omega)$ or $g(\underline{x})$, of the special generating tuple $\underline{x}$ acting on the set Ω is the genus of X, X as above.*

Definition *The* signature $\mathrm{sig}(\underline{x})$ *of an r-tuple $\underline{x} = (x_1, x_2, \ldots, x_r)$ of group elements is the r-tuple $(o(x_1), o(x_2), \ldots, o(x_r))$ of positive integers.*

Definition *For x a permutation of the finite set Ω, let $\mathrm{Fix}_\Omega(x)$ (or $\mathrm{Fix}(x)$) denote the fixed points of x on Ω and let $\mathrm{fpr}_\Omega(x)$ (or $\mathrm{fpr}(x)$) denote the* fixed point ratio *of this permutation. That is,* $\mathrm{fpr}(x) = |\mathrm{Fix}(x)|/|\Omega|$.

Definition *Let V be a vector space and let $x \in GL(V)$. If x acts as a permutation on Ω then the triple (x, V, Ω) satisfies Grassmann Condition ϵ provided*

$$\underset{\Omega}{\mathrm{fpr}}(x) < \frac{|W|}{|V|} + \epsilon$$

for some eigenspace W for the action of x on V.

A classical group G with natural module V acting as a permutation group on the set Ω satisfies Grassmann Condition ϵ provided (x, V, Ω) sastifies Grassmann Condition ϵ for every $x \in G$.

[If V is a vector space over a field K and $x \in GL(V)$, then W is an eigenspace for x provided W is an F-subspace for V_F for some subfield F of K and $x|_W$ acts as a scalar.

If V is a K-vector space and F is a subfield of K, let V_F denote V with the induced F-vector space structure. (Thus, $\dim_F(V_F) = [K : F] \dim_K(V)$.)]

We prove here the following results.

Theorem 1 *Let G be a classical group with natural module V where V contains at least 10^4 projective points. If $\underline{x}$ is a special generating r-tuple for G in some primitive permutation action, then one of the following is true.*

Table 1: Violators of Grassmann Condition 10^{-2}

$\dim V$	q	type of V	action of x
6	$11, 13$	orthogonal	fixes complementary $n/2$-spaces
8	5	orthogonal	fixes complementary $n/2$-spaces
8	2^2	unitary	fixes complementary $n/2$-spaces
6	16	orthogonal	field automorphism
10	4	orthogonal	field automorphism
4	$7^2, 8^2, 9^2$	unitary	field automorphism
6	3^2	unitary	field automorphism
8	2^2	unitary	field automorphism

1. *$g(\underline{x}) > 2$.*

2. *Some element of $\underline{x}$ does not satisfy Grassmann Condition $1/100$.*

3. *The characteristic of V, the dimension of V over its prime field, and the signature of $\underline{x}$ are given in Table 2.*

Theorem 2 *Let G be a classical group with natural module V. Let Ω be a primitive point action for G with $|\Omega| \geq 10^4$ and assume that $x \in G$ does not satisfy Grassmann Condition $1/100$. Then Ω consists of singular points, and either x is an inner-diagonal element of G that fixes pointwise two complementary totally singular subspaces or x acts as a field automorphism of order 2. Furthermore,* $\dim V$*, q, the type of V, and the action of x are listed in Table 1.*

Theorem 3 *Let G be a classical group with natural module V Assume $\underline{x}$ is a special generating tuple for G and that Ω is a primitive point action for G with $|\Omega| \geq 10^4$. If the characteristic of V, the dimension of V over its prime field, and the signature of $\underline{x}$ are given in Table 2 then $g(\underline{x}) > 2$.*

Theorem 4 *Let G be a classical group with natural module V and that Ω is a primitive point action for G with $|\Omega| \geq 10^4$. If $\underline{x}$ is a special generating system for G then either $g(\underline{x}, \Omega) > 2$ or Ω consists of singular points of V and there is a $y \in \underline{x}$ such that some power of y is as described in Table 1.*

Corollary 5 *If (G, Ω) is a primitive classical point action of degree at least 10^5, then the action has genus larger than 2.*

Definition *The almost simple classical group G has a* point action *on Ω provided G has a natural module V of dimension n over $\mathbf{F}_q$ where (G, Ω, n, V) satisfy one of the following conditions.*

L : $F^*(G) \cong L_n(q)$, and Ω is the set of all points in V. $n \geq 2$.

Table 2: Characteristic, Dimension and Signature of Exceptional Cases in Theorem 1

p	$\dim_{\mathbf{F}_p}(V)$	$\mathrm{sig}(\underline{x})$
11	$5, 6$	$(2,3,7)$
7	6	$(2,3,7)$
5	$7, 8, 9$	$(2,3,7)$
3	12	$(2,3,7)$
2	$14, 15, \ldots, 21$	$(2,3,7)$
3	10	$(2,3,8)$
2	4	$(2,4,5)$

$O^\epsilon, \mathbf{s}$: *$F^*(G) \cong O^\epsilon_n(q)$, V is an orthogonal space of type ϵ, and Ω is the set of singular points in V. n is even, $n \geq 6$, $\epsilon = +$ or $-$.*

$O^\epsilon, \mathbf{n}$: *$F^*(G) \cong O^\epsilon_n(q)$, V is an orthogonal space of type ϵ, and Ω is the set of $+$-type points in V. n is even, $n \geq 6$, $\epsilon = +$ or $-$.*

$O, \mathbf{s}$: *$F^*(G) \cong O_n(q)$, V is an orthogonal space, and Ω is the set of singular points in V. n is odd, $n \geq 5$, and q is odd.*

O, δ : *$F^*(G) \cong O_n(q)$, V is an orthogonal space, and Ω is the set of δ-type points in V. n is odd, $n \geq 5$, $\delta = +$ or $-$, and q is odd.*

Sp : *$F^*(G) \cong Sp_n(q)$, V is a symplectic space and Ω is the set of points in V. n is even, $n \geq 4$.*

Sp, δ : *$F^*(G) \cong Sp_n(q)$, V is a symplectic space, and $\widetilde{V}$ is an orthogonal space of dimension $n+1$ such that $\mathrm{rad}\,\widetilde{V}$ is anisotropic of dimension 1 and $V \cong \widetilde{V}/\mathrm{rad}\,\widetilde{V}$, and Ω is the set of all complements to $\mathrm{rad}\,\widetilde{V}$ in $\widetilde{V}$ of type δ. n is even, $n \geq 4$, $\delta = +$ or $-$, and q is even.*

$U, \mathbf{s}$: *$F^*(G) \cong U_n(q^{1/2})$, V is a hermitian space, and Ω is the set of singular points in V. $n \geq 3$, q is a square,*

$U, \mathbf{n}$: *$F^*(G) \cong U_n(q^{1/2})$, V is a hermitian space, and Ω is the set of nonsingular points in V. $n \geq 3$, q is a square,*

We prove Theorems 1, 2, and 3 in the subsequent sections.

Theorem 4 follows immediately from Theorems 1, 2, and 3 since V contains at least $|\Omega|$ projective points.

Corollary 5 follows immediately from Theorem 4 because all actions listed in Table 1 have degree at most 87637.

2 Proof of Theorem 1

2.1 Notation and basic results

Let G be an almost simple classical group with natural module V of dimension n_q over $\mathbf{F}_q$, and let Ω be a primitive G-set of order N. Assume that V contains at least 10000 projective points and that $\underline{x}$ be a special generating set for G.

Let p be the characteristic of $\mathbf{F}_q$. Then $V_{\mathbf{F}_p}$ is an $\mathbf{F}_p$-vector space and all elements of G correspond to $\mathbf{F}_p$-linear maps. Setting $n_p = \dim_{\mathbf{F}_p} V_{\mathbf{F}_p}$, we have $n_p = n_q \log_p(q)$.

Thus

$$G = \langle x_1, \ldots, x_r \rangle$$

Let $g = g(\underline{x})$, and let

$$\underline{d} = (d_1, \ldots, d_r)$$

be the signature of $\underline{x}$, so that $d_i = o(x_i), i = 1, \ldots, r$.

Definition *For $y \in G$,*

$F(y)$ is the number of fixed points of y on Ω.

$f(y) = \frac{F(y)}{N}$, the fixed point ratio of y acting on Ω.

$\mathrm{Ind}(y)$ is the permutation index of y on Ω.

$v_q(y)$ [resp., $v_p(y)$] is the codimension of the largest eigenspace of the action of an associate of y on V [resp., $V_{\mathbf{F}_p}$].

When the context is clear, we will write n instead of n_q or n_p and v instead of v_q or v_p.

By the Riemann-Hurwitz Formula,

$$\sum \mathrm{Ind}(x_i) = 2(N + g - 1).$$

Let x be a generic element of G with $o(x) = d$. The Cauchy-Frobenius Formula says that

$$\mathrm{Ind}(x) = N - \frac{1}{d} \sum_{y \in \langle x \rangle} F(y).$$

It follows that

$$\sum_{i=1}^{r} \frac{1}{d_i} \left(1 + \sum_{y \in \langle x_i \rangle^\sharp} f(y) \right) = r - 2 - 2 \left(\frac{g-1}{N} \right). \tag{1}$$

For $x \in G$, with $o(x) = d$, set

$$\kappa(x) = \frac{1}{d} \left(1 + \sum_{y \in \langle x \rangle^\sharp} p^{-v(y)} \right)$$

$$\epsilon_0 = \frac{2(g-1)}{N}$$

$$A(\underline{d}) = \sum \frac{d_i - 1}{d_i}$$

2.2 The Grassmann condition and preliminary inequalities

The following key inequality is an immediate consequence of the definitions and the previously stated equalities. It will be used extensively.

6 *If G satisfies Grassman condition ϵ then*

$$\sum \kappa(x_i) > r - 2 - A(\underline{d})\epsilon - \epsilon_0$$

The significance of this result can be seem from the main result of [FM00].

Theorem 7 (Grassmann Theorem) *There is a function $\hat{\epsilon} : \mathbf{N} \to \mathbf{R}^+$ such that*

1. *(G, Ω) satisfies Grassman condition $\hat{\epsilon}(N)$ whenever (G, Ω) is a classical subspace action of degree N, and*

2. $\lim_N \hat{\epsilon}(N) = 0$.

In the balance of this subsection we obtain upper bounds for $\kappa(x)$ that will be used in the analysis to prove Theorem 1

Set

$$\zeta(d) = \zeta(d,p) = \frac{1}{d}\left(1 + \sum_{m|d, m>1} \phi(m)p^{-1}\right).$$

Since

$$\kappa(x) = \frac{1}{d}\left(1 + \sum_{m|d, m>1} \phi(m)p^{-v(x^{d/m})}\right),$$

it follows that if x has order d, then

$$\kappa(x) \le \zeta(d) = \frac{1}{d} + \frac{1}{p} - \frac{1}{dp}. \tag{2}$$

Note that ζ is a decreasing function of both d and p.

For each positive integer $s \ge 1$, set

$$\zeta_s(d) = \frac{1}{d}\left(1 + \phi(d) \cdot p^{-s} + \sum_{m|d, m>1, m/d>1} \phi(m)p^{-1}\right).$$

More generally, for a finite sequence $s_1, s_2, \ldots, s_l$ of positive integers, let

$$\zeta_{s_1,s_2,\ldots,s_l}(d) = \frac{1}{d}\left(1 + \sum_{i=1}^{l} \phi(d/i)p^{-s_i} + \sum_{m|d, 1<m<d/l} \phi(m)p^{-1}\right)$$

where ϕ is the Euler ϕ-function on integers, and we take $\phi(a) = 0$ when a is not an integer.

The following statement is evident.

8 *If x has order d and $v(x^i) \geq s_i, i = 1, \ldots, l$, then $\kappa(x) \leq \zeta_{s_1,\ldots,s_l}(d)$.*

The estimates for $\kappa(x)$ can be refined by taking into consideration the possible actions of elements of a given order on a vector space over $\mathbf{F}_p$.

Definition *For each prime p and integer $d \geq 2$ let $\mu_*(d,p)$ be the smallest positive integer μ such that $\mu = \dim[V,x]$ for some linear operator x of order d acting on a vector space V over $\mathbf{F}_p$.*

If $y \in G$ has order m, then $v(y) \geq \mu_*(m,p)$. Consequently, $\kappa(x) \leq \zeta^*(d)$ where

$$\zeta^*(d) = \zeta^*(d,p) = \frac{1}{d}\left(1 + \sum_{m|d, m>1} \phi(m)p^{-\mu_*(m,p)}\right). \tag{3}$$

Setting

$$\zeta^*_{s_1,\ldots,s_l}(d) = \frac{1}{d}\left(1 + \sum_{m|d, m>1} \phi(m)p^{-\alpha(d/m)}\right), \quad \alpha(i) = \max(s_i, \mu_*(d/i,p)), \tag{4}$$

we have $\kappa(x) \leq \zeta^*_{s_1,\ldots,s_l}(d)$ whenever $v(x^i) \geq s_i$, $i = 1, \ldots, l$.

Lemma 9 *1. If $p > 2$ then $\zeta^*(d) < \frac{3}{d} + .04$.*

2. If $p = 2$ then $\zeta^*(d) < \frac{4}{d} + .032$.

Proof. Suppose $p > 3$. Then $\mu_*(d) = 1$ if and only if $d = p$ or $d|p-1$, and $\mu_*(d) > 1$ for all other d. Since at most $p-1$ nontrivial powers of an element have order p and at most $p-2$ nontrivial powers of an element have order dividing $p-1$ this implies that $\zeta^*(d,p) \leq \frac{1}{d}(1 + (2p-3)p^{-1} + (d-1-(2p-3))p^{-2}) < \frac{1}{d}(1 + 2 + d/p^2) = 3/d + 1/p^2 < 3/d + 1/5^2$. If $p = 3$, then $\mu_*(m,p) = 1$ if and only if $m = 2$ or 3, and $\mu_*(m,p) = 2$ if and only if $m = 4, 6$, or 8. This implies that $\sum_{m|d, \mu_*(m,p)=2} \phi(m) \leq \phi(4) + \phi(6) + \phi(8) = 8$. Therefore $\zeta^*(d,3) \leq \frac{1}{d}(1 + 3 \cdot 3^{-1} + 8 \cdot 3^{-2} + (d-12) \cdot 3^{-3}) < 3/d + 1/27$.

For $p = 2$, we note that $\mu_*(m,2) = 1$ if and only if $m = 2$; $\mu_*(m,2) = 2$ if and only if $m = 3$ or 4; $\mu_*(m,2) = 3$ if and only if $m = 6$ or 7; and $\mu_*(m,2) = 4$ if and only if $m = 5, 8, 12, 14$, or 15. It follows from this that $\zeta^*(d,2) \leq 4/d + 1/32$. □

Corollary 10 *Let $x \in G$ have order d.*

1. *If $p > 2$ and $\zeta(d) \geq k > .04$ for some real number k then $d \leq \dfrac{3}{k - .04}$.*
2. *If $p = 2$ and $\zeta(d) \geq k > .032$ for some real number k then $d \leq \dfrac{4}{k - .032}$.*

Combining **6** with the inequality $\kappa(x_i) \leq \frac{1}{d_i} + \frac{1}{p} - \frac{1}{d_i p}$, we have the following useful inequalities.

11 *$A(\underline{d}) > (.99A(\underline{d}) - 2.0002)p$. Consequently*

1. *$p < \dfrac{A(\underline{d})}{.99A(\underline{d}) - 2.0002}$*
2. *$A(\underline{d}) < \dfrac{2.0002p}{.99p - 1}$*

The precise value of $\mu_*(d, p)$, the smallest possible commutator dimension for an element of order d over $\mathbf{F}_p$, can be computed using the following statement.

12

1. *If d_p is the largest power of p dividing d and $d_{p'} = d/d_p$, then $\mu_*(d, p) = \mu_*(d_p, p) + \mu_*(d_{p'}, p)$.*
2. *For $a \geq 1$, $\mu_*(p^a, p) = p^{a-1}$.*
3. *If $(d, p) = 1$ then either $\mu_*(d, p)$ is the exponent of $p \pmod d$ or $\mu_*(d, p) = \mu_*(a, p) + \mu_*(b, p)$ for some integers a, b with $ab = d$, $a, b > 1$, and $(a, b) = 1$.*

Proof. We may assume that $d > 1$. Suppose x is an operator of order d on V that achieves the minimum commutator dimension. Without loss, assume that $\dim V$ is minimal. Then V is a direct sum of indecomposable $\mathbf{F}_p\langle x\rangle$-submodules V_i. Setting $x_i = x|_{V_i}$ we have $o(x) = \gcd(\{o(x_i)\})$ and $\dim[V, x] = \sum \dim[V_i, x_i]$. Since $\dim[V_i, x_i^m] \leq \dim[V_i, x_i]$ for all $m \in \mathbf{N}$, by minimality of $\dim[V, x]$ we may assume that $o(x_i)$ is relatively prime to $o(x_j)$ when $i \neq j$.

In particular, if $d = p^a$, then V consists of a single Jordan block with eigenvalue 1. In general, if W is a single Jordan block of dimension b for the operator y then $(y-1)^{b-1} \neq 0$ and $(y-1)^b = 0$. This implies that $y^{p^k} = 1$ if and only if $p^k \geq b$. It follows that $p^a \geq \dim V > p^{a-1}$, whence $\dim V = p^{a-1} + 1$ by minimality. Statement 2 now follows since $\dim[V, x] = \dim V - 1$ here.

To prove 1, note that since $ab \geq a - 1 + b$ for positive integers a and b, for unipotent u and semisimple s the commutator dimension of $u \otimes s$ is always at least as large as the commutator dimension of $u \oplus s$.

The last statement follows easily from the fact that if x acts irreducibly and semisimply on V then $\dim V$ is the exponent of $p \pmod d$. This completes the proof of (**12**). □

2.3 Consequences of Scott's Theorem

We use a result of L. Scott [Sco77] on linear groups to control the contributions of elements with large fixed point ratios to the index sum.

Theorem 13 (Scott's Theorem) *If $\widehat{G}$ is a group of linear operators on V with $[V, \widehat{G}] = V$ and $\widehat{G} = \langle g_i \rangle$ where $\prod g_i = 1$, then $\sum \dim[V, g_i] \geq 2 \dim V$.*

As in [FM01], we use Scott's Theorem (Theorem 13) in the following form which provides bounds not merely for $v(x_i)$ but also in some cases for $v(x_i^d)$ for small integers d.

Lemma 14 *Assume that $\underline{e}$ is an ordered r-tuple that is a permutation of one of the following.*

1. $(m, m, 1, \ldots, 1)$, $m \geq 1$.
2. $(2, 2, m, 1, \ldots, 1)$, $m \geq 2$.
3. $(2, 3, m, 1, \ldots, 1)$, $m = 3, 4,$ *or* 5.

Set $C_i = C_i(\underline{e}) = \dfrac{2}{e_i(2 - A(\underline{e}))}$. *Then, for each* i^*,

$$(C_{i^*} - 1)v(x_{i^*}^{e_{i^*}}) + \sum_{i \neq i^*} C_i v(x_i^{e_i}) \geq n.$$

If $p = 2$, *then*

$$\sum C_i v(x_i^{e_i}) \geq 2n.$$

Note that if $\underline{C}(\underline{e}) = (C_1, \ldots, C_r)$ then

$$\underline{C}(m, m, 1, \ldots, 1) = (1, 1, m, \ldots, m)$$

$$\underline{C}(2, 2, m, 1, \ldots, 1) = (m, m, 2, 2m \ldots, 2m)$$
$$\underline{C}(2, 3, 3, 1, \ldots, 1) = (6, 4, 4, 12, \ldots, 12)$$
$$\underline{C}(2, 3, 4, 1, \ldots, 1) = (12, 8, 6, 24, \ldots, 24)$$
$$\underline{C}(2, 3, 5, 1, \ldots, 1) = (30, 20, 12, 60, \ldots, 60)$$

15 *Suppose* $r = 3$.

1. *If* $n > d_1$, *then* $v(x_i) \geq 2$ *for* $i \geq 2$.
2. *If* $n > d_2$, *then* $v(x_1) \geq 2$ *for all* i.
3. *If* $d_1 \leq 3$, *then* $\kappa(x_i) < \zeta_2(d_i)$ *for* $i > 1$.
4. *If* $d_2 \leq 3$, *then* $\kappa(x_i) < \zeta_2(d_i)$ *for all* i.

Proof. Use Lemma 14 with $\underline{e} = (d_1, 1, d_1), i^* = 3$; $\underline{e} = (d_1, d_1, 1), i^* = 2$; and $\underline{e} = (1, d_2, d_2), i^* = 3$ for the first two statements. The others follow from **8**. □

Set $\zeta^t(d) = \frac{1}{d}\left(1 + \sum_{m|d, m<d} \phi(d/m) p^{-\max(1, n-mt)}\right)$.

Note that

$$\zeta^t(d) = \zeta_{n-t, n-2t, \ldots}(d).$$

Lemma 16 *If $j \neq k$ and $\sum_{i \neq j,k} v(x_i) \leq t$, then $\kappa(x_j) \leq \zeta^t(d_j)$ and $d_j \geq n/t$.*

Proof. Without loss, $j = 1$ and $k = 2$. From Lemma 14 with $\underline{e} = (m, m, 1, \ldots, 1)$ and $i^* = 2$ we have $v(x_1^m) \geq n - mt$. The total contribution of the $\phi(d_1/m)$ generators of $\langle x_1^m \rangle$ to $\kappa(x_1)$ is therefore at most $\phi(d_1/m) \cdot \frac{1}{d_1} \cdot p^{-\max(1, n-mt)}$. This implies the inequality for $\kappa(x_j)$. Since $v(x_1^{d_1}) = 0$, it also follows that $d_1 \geq n/t$. □

Lemma 17 *If j, k, l are distinct, $d_k = d_l = 2$, and $\sum_{i \neq j,k,l} v(x_i) \leq t$, then $\kappa(x_j) \leq \zeta^{2t}(d_j)$ and $d_j \geq n/2t$.*

Proof. Argue as in the proof of Lemma 16. Assume $j = 1$, $k = 2$, $l = 3$, and use Lemma 14 with $\underline{e} = (m, 2, 2, 1, \ldots, 1)$ and $i^* = 1$ to get $v(x_1^m) \geq n - 2mt$. □

Lemma 18 *Suppose $\underline{d} = (2, d_2, d_3)$ and $v(x_2^2) = v$.*

1. *$\kappa_3 \leq \zeta^v(d_3)$ and $d_3 \geq n/v + 1$.*
2. *If $p = 2$ then $\kappa_3 \leq \zeta^{v/2}(d_3)$ and $d_3 \geq 2n/v$.*

Proof. Using $\underline{e} = (2, 2, k)$, $i^* = 3$, in Lemma 14, we have $v(x_3^k) \geq n - kv$ in general, and $v(x_3^k) \geq n - kv/2$ when $p = 2$. Also, $(d_3 - 1)v \geq n$. □

Lemma 19 *If $r = 3$ and $i \neq j$, then $d_i v(x_j) \geq n$. In particular, $\kappa_j \leq \zeta_{\lceil n/d_i \rceil}(d_j)$.*

Proof. Without loss, $i = 1$ and $j = 2$. Setting $d = d_1$, the first statement follows from Lemma 14 with $\underline{e} - (d, 1, d)$ and $i^* = 3$. The second statement follows from the first. □

Lemma 20 *Assume that $\underline{d} = (2, 3, d)$. If p is odd, set $s_2 = \lceil n/2 \rceil$ $s_3 = \lceil n/3 \rceil$ $s_4 = \lceil n/5 \rceil$ $s_5 = \lceil n/11 \rceil$. If $p = 2$, set $s_2 = \lceil 2n/3 \rceil$ $s_3 = \lceil n/2 \rceil$ $s_4 = \lceil n/3 \rceil$ $s_5 = \lceil n/6 \rceil$. Then*

$$v(x_3^k) \geq s_k, \quad d = 2, 3, 4, 5.$$

In particular $\kappa_3 \leq \zeta^*_{s_2, s_2, s_3, s_4, s_5}(d)$.

Proof. Use Lemma 14 with $\underline{e} = (2, 3, e)$ and $e = 2, 3, 4, 5$, with $i^* = 3$ for the general case. We have $C_3(\underline{e}) = 3, 4, 6, 12$ in the respective situations. □

Lemma 21 *Assume that* $\underline{d} = (2,4,d)$. *If* p *is odd, set* $s_2 = \lceil n/3 \rceil$ *and* $s_3 = \lceil n/7 \rceil$. *If* $p = 2$, *set* $s_2 = \lceil n/2 \rceil$ *and* $s_3 = \lceil n/4 \rceil$. *Then*

$$v(x_3^k) \geq s_k, \quad d = 2,3.$$

In particular $\kappa_3 \leq \zeta^*_{s_2,s_2,s_3}(d)$.

Proof. Use Lemma 14 with $\underline{e} = (2,4,e)$ and $e = 2,3$, with $i^* = 3$ for the general case. We have $C_3(\underline{e}) = 4, 8$ in the respective situations. □

Lemma 22 *Suppose* $p = 2$, $r = 3$, *and* $\{i,j,k\} = \{1,2,3\}$. *Then*

1. $v(x_i^2) + v(x_j^2) \geq 28/d_k$.
2. *If* $d_i = d_j = 3$, *then* $v(x_k^2) \geq 5$.
3. *If* $d_i = 3$ *and* $d_j = 4$, *then* $v(x_k^2) \geq 3$.

Proof. Without loss, $i = 1$, $j = 2$, and $k = 3$. Use Lemma 14 with $\underline{e} = (2,2,d_3)$, $(3,3,2)$, and $(3,4,2)$, respectively. □

2.4 Initial reductions

Assume, unless stated otherwise, that $d_1 \leq d_2 \leq \ldots \leq d_r$.

Set $S = S(\underline{d}) = r - 2 - .01A(\underline{d}) - .0002$, the right hand side of the inequality in statement **6**.

Lemma 23 $n \geq 3$.

1. *If* $p \leq 97$ *then* $n \geq 4$.
2. *If* $p \leq 19$ *then* $n \geq 5$.
3. *If* $p = 7$ *then* $n \geq 6$.
4. *If* $p = 5$ *then* $n \geq 7$.
5. *If* $p = 3$ *then* $n \geq 10$.
6. *If* $p = 2$ *then* $n \geq 14$.

Proof. The enumerated statements are immediate consequences of the inequality $(p^n - 1)/(p - 1) \geq 10000$.

If $n = 2$, then $F^*(G) \cong L_2(p)$, and $F(x) \leq 2$ for all $x \in G^\sharp$. It follows that $f(x) \leq 1/5000$ for all $x \in G^\sharp$, so equation (1) cannot hold for $g \leq 2$. □

Lemma 24

1. *If* $p \geq 17$, *then* $r = 3$.
2. *If* $p \geq 7$, *then* $r \leq 4$.
3. *If* $p = 7$, *then* $r \leq 4$ *and* $S \geq (r - 3) + .9761$.

4. If $p = 5$, then $r \leq 5$ and $S \geq (r-3) + .9744$.

5. If $p = 3$, then $r \leq 6$ and $S \geq (r-3) + .9693$.

6. If $p = 2$, then $r \leq 8$ and $S \geq (r-3) + .9589$.

Proof. Since ζ is a decreasing function, we have $\zeta(d) \leq \zeta(2) = (p+1)/2p$, so $\kappa(x_i) \leq (p+1)/2p$ for all i. Therefore $r \cdot \frac{p+1}{2p} > r - 2 - .01A(\underline{d}) - .0002 > .99r - 2.0002$, whence

$$r < \frac{4.0004p}{.98p - 1}.$$

All assertions about r, except the first, follow from this.

If $r = 4$, then $A(\underline{d}) \geq 13/6$, so $p < 17$ by **11.1**.

The statements concerning S follow from **11.2** . □

Lemma 25 *1. If $r = 3$, then $S \geq .9698$.*

2. If $r = 3$ and $d_1 = 2$ then $S \geq .9748$.

3. If $r = 3$, $d_1 = 2$, and $d_2 = 3$ then $S \geq .9781$.

4. If $\underline{d} = (2, 3, 7)$, then $S \geq .9795$.

Proof. These statements follow from straightforward computations. □

2.5 Completion of the Proof

The proof of Theorem 1 is based on routine calculations that use the results of the previous section. We include in Appendix A GAP4 code confirming these calculations.

For $i = 1, \ldots, r$, set $\kappa_i = \kappa(x_i)$. Set $\Sigma = \sum \kappa_i$ and $S = r - 2.0002 - .99A(\underline{d})$. Then $\Sigma > S$ by **6**.

Unless stated otherwise, we assume that $d_1 \leq d_2 \leq \ldots \leq d_r$ and that $v(x_i) \leq v(x_{i+1})$ whenever $d_i = d_{i+1}$.

Lemma 26 *$n \geq 4$.*

Proof. Suppose $n = 3$. Then Ω is the set of points in the natural module for $F^*(G) \cong L_3(p)$. We have $N = p^2 + p + 1$. By Lemma 23, $p > 100$, so $A(\underline{d}) < 2.02$ by **11.2**. It follows that $\underline{d} = (2, 3, 7)$.

Since x_1 is an involution in G, we have $\mathrm{Fix}(x_1) \leq p + 2$, and $\mathrm{Ind}(x_1) \geq \frac{1}{2}(p^2 - 1)$. By Lemma 19, $v(x_i) \geq 2$, for $i = 2, 3$. This implies that $\mathrm{Fix}(x_i) \leq 3$, $i = 2, 3$, whence $\mathrm{Ind}(x_i) \geq (d_i - 1)/d_i \cdot (p^2 + p - 2)$. It follows from the Riemann-Hurwitz equation that $g > 2$, a contradiction. □

Lemma 27 *$p \leq 19$*

Proof. Suppose $p \geq 23$. Then $A(\underline{d}) \leq 2.0002p/(.99p-1) < 2.114$ by **11.2** . This implies that $\underline{d}$ is one of the following: $(2,3,d)$, $(2,4,\leq 7)$, $(2,5,5)$, or $(3,3,4)$. Also, $S > .9787$ by **6**.

If $\underline{d} = (2,3,d)$, $d \geq 8$, then **15** implies that $\sum \kappa(x_i) \leq \zeta_2(2) + \zeta_2(3) + \zeta_2(d)$. Since $\phi(d) \geq 4$, it follows that $\zeta_2(d) \leq \frac{1}{d}(1 + (d-5)/p + 4/p^2) = \frac{1}{p} + (1 + \frac{4}{p^2} + \frac{5}{p}) \cdot \frac{1}{d} \leq \zeta_2(8) < .1423$, whence $\sum \zeta_2(d_i) \leq .9778$, a contradiction.

In the remaining six cases, we have $\kappa_i \leq \zeta_2(d_i), i = 2,3$ and $\kappa_1 \leq \zeta(d_1)$ in all cases, and $\kappa_1 \leq \zeta_2(2)$ in the $(2,3,7)$ case. By inspection, either $\Sigma < S$ or $p = 23$ and $\underline{d} = (2,4,5)$ or $(2,3,7)$.

Suppose $\underline{d} = (2,4,5)$. Then $v(x_3) \geq \mu_*(5,23) = 4$. If $v(x_1) \geq 2$, then $\sum \kappa_i < .9628$, so we must have $v(x_1) = 1$. Therefore $v(x_2) \geq n - v(x_1) \geq 3$. Furthermore, $n = 4$, by Lemma 19. This implies that $v(x_2^2) \geq 2$ since every involution t in $PGL(4,23)$ with $v(t) = 1$ is not a square in that group. It follows that $\sum \kappa_i < .974$, a contradiction.

We must have $\underline{d} = (2,3,7)$, whence $v(x_3) \geq \mu_*(7,23) = 3$, and $\kappa_3 \leq \zeta_3(7)$. This implies that $\sum \kappa_i < .9786$, which is not so. □

Proposition 28 *If $p > 7$ then $p = 11$, $\underline{d} = (2,3,7)$, and $n = 5$ or 6.*

Proof. By Lemmas 23 and 27, $n \geq 5$. Suppose $p > 7$. Then $p \geq 11$, and for purposes of estimation with $\zeta(d)$ and $\zeta_k(d)$ we may assume that $p = 11$.

Since $A(\underline{d}) \leq 2.2246$ by **11**, we have $S \geq (r-3) + .9775$.

If $r > 3$, then $\underline{d} = (2,2,2,3)$ by the condition on $A(\underline{d})$. Since $\sum_{i \neq j} v(x_i) \geq n \geq 5$ for $j = 3,4$, we have $v(x_3) > 1$ and either $v(x_2) > 1$ or $v(x_4) > 1$. Therefore $\sum \kappa_i \leq \max(2\zeta(2) + \zeta_2(2) + \zeta_2(3), \zeta(2) + 2\zeta_2(2) + \zeta(3)) < 1.95$, a contradiction.

Thus $r = 3$. Since $\zeta(d_1) \geq S/3 > \zeta(4)$, it follows that $d_1 = 2$ or 3.

Suppose $d_1 = 3$. Then $\zeta(d_2) > (S - \zeta(3))/2 > \zeta(5)$, so $d_2 \leq 4$, and $\kappa_1 \leq \zeta_2(3)$ by **15**. Since $\zeta_2(4) < \zeta_2(3)$, this implies that $\kappa_3 > S - 2\zeta_2(3) > \zeta_2(4) > \zeta(5)$, whence $d_3 = 3$, which is impossible because $\underline{d} \neq (3,3,3)$. This shows that $d_1 = 2$.

Since $\kappa_3 \leq \zeta(d_3) \leq \zeta(d_2)$ and $\kappa_2 \leq \zeta(d_2)$, we must have $\zeta(d_2) > (S - \zeta(2))/2 > \zeta(8)$ so $d_2 \leq 7$. If $d_2 = 5, 6$, or 7, then $\kappa_2 \leq \max_{5 \leq d \leq 7}(\zeta_2(d)) \leq \zeta_2(6)$. [Recall that $p = 11$ for the purpose of calculation.] Since $\zeta(8) < \zeta_2(6)$ and $\zeta(d) < \zeta(8)$ for $d > 8$, we have $\kappa_3 \leq \zeta_2(6)$ and $\sum \kappa_i \leq \zeta(2) + 2\zeta_2(6) < S$. Therefore $d_2 \leq 4$.

Suppose $d_2 = 4$. Then $\kappa_3 \geq S - \zeta_2(2) - \zeta_3(4) > .2002 > \zeta_3(d)$ for $d > 6$, so $d_3 \leq 6$. If $d_3 = 5$, then $A(\underline{d}) = 2.05$, so $S \geq .9793$ and $\Sigma \leq \zeta_2(2) + \zeta_3(4) + \zeta_3(5) < .9781$. It follows that $d_3 = 6$. From Lemma 14 with $\underline{e} = (2,2,2)$, either $v(x_2^2) > 1$ or $v(x_3^2) > 1$. If $v(x_2^2) > 1$, then $\kappa_2 \leq \zeta_{3,2}(4)$. If $v(x_3^2) > 1$, then $\kappa_3 \leq \zeta_{3,2}(6)$. In either case, $\Sigma < .97 < S$.

Suppose $d_2 = 3$. Then $S \geq .9781$ and $\kappa_1 + \kappa_2 \leq \zeta_2(2) + \zeta_3(3) < .8426$, so $\kappa_3 > .1355$. If $d \geq 21$, then $\zeta(d) < \zeta(21) < .135$. Therefore $d_3 \leq 20$. By inspection, if $d_3 = 9$ or $d_3 \geq 11$, then $\zeta_3(d_3) < .137$, and the inquality cannot hold. Therefore d_3 is one of $7, 8$, or 10. If $d_3 = 8$, or 10, then $\kappa_3 \leq \zeta_{3,3}(d_3)$ by Lemma 20 and $\sum \kappa_i < S$.

Therefore $d_3 = 7$, so $S \geq .9795$ and the condition $\zeta_2(2) + \zeta_3(3) + \zeta_3(7) \geq S$ implies that $p = 11$ or 13. If $p = 13$, then $v(x_3)$ is necessarily even, so $\kappa_3 \leq \zeta_4(7)$ and $\sum \kappa_i < S$. Therefore $p = 11$. It follows that κ_2 is even, so $\kappa_2 \leq \zeta_4(3)$. If $v(x_1) > 2$, then $\sum \kappa_i \leq \zeta_3(2) + \zeta_4(3) + \zeta_3(7) < S$. Therefore $v(x_1) = 2$ and $n = 5$ or 6. □

Proposition 29 *If $p = 7$, then $n = 6$ and $\underline{d} = (2, 3, 7)$.*

Proof. By Lemma 24 $n \geq 6$, $r \leq 4$, and $S \geq (r - 3) + .9761$.

Suppose $r = 4$. If $v(x_1) + v(x_2) = 2$, then $d_j \geq 3$, $j > 2$, and $\sum \kappa_i \leq 2\zeta(2) + 2\zeta^2(3) < 1.9$ by Lemma 16. Therefore $v(x_1) + v(x_2) \geq 3$, and in fact $v(x_i) \geq 2$ for at least 3 choices of i. It follows from inspection of values of $\zeta(d)$ and $\zeta_2(d)$ that $\sum \kappa_i < S$, a contradiction.

Therefore $r = 3$. If $v(x_1) = 1$, then $\kappa_2, \kappa_3 \leq \zeta^1(d) < .168$ by Lemma 16. Since $\kappa_1 \leq \zeta(2) < .572$, we have $\sum \kappa_i < S$, a contradiction. Therefore $v(x_i) \geq 2$ and $\kappa_i \leq \zeta_2(d_i)$ for all i. Since $\zeta_2(d) < .3$ for $d > 3$, we have $d_1 \leq 3$.

Suppose $d_1 = 3$. Then, by inspection of $\zeta_2(d)$, $d \geq 3$, we have $\underline{d} = (3, 3, 4)$. Either $v(x_1) = 2$, in which case $\sum \kappa_i \leq \zeta_2(3) + \zeta_4(3) + \zeta_4(4)$, or $v(x_1) \geq 3$, in which case $\sum \kappa_i \leq 2\zeta_3(3) + \zeta_2(4)$. In either case, $\sum \kappa_i < .97$, a contradiction. We conclude that $d_1 = 2$.

We have $\kappa_2 + \kappa_3 \geq S - \zeta_2(2) \geq .465$. Also $\kappa_i \leq \zeta_3(d_i), i > 1$ by Lemma 19. By inspection, $\zeta_3(d) < .2$ for $d > 6$, so $d_2 \leq 6$.

Suppose $v(x_1) = 2$. Then $\kappa_j \leq \zeta^2(d_j), j \geq 2$, whence $d_2 \leq 4$ because $\zeta^2(d) < .21$ for $d > 4$. If $d_2 = 4$, then $d_3 \geq 5$ because $A(\underline{d}) > 2$, so $\sum \kappa_i \leq \zeta_2(2) + \zeta^2(4) + \zeta^2(5) < .97$, a contradiction. Therefore $d_2 = 3$ and $d_3 \geq 7$, so $S \geq .9781$, and $\kappa_2 + \kappa_3 \geq .4678$, whence $\kappa_3 \geq .4678 - \zeta^2(3) > .1341$. By inspection, $d_3 \in \{7, 8, 9, 12\}$. By Lemma 20, $\kappa_3 < \zeta_{4,3,2,2}(d_3)$, and we conclude that $\underline{d} = (2, 3, 7)$. Note that $n = 6$ by Lemma 19.

We may assume henceforth that $v(x_1) \geq 3$, so $\kappa_1 \leq .5015$ and $\kappa_2 + \kappa_3 > .4747$.

If $d_2 = 6$, then $d_3 = 6$ by inspection of the values of $\zeta_3(d)$, $d \geq 6$. From Lemma 14 with $\underline{e} = (2, 2, 2)$ we have $v(x_j^2) > 1$ for some $j > 1$, so $\kappa_2 + \kappa_3 \leq \zeta_3(6) + \zeta_{3,2}(6) < S - \kappa_1$. This implies that $d_2 < 6$.

By inspection, $d_2 \neq 5$. If $d_2 = 4$, then $\kappa_2 \leq \zeta_3(4) < .2872$, so $\kappa_3 > .1875$. This implies that $d_3 \leq 6$. From Lemma 14 with $\underline{e} = (2, 2, d_3)$ we have $v(x_2^2) > 1$, so $\kappa_2 \leq \zeta_{3,2}(4) < .257$. When $d_3 = 6$, the same argument shows that $\kappa_3 \leq \zeta_{3,2}(6) < .2$. In each case, $\sum \kappa_i < S$.

So $d_2 \neq 4$, and we have $d_2 = 3$. Also, $\kappa_1 + \kappa_2 \leq \zeta_3(2) + \zeta_3(3) < .8368$. so $\kappa_3 \geq S - \kappa_1 - \kappa_2 > .1413$. By inspection of $\zeta_3(d)$, we have $d_3 \leq 18$. By Lemma 20, $\kappa_3 \leq \zeta_{3,3,2,2}(d_3)$, so by inspection $d_3 = 7$. If $n > 6$, then $v(x_1) \geq 3$ and $v(x_j) \geq 4$, $j > 1$, so $\sum \kappa_i \leq \zeta_3(2) + \zeta_4(3) + \zeta_4(7) < .9781 < S$. Therefore $n = 6$. □

Proposition 30 *If $p = 5$, then $\underline{d} = (2, 3, 7)$, $n = 7$, 8, or 9, $v(x_1) = 3$, and $v(x_3) = 6$.*

Proof. By Lemma 24, $n \geq 7$, $r \leq 5$, and $S \geq (r-3) + .9744$.

If $r = 5$, then $\sum \kappa_i \leq 3\zeta(2) + 2\zeta_2(2) < S$ because $v(x_i) > 1$ for at least two choices of i. Therefore $r \leq 4$.

Suppose $r = 4$. If $v(x_1)+v(x_2) \leq 3$, then Lemma 16 implies that $d_i \geq 7/3 > 2$ for $i = 3, 4$, and $\kappa_i \leq \zeta^3(d_i) \leq \zeta^3(3) = .3344$. Since $\kappa_1 + \kappa_2 \leq 2\zeta(2) = 1.2$, it follows that $\Sigma < S$. Therefore $v(x_1) + v(x_2) \geq 4$. Moreover, $v(x_i) + v(x_j) \geq 4$ whenever $i \neq j$. If $v(x_1) = 1$, then $\sum \kappa_i \leq \zeta(2) + 2\zeta_3(2) + \zeta_3(3) < 1.95$. If $v(x_1) = 2$, then $\sum \kappa_i \leq 3\zeta_2(2) + \zeta_2(3) = 1.92$. Therefore $v(x_1) \geq 3$. If $d_3 > 2$, then we have $\sum \kappa_i < 2\zeta_3(2) + 2\zeta(3) < 1.95$, noting that $\kappa_1, \kappa_2 \leq \zeta_3(2)$ since $\zeta(3) < \zeta_3(2)$. So $d_3 = 2$ and $\kappa_1+\kappa_2+\kappa_3 \leq 3\zeta_3(2) = 1.512$. From Lemma 14 with $\underline{e} = (2, 2, 2, 1)$ we have $v(x_4) \geq 3$, so $\kappa_4 \leq \zeta_3(d)$ and $\sum \kappa_i \leq 3\zeta_3(2)+\zeta_3(d) < 1.9$. We conclude that $r \neq 4$. Thus, $r = 3$.

If $v(x_1) = 1$, then Lemma 16 shows that $d_2 \geq 7$ and $\kappa_i \leq \zeta^1(d_i)$, $i = 2, 3$. So $\sum \kappa_i \leq \zeta(2) + 2\zeta^1(d) < .9$. Therefore $v(x_1) \geq 2$, and in fact $\kappa_i \leq \zeta_2(d_i)$ for all i. Since $\zeta^*(d) \leq .29$ for $d \geq 12$ by Lemma 9 and $\zeta_2(d) \leq .32$ for $4 \leq d \leq 11$ by inspection it follows that $d_1 \leq 3$, whence $\kappa_i \leq \zeta_3(d_i)$, $i = 2, 3$ by Lemma 19.

Suppose $d_1 = 3$. Then $\kappa_1 \leq \zeta_2(3) = .36$. If $d_2 \geq 4$, then $\kappa_i \leq \zeta_3(d) \leq .304$ for $i > 1$, and $\sum \kappa_i \leq .968$. Therefore $d_2 = 3$, so $v(x_1) \geq 3$ and $\kappa_1 + \kappa_2 \leq 2\zeta_3(3) < .6774$. If $d_3 > 4$, then $\kappa_3 \leq \zeta_3(d_3) \leq .27$, so $d_3 = 4$. From Lemma 14 with $\underline{e} = (3, 3, 2)$ we have $v(x_3^2) \geq 2$ so $\kappa_3 \leq \zeta_{3,2}(4) = .264 < S - \kappa_1 - \kappa_2$. We conclude that $d_1 = 2$.

We have shown that $v(x_1) > 1$. If $d_3 > 23$, then $\kappa_3 < \zeta^*(d_3) < .165$. Suppose $v(x_1) = 2$. Then $d_2 \geq 4$ and $\kappa_i \leq \zeta^2(d_i)$, $i = 2, 3$. Since $\kappa_1 \leq .52$ and $\zeta^2(d) \leq .203$ for $d > 4$, we must have $d_2 = 4$, whence $d_3 > 4$. If $d_3 \neq 6$, then $\sum \kappa_i \leq \zeta_2(2) + \zeta^2(4) + \zeta^2(5) < .973 < S$. Therefore $d_3 = 6$ and $A(\underline{d}) < 2.09$, so $S > .9789$. We have $\sum \kappa_i \leq \zeta_2(2) + \zeta^2(4) + \zeta^2(6) \leq .975$, a contradiction. This shows that $v(x_1) > 2$.

We have $\kappa_1 \leq \zeta_3(2) = .504$ so $\kappa_2+\kappa_3 > .47$. Also, $\kappa_i \leq \zeta_4(d_i), i = 1, 2$, Since $\zeta^*(d) < .2$ for $d \geq 20$ and $\zeta_4(d) < .21$ for $6 < d < 20$, we have $d_2 \leq 6$. From Lemma 14 with $\underline{e} = (2, d_2, 2)$ and $i^* = 3$ it follows that $v(x_3^2) \geq 7/(d_2 - 1) > 1$. This implies that $\kappa_3 \leq \zeta_{4,2}(d_3)$. If $d_2 = 6$, then $\kappa_2 \leq .268$. We have $d_3 = 6$ as otherwise $\kappa_3 \leq \min(\zeta^*(d_3), \zeta_{4,2}(d_3)) < .174$, so $\kappa_2, \kappa_3 \leq \zeta_{4,2}(6) < .214$, and $\Sigma < S$. Therefore $d_2 < 6$. If $d_2 = 5$, then $\kappa_2 + \kappa_3 \leq \zeta_4(5) + \zeta_{4,2}(6) \leq .47$. If $d_2 = 4$, then $\kappa_2 \leq \zeta_4(4) = .3008$. By Lemma 21, $\kappa_3 \leq \zeta_{4,3}(d_3)$. If $d_3 > 23$, then $\kappa_3 < \zeta^*(d_3) < .165$. It follows from inspection that $\zeta_{4,3}(d) \leq .214$ for $6 < d < 24$. Therefore $d_3 \leq 6$, whence $v(x_2^2) \geq 2$ and $\kappa_2 \leq \zeta_{4,2}(4) = .2608$. If $d_3 = 5$, then $\sum \kappa_i < S$. If $d_3 = 6$, then $\kappa_3 \leq .2032$, and $\sum \kappa_i < S$. It follows from this paragraph that $d_2 \neq 4$. Therefore $d_2 = 3$.

We have $\kappa_1 + \kappa_2 \leq \zeta_3(2) + \zeta_4(3) = .8384$. By Lemma 24, $S \geq .9781$, so $\kappa_3 \geq .1397$. Since $\kappa_3 < 3/d_3 + .04$ we may assume that $d_3 \leq 30$. By Lemma 20, $\kappa_3 \leq \zeta_{4,4,3,2}(d_3)$. By inspection, $d_3 = 7$.

If $v(x_1) \geq 4$, then $\sum \kappa_i \leq \zeta_4(2) + \zeta_4(3) + \zeta_4(7) < S$. So $v(x_1) = 3$ and $n \leq d_2 v(x_1) = 9$. □

Proposition 31 *If $p = 3$, then either*

1. $\underline{d} = (2,3,7)$, $n = 12$, $v(x_1) = 4$, $v(x_2) = 8$, *and* $v(x_3) = 12$ *or*

2. $\underline{d} = (2,3,8)$, $n = 10$, $v(x_1) = 4$, $v(x_2) = 6$, *and* $v(x_3^4) = 2$.

Proof. By Lemma 24, $n \geq 10$, $r \leq 6$, and $S \geq (r-3) + .9693$.

We note that $\zeta^*(d) < .11$ for $d > 42$ by Lemma 9 and $\zeta^*(d) < .11$ by direct computation for $24 < d \leq 42$. Also, $\zeta^*(d) < .2$ for $d > 12$. Thus, statements bounding κ_i with weaker bounds need only be verified for a finite number of possible values of d_i. We shall use this implicitly in the following argument.

Since $n > r$, we have $\kappa_i \leq \zeta_2(d_i)$ for at least two choices of i. If $r = 6$, then $\sum \kappa_i \leq 4\zeta(2) + 2\zeta_2(2) < 3.8$, a contradiction, so $r \leq 5$.

Suppose $r = 5$. If $v(x_1)+v(x_2)+v(x_3) = 3$, then $d_i \geq 4$ and $\kappa_i \leq \zeta^3(d_i) < .3$ for $i = 4,5$ by Lemma 16. If $v(x_1) + v(x_2) + v(x_3) = 4$, then $d_i \geq 3$ and $\kappa_i \leq \zeta^4(d_i) < .35$ for $i = 4,5$ Since $\kappa_1 + \kappa_2 + \kappa_3 \leq 3\zeta(2) = 2$ we have $\Sigma < S$ in this case. Therefore $v(x_i) + v(x_j) + v(x_k) \geq 5$ for any choice of distinct i,j,k. If $v(x_i) = 1$ for two values of i, then $v(x_i) \geq 3$ for three values and $\sum \kappa_i \leq 2\zeta(2) + 3\zeta_3(2) < 2.9$. Therefore $v(x_i) = 1$ for at most one value of i, and $\sum \kappa_i \leq \zeta(2) + 4\zeta_2(2) < 2.9$. We conclude that $r \leq 4$.

Suppose $r = 4$. If $v(x_1) + v(x_2) = 2,3,4$, respectively, then $\kappa_1 + \kappa_2$ is respectively at most 1.3334, 1.2223, 1.1852, while Lemma 16 implies that for $i = 3$ or 4, $\kappa_i \geq \zeta^2(d_i)$ and $d_i \geq 5$, $\kappa_i \geq \zeta^3(d_i)$ and $d_i \geq 4$, $\kappa_i \geq \zeta^4(d_i)$ and $d_i \geq 3$, in the respective cases. By inspection, $\kappa_3 + \kappa_4$ is respectively at most .401, .511, .67, whence $\sum \kappa_i < S$. It follows that $v(x_1) + v(x_2) \geq 5$. Since the same is true of $v(x_i) + v(x_j)$, $i \neq j$, it follows that $v(x_i) \geq 3$ for at least 3 choices of i. Since $\zeta(2) < .67$, $\zeta_3(2) < .52$, and $\zeta(d) < .56$, $\zeta_3(d) < .36$ when $d > 2$, we have $d_3 = 2$, else $\sum \kappa_i < 1.96 < S$. Set $v = v(x_1)$. If $v = 1$, then $\kappa_1 + \kappa_2 + \kappa_3 \leq \zeta(2) + 2\zeta_4(2) < 1.68$ and, by Lemma 17, $\kappa_4 \leq \zeta^2(d_4)$ where $d_4 \geq 5$, so $\kappa_4 < .21$. If $v = 2$, then $\kappa_1 + \kappa_2 + \kappa_3 \leq \zeta_2(2) + 2\zeta_3(2) < 1.6$ and, by Lemma 18, and inspection of ζ^2 values, $\kappa_4 \leq \zeta^4(d_4)$ where $d_4 \geq 4$, so $\kappa_4 < .28$. If $v > 2$, then $\kappa_1+\kappa_2+\kappa_3 \leq 3\zeta_3(2) < 1.56$. From Lemma 14 with $\underline{e} = (2,2,2,1)$ and $i^* = 4$ we have $v(x_4) \geq 4$ and $\kappa_4 \leq \zeta_4(3) < .35$. In all cases, $\sum \kappa_i < S$. Therefore $r \neq 4$.

We have $r = 3$. By inspection, $d > 6$ implies $\zeta^*(d) \leq .25$. Therefore $d_1 \leq 6$ and $A(\underline{d}) \leq 3 \cdot 5/6 < 2.84$, so $S > .9714$. By inspection, $\kappa_1 \leq \zeta(2) < .67$.

If $v(x_1) = 1$, then, by Lemma 16, $d_i \geq 10$ and $\kappa_i \leq \zeta^1(d_i) < .11$, $i = 2,3$. If $v(x_1) = 2$, then $\kappa_1 \leq \zeta_2(2) < .556$. Also, by Lemma 16, $d_i \geq 5$ and $\kappa_i \leq \zeta^2(d_i) < .201$, $i = 2,3$. It follows that $v(x_1) > 2$, so $\kappa_i \leq \zeta_3(d_i)$ for all i.

Suppose $d_1 \geq 4$. Then $\kappa_i \leq \zeta_3(d_i) \leq \zeta_3(4) < .3519$ for all i, so $\sum_{j \neq i} \kappa_j \geq S - \zeta_3(4) > .619$, $i = 1,2,3$. If $v(x_i) = 3$ for some i, then Lemma 16 shows that $\kappa_j \leq \zeta^3(d_j) \leq .254$ for $j \neq i$. If $v(x_i) = 4$ for some i, then $\kappa_i \leq \zeta_4(d_i) \leq \zeta_4(4) < .34$ and $\kappa_j \leq \zeta^4(d_j) < .28$, $j \neq i$. It follows that $v(x_i) \geq 5$ for all i, so $\kappa_i \leq \zeta_5(4) < .336$. Since $\zeta^*(d_3) \leq .25$ for all $d > 4$ with $d \neq 6$ we conclude that $d_i = 4$ or 6 for all i. From Lemma 14 with $\underline{e} = (2,2,2)$ we have $v(x_i^2) \geq 2$ for some i. Therefore $\kappa_i \leq \zeta_{5,2}(d_i) < .28$ for some i. Since $\kappa_j \leq \zeta_5(d_j) < .34$ for all j it follows that $\Sigma < .96 < S$.

Suppose $d_1 = 3$. Then $\kappa_1 \leq \zeta_3(3) < .36$. If $v(x_1) = 3$, then $d_i \geq 4$ and $\kappa_i \leq \zeta^3(d_i) < .26$, $i = 2,3$, by Lemma 16, whence $\sum \kappa_i < S$. Therefore

$v(x_1) \geq 4$ and $\kappa_1 \leq \zeta_4(3) < .342$, so $\kappa_2 + \kappa_3 \geq S - \kappa_1 > .6295$. If $d > 3$ and d is odd, then $\zeta^*(d) < .21$. For all $d \geq 3$ we have $\zeta_4(d) \leq \zeta_4(3) < .342$. It follows that d_i is even whenever $d_i > 3$. If $d_2 > 3$, then Lemma 14 with $\underline{e} = (3, 2, 2)$ implies that $v(x_i^2) > 1$ for some $i > 1$. Therefore $\kappa_2 + \kappa_3 \leq \zeta_{4,2}(d_i) + \zeta_4(d_{5-i}) \leq \zeta_{4,2}(4) + \zeta_4(4) < S - \kappa_1$. This implies that $d_2 = 3$, so $d_3 > 3$. From Lemma 14 with $\underline{e} = (3, 3, 2)$ and $i^* = 3$ we have $v(x_3^2) \geq 2$. Therefore $\sum \kappa_i \leq 2\zeta_4(3) + \zeta_{4,2}(d_3) \leq 2\zeta_4(3) + \zeta_{4,2}(4) < S$, a contradiction.

We have $d_1 = 2$ and $\kappa_1 \leq \zeta_3(2) < .5186$. Since $\zeta^*(d) < .22$ for $d > 8$ it follows that $d_2 \leq 8$. By Lemma 19, $v(x_i) \geq 5$ for $i = 2, 3$.

We claim that if $i = 2$ or 3 and $d_i > 4$, then $\kappa_i \leq .236$ and furthermore, either $\kappa_i < .204$ or $d_i = 6$ and $v(x_i^2) \geq 3$. Since $\zeta^*(d) < .2$ for $d \geq 13$ and $\zeta_5^*(d) < .204$ for d odd with $4 \leq d < 12$, it suffices to assume that d_i is even and $d_i \leq 12$. We have $\kappa_i \leq \zeta_5(d_i) < .236$. Suppose $v(x_i^2) = 1$. By Lemma 18, $\kappa_{5-i} \leq \zeta^1(d_{5-i})$ and $d_{5-i} \geq 11$, so $\kappa_{5-i} < .11$. It follows that $\sum \kappa_i < .97 < S$. Therefore $v(x_i^2) > 1$. Suppose $v(x_i^2) = 1$. Then $\kappa_i \leq \zeta_{5,2}(d_i) < .281$. By Lemma 18, $\kappa_{5-i} \leq \zeta^2(d_{5-i})$ and $d_{5-i} \geq 6$, so $\kappa_{5-i} < .17$. This also implies that $\sum \kappa_i < .97 < S$. Therefore $v(x_i^2) \geq 3$ and $\kappa_i \leq \zeta_{5,3}^*(d_i)$. The claim follows.

It follows from the claim that if $d_2 > 4$ then $\underline{d} = (2, 6, 6)$ and $v(x_i^2) \geq 3$ for $i = 2, 3$. By Lemma 14 with $\underline{e} = (2, 3, 3)$ we have $v(x_i^3) > 1$ for some $i > 1$, so $\kappa_2 + \kappa_3 \leq \zeta_{5,3}(6) + \zeta_{5,3,2}(6) < .435 < S - \kappa_1$. This shows that $d_2 \leq 4$.

Suppose $d_2 = 4$. Set $v = v(x_2^2)$. If $v = 1$, then $\kappa_2 \leq \zeta_5(4) < .336$ and, as above, $\kappa_3 < .11$. If $v = 2$, then $\kappa_2 \leq \zeta_{5,2}(4) \leq .28$ and $\kappa_3 \leq \zeta^2(d_3) \leq \zeta^2(6) < .17$. In either case, $\kappa_2 + \kappa_3 < .45 < S - \kappa_1$. Therefore $v \geq 3$ and we have $\kappa_2 \leq \zeta_{5,3}(4) < .2614$. If $d_3 \neq 5, 6, 8, 9, 12$, then $\kappa_3 < \zeta^*(d_3) < .15$, so we may assume that $d_3 \in \{5, 6, 8, 9, 12\}$. By Lemma 21 and the condition that $v(x_3) \geq 5$, $\kappa_3 \leq \zeta_{5,4,2}(d_3)$. By inspection, this is at most .191 for $d_3 > 5$, so $\sum \kappa_i < .971 < S$ in this case. We must have $\underline{d} = (2, 4, 5)$. Thus, $A(\underline{d}) = 2.05$ and $S = .9793$. If $v(x_1) = 3$, then $\Sigma \leq \zeta_3(2) + \zeta^3(4) + \zeta^3(5) < .975 < S$. Therefore $v(x_1) \geq 4$ and $\kappa_1 \leq \zeta_4(2) < .507$. We have $\kappa_2 \leq \zeta_{5,3}(4) < .262$ and $\kappa_3 \leq \zeta_5(5) \leq .204$, so $\sum \kappa_i < .973 < S$, a contradiction. This shows that $d_2 \neq 4$, so $d_2 = 3$.

We have $S > .9781$ by Lemma 25. By Lemma 19, $v(x_1) \geq 4$ and $v(x_2) \geq 5$. Since $v(x_1) + v(x_2) \geq 10$, we have $\kappa_1 + \kappa_2 \leq \max(\zeta_4(2) + \zeta_6(3), \zeta_5(2) + \zeta_5(3)) < .8405$. By Lemma 20, $\kappa_3 \leq \zeta_{5,5,4,2}(d_3)$. If $d_3 > 8$, then $\kappa_3 < .137 < S - \kappa_1 - \kappa_2$. Therefore $d_3 = 7$ or 8.

If $d_3 = 7$, then $S > .9795$. If $n > 12$, then $v(x_1) \geq 5$, $v(x_2) \geq 7$, and $v(x_3) \geq 7$, so $\sum \kappa_i \leq \zeta_5(2) + \zeta_7(3) + \zeta_7(7) < S$. Therefore $n \leq 12$. Since $\zeta_6(2) + 1/3 + 1/7 > S([2, 3, 7])$ we must have $v(x_1) \leq 5$. We have $n \geq 10$. Therefore $v(x_1) \leq n - v(x_1)$. From the strong form of Scott's Theorem we have $\max(v(x_1), n - v(x_1)) + v(x_2) + v(x_3) \geq 2n$. Therefore $v(x_2) + v(x_3) \geq n + v(x_1) \geq 4n/3$. Since $p = 3$, we have $v(x_2) \leq 2n/3$, so $v(x_3) \geq 2n/3$. Since 3 has multiplicative order 6 modulo $d_3 = 7$, $v(x_3)$ is necessarily a multiple of 6. Since $10 \leq n \leq 12$ we must have $v(x_3) = n = 12$. If $v(x_1) \geq 5$, then $v(x_2) \geq 7$ and $\sum \kappa_i \geq \zeta_5(2) + \zeta_7(3) + \zeta_{12}(7) > S([2, 3, 7])$, a contradiction. Therefore $v(x_1) = 4$ and $v(x_2) = 8$.

Suppose $d_3 = 8$. If $n > 10$, then $v(x_1) \geq 4$, $v(x_2) \geq 6$, $v(x_2^2) \geq 6$, and $v(x_2^4) \geq 3$, so $\Sigma \leq \zeta_4(2) + \zeta_5(3) + \zeta_{6,6,1,3}(8) < S$. Therefore $n = 10$. If $d_1 > 4$, then $d_1 = 5$, $5 \leq d_2 \leq 6$, and $d_3 \geq 8$ by the strong form of Scott's Theorem, so $\Sigma \leq \zeta_5(2) + \zeta_5(3) + \zeta_{8,5,1,2}(8) < S$. Therefore $d_1 = 4$, whence $d_2 = 6$. Since $\Sigma \leq \zeta_4(2) + \zeta_6(3) + \zeta_{6,5,1,3}(8) < S$, we also have $v(x_3^4) = 2$. □

Proposition 32 *If $p = 2$, then one of the following is true.*

1. $\underline{d} = (2, 3, 7)$
2. $n = 16$, $\underline{d} = (2, 4, 5)$, $v(x_1) = 4$, $v(x_2) = 12$, *and* $v(x_3) = 16$.

Proof. Assume that $p = 2$. By Lemma 24, $n \geq 14$, $r \leq 8$, and $S > (r-3) + .9589$.

Step 1 *1.* $\zeta^*(2) = .75$.

2. *If* $d > 2$, *then* $\zeta^*(d) \leq .5$.
3. *If* $d > 4$, *then* $\zeta^*(d) \leq .375$.
4. *If* $d > 6$, *then* $\zeta^*(d) < .282$.
5. *If* $d > 8$, *then* $\zeta^*(d) \leq .25$.
6. *If* $d > 12$, *then* $\zeta^*(d) < .19$.
7. *If* $d > 14$, *then* $\zeta^*(d) \leq .15$.
8. *If* $d > 30$, *then* $\zeta^*(d) < .094$.
9. *If* $d > 42$, *then* $\zeta^*(d) < .08$.

In view of Lemma 9, the assertions follows immediately from inspection of the values of $\zeta^*(d)$ for $d < 100$.

Step 2 $r < 5$.

If $r = 8$, then $v(x_i) \geq 2$ for at least 2 choices of x_i, so $\sum \kappa_i \leq 6\zeta(2) + 2\zeta_2(2) = 5.75$. If $r = 7$, then $v(x_i) \geq 3$ for at least 2 choices of x_i since $v(x_1) + \ldots + v(x_6) \geq 14 > 6 \cdot 2$. Therefore $\sum \kappa_i \leq 5\zeta(2) + 2\zeta_3(2) \leq 4.875 < S$. This shows that $r \leq 6$.

Suppose $r = 6$. Set $w = v(x_1) + v(x_2) + v(x_3) + v(x_4)$. If $w \leq 6$, then Lemma 16 implies that $d_5, d_6 > 2$ and $\kappa_i \leq \zeta^6(d_i) < .34$, $i = 5, 6$, so $\sum \kappa_i \leq 4\zeta(2) + 2 \cdot .34 < 3.7$. Therefore $v(x_1) + v(x_2) + v(x_3) + v(x_4) \geq 7$, and the same is true for any other choice of 4 distinct subscripts. If $v(x_i) = 1$ for 3 values of i, then $v(x_j) \geq 4$ for all other values and $\sum \kappa_i \leq 3\zeta(2) + 3\zeta_4(2) < 3.9$. If $v(x_i) = 1$ for exactly 2 values of i, then $v(x_j) \geq 3$ for at least 3 values of j and $\sum \kappa_i \leq 2\zeta(2) + \zeta_2(2) + 3\zeta_3(2) < 3.9$. It follows that $v(x_i) = 1$ for at most 1 choice of i, and $\sum \kappa_i \leq \zeta(2) + 5\zeta_2(2) < 3.9$. Therefore $r < 6$.

Suppose $r = 5$. We claim that if i, j, and k are distinct, then $v(x_i) + v(x_j) + v(x_k) \geq 7$. Assume that $v(x_i) + v(x_j) + v(x_k) \leq 6$. Then, by Lemma 16, $d_l > 2$

for $l \neq i, j, k$ and $\kappa_l \leq \zeta^6(d_l)$. If $d_l > 6$, then $\kappa_l < .3$ by Step 1. If $3 \leq d_l \leq 6$, then $\zeta^6(d_l) < .34$ by inspection. This implies that $\sum \kappa_i < 3\zeta(2) + 2 \cdot .34 = 2.93 < S$, and the claim follows.

We claim further that if $v(x_i) + v(x_j) \leq 4$ for distinct i, j, then $d_k = 2$ for all $k \neq i, j$. For the purpose of establishing this claim we remove the running assumption on the ordering of x_i for the balance of this paragraph and show that if $v(x_1) + v(x_2) \leq 4$ then $d_k = 2$ for $k > 2$. If $v(x_1) + v(x_2) = 2$, then $v(x_k) \geq 5$ for $k > 2$ by the previous paragraph, and $\sum \kappa_i \leq 2\zeta(2) + \sum_{k>2} \zeta_5(d_k)$. Since $\zeta_5(2) < .52$ and $\min(\zeta_5(d), \zeta^*(d)) < .4$ for $d > 2$, we have either $\sum \kappa_i \leq 2\zeta(2) + 2\zeta_5(2) + .4 < 2.94$ or $d_k = 2$ for all $k > 2$. If $v(x_1) + v(x_2) = 3$, then $\kappa_1 + \kappa_2 \leq \zeta(2) + \zeta_2(2) = 1.4$ and $\sum \kappa_i \leq \kappa_1 + \kappa_2 + \sum_{k>2} \zeta_4(d_k)$. Since $\zeta_4(2) < .54$ and $\min(\zeta_4(d), \zeta^*(d)) < .41$ when $d > 2$, either $\sum \kappa_i < 2.9$ or $d_k = 2$ for all $k > 2$. Finally, if $v(x_1) + v(x_2) = 4$, then $\kappa_1 + \kappa_2 \leq \zeta(2) + \zeta_3(2) < 1.32$. Considering that $\zeta_3(2) < .57$ and $\min(\zeta_3(d), \zeta^*(d)) < .44$ for $d > 2$, either $\sum \kappa_i < 2.9$ or $d_k = 2$ for all $k > 2$. Since $\sum \kappa_i \geq S$ we conclude in every case that $d_k = 2$ for all $k > 2$. This completes the argument that if $v(x_i) + v(x_j) \leq 4$ for some $i \neq j$, then $d_k = 2$ whenever $k \neq i, j$.

Reverting to the ordering of x_i, so that d_5 is the largest value of d_i, the previous paragraph implies that if $d_5 > 2$, then $v(x_i) + v(x_j) \geq 5$ for every pair of distinct $i, j < 5$. In that case, $\sum_{i<5} \kappa_i \leq \max(\zeta(2) + 3\zeta_4(2), \zeta_2(2) + 3\zeta_3(2)) < 2.4$, and $\kappa_5 \leq \zeta^*(d_5) \leq .5$, whence $\sum \kappa_i < S$. We conclude that $d_i = 2$ for all i. From Lemma 14 with $\underline{e} = (2, 2, 2, 1, 1)$ it follows that $v(x_i) + v(x_j) \geq 7$ whenever $i \neq j$, whence $\kappa_i \leq \max(\{\zeta_a(2) + 4\zeta_{7-a}(2) \ : \ a = 1, 2, 3\}) < 2.8 < S$. This completes the argument that $r \neq 5$.

Step 3 $r = 3$.

Suppose $r = 4$. Since $v(x) + v(x') + v(x'') \geq 14$ for every set of 3 generators $\{x, x', x''\}$ it follows that $v(x) \geq 5$ for at least two of the four generators, so $\kappa_i \leq \zeta_5(d_i)$ for at least two values of i.

We claim that $A(\underline{d}) \leq 3$. Suppose $A(\underline{d}) > 3$. Then $\sum 1/d_i < 1$. The ordering assumption on d_i implies that $d_2 > 2$ and $d_4 > 4$, so $\kappa_i \leq .5$ for $i > 1$ and $\kappa_4 \leq .375$. If $d_1 > 2$, then $\sum \kappa_i \leq 1.875$, which is not the case, so $d_1 = 2$. It follows that $d_3 > 4$, since otherwise $1/d_1 + 1/d_2 + 1/d_3 \geq 1$. This implies that $\kappa_1 + \kappa_2 + \kappa_3 \leq 1.625$. Therefore $\kappa_4 > .3$, so $d_4 \leq 6$ and $A(\underline{d}) \leq A(2, 4, 6, 6) < 3$, a contradiction. This establishes the claim, and we conclude that $S \geq 1.9698$.

Set $w = v(x_1) + v(x_2)$. Then, by Lemma 16, $\kappa_3 \leq \zeta^w(d_3)$, $\kappa_4 \leq \zeta^w(d_4)$, and $d_3 \geq 14/w$. If $w \leq 3$, then $\kappa_1 + \kappa_2 \leq 1.5$, $d_i \geq 5$, and $\kappa_i < \zeta^3(d_i) < .201$ for $i > 2$. If $w = 4$, then $\kappa_1 + \kappa_2 \leq \zeta(2) + \zeta_3(2) < 1.32$, $d_i \geq 4$, and $\kappa_i < \zeta^4(d_i) < .26$ for $i > 2$. If $w = 5$, then $\kappa_1 + \kappa_2 \leq \zeta(2) + \zeta_4(2) < 1.282$, $d_i \geq 3$, and $\kappa_i < \zeta^5(d_i) < .335$ for $i > 2$. If $w = 6$, then $\kappa_1 + \kappa_2 \leq \zeta(2) + \zeta_5(2) < 1.266$, $d_i \geq 3$, and $\kappa_i < \zeta^6(d_i) < .336$ for $i > 2$. In each case, $\sum \kappa_i \leq 1.96 < S$. This implies that $v(x_1) + v(x_2) \geq 7$. More generally, $v(x_i) + v(x_j) \geq 7$ whenever $i \neq j$.

Suppose $d_3 > 2$ and set $v = v(x_1)$. We claim that $v = 1$. If $v = 2$, then $\kappa_i \leq \zeta_5(d_i)$ for all $i > 1$. Since $\zeta^*(d) < \zeta_5(4) < .5$ when $d > 4$ and $\zeta_k^*(3) \leq \zeta_k^*(4)$

for all k it follows that $\Sigma \leq \zeta_2(2)+\zeta_5(2)+2\zeta_5^*(4) < 1.93$. Similarly, if $v = 3$, then $\Sigma \leq \zeta_3(2) + \zeta_4(2) + 2\zeta_4^*(4) < 1.91$. If $v = 4$, then $\Sigma \leq 2\zeta_4(2) + \zeta_3^*(4) + \zeta_4^*(4) < 1.91$. If $v = 5$, then $\Sigma \leq 2\zeta_5(2) + \zeta_2^*(4) + \zeta_5^*(4) < 1.93$. If $v \geq 6$, then $\kappa_1 + \kappa_2 \leq 2\zeta_6(2) < 1.02$ and $\kappa_3 + \kappa_4 \leq \max_{t=1,2,3}(\zeta_t^*(4) + \zeta_{7-t}^*(4)) < .9$. In all cases, $\Sigma < S$.

Therefore $v(x_1) = 1$ and $v(x_i) \geq 6$ for $i > 1$. We have $\kappa_1+\kappa_2 \leq \zeta(2)+\zeta_6(2) < 1.26$. Also, $\kappa_i \leq \zeta_6^*(d_i)$ when $i > 2$. If $d > 4$, then $\zeta_6^*(d) < .34$. Therefore $d_3 \leq 4$ and $\kappa_3 < .39$. From Lemma 14 with $\underline{e} = (1, 2, d_3, 2)$ we have $2d_3 + d_3 v(x_4^2) \geq 28$, whence $v(x_4^2) \geq 28/d_3 - 2 \geq 5$. Therefore $\kappa_4 \leq \zeta_{6,5}^*(d_4)$. If $d_4 \geq 4$, then $\kappa_4 < .3$ and $\Sigma < 1.95$. Therefore $d_4 = 3$, whence $d_3 = 3$, and $\kappa_i \leq \zeta_6(3) < .35$ for $i = 3$ or 4. Once again, $\Sigma < S$. This shows that $d_3 = 2$.

We have $A(\underline{d}) < 2.5$, so $S > 1.9748$. As before, set $v = v(x_1)$. From Lemma 17, $\kappa_4 \leq \zeta^{2v}(d_4)$ and $d_4 \geq 7/v$. If $v = 1$, then $\kappa_1 + \kappa_2 + \kappa_3 \leq \zeta(2) + 2\zeta_6(2) < 1.766$ and $\kappa_4 \leq \zeta^2(d_4) < .144$ because $d_4 \geq 7$. If $v = 2$, then $\kappa_1 + \kappa_2 + \kappa_3 \leq \zeta_2(2) + 2\zeta_5(2) < 1.657$ and $\kappa_4 \leq \zeta^4(d_4) < .255$ because $d_4 \geq 4$. If $v = 3$, then $\kappa_1 + \kappa_2 + \kappa_3 \leq \zeta_3(2) + 2\zeta_4(2) = 1.625$ and $\kappa_4 \leq \zeta^6(d_4) < .336$ because $d_4 \geq 3$. This shows that $\Sigma < S$ when $v \leq 3$. Therefore $v \geq 4$ and $\kappa_1 + \kappa_2 + \kappa_3 \leq 3\zeta_4(2) < 1.594$. From Lemma 14 with $\underline{e} = (2, 2, 2, 1)$ we have $v(x_4) \geq 7$, so $\kappa_4 \leq \zeta_7(d_4) < .379$ because $d_4 > 2$. In this case as well, $\sum \kappa_i < S$. This shows that $r < 4$.

Step 4 $v(x_i) \geq 4$ *for all* i.

Since $A(\underline{d}) < r = 3$, $S > .9698$. Set $v = v(x_1)$. We apply Lemma 16 once again to bound v from below. If $v = 1, 2$, or 3, then $\kappa_1 \leq \zeta_v(2) \leq .75$, .625, .563, respectively. For $i > 1$, $\kappa_i \leq \zeta^v(d_i)$ where $d_i \geq 14, 7, 5$ in the respective cases. Using Step 1 and inspection, we have $\kappa_i < .08, .15, .201$ in the respective cases. It follows that $\sum \kappa_i < S$ whenever $v(x_1) < 4$. Therefore $v(x_1) \geq 4$. More generally, since the argument that established this does not use the ordering assumption on x_i, it follows that $v(x_i) \geq 4$ for all i.

Step 5 $d_1 = 2$.

Assume that $d_1 > 2$. It follows from Step 1 that $d_1 \leq 6$, so $A(\underline{d}) < 2.84$ and $S > .9714$. Since $v(x_1) \geq 4$, we have $\kappa_1 \leq \zeta_4^*(d_1)$. It follows from Step 1 and inspection that $\kappa_i < .41$ for all i.

If $v(x_1) = 4$, then $d_i \geq 4$, $i = 2, 3$ and $\kappa_i \leq \zeta^4(d_i) < .255$, which implies that $\sum \kappa_i < S$. Therefore $v(x_1) \geq 5$, and, similarly, $v(x_i) \geq 5$ for all i. Thus $\kappa_i \leq \zeta_5^*(d_i)$ for all i. In particular, $\kappa_i < .3907$ for all i.

Suppose $d_1 > 4$. Since $\zeta_5^*(d) < .27$ when $d > 4$, $d \neq 6$ and $\zeta_5^*(6) < .35$, it follows that $d_i = 6$ for all i. Lemma 22 implies that $v(x_i^2) \geq 3$ for at least two choices of i, so $\sum \kappa_i \leq \zeta_5^*(6) + 2\zeta_{5,3}^*(6) < .95$. Therefore $d_1 \leq 4$.

Suppose $d_1 = 4$. Then $d_2 \leq 6$ since otherwise $\kappa_i \leq \zeta_5^*(d_i) < .27$ for $i = 2, 3$ and $\Sigma < \zeta_5^*(4) + 2 \cdot .27 < S$. It follows from Step 1 that $d_3 \leq 12$ as otherwise $\Sigma < S$. This implies that $A(\underline{d}) \leq A(4, 6, 12) = 2.5$, so $S \geq .9748$. Also, Lemma 22 implies that $v(x_1^2) + v(x_2^2) \geq 3$. We claim that $d_3 \leq 8$. If $d_2 = 5$ or 6, then

$\kappa_1+\kappa_2 < \zeta_5^*(4)+\zeta_5^*(6) < .7345$, so $\zeta_5^*(d_3) \geq \kappa_3 > .24$. It follows from inspection that $d_3 \leq 8$ in this case. If $d_2 = 4$, then $\kappa_1 + \kappa_2 \leq \zeta_5^*(4) + \zeta_{5,2}^*(4) < .7188$, so $\zeta_5^*(d_3) > .25$ and $d_3 \leq 8$ in this case as well.

From Lemma 22.1 we have $v(x_1^2) + v(x_2^2) \geq 4$ and $v(x_1^2) + v(x_3^2) \geq 5$. If $v(x_1^2) = 1$, then $v(x_2^2) \geq 3$ and $v(x_3^2) \geq 4$. so $\kappa_2 \leq \zeta_{5,3}^*(d_2) < .3021$, $\kappa_3 \leq \zeta_{5,4}^*(d_3) < .2813$, and $\sum \kappa_i < .974 < S$. If $v(x_1^2) = 2$, then $\kappa_1 \leq \zeta_{5,2}^*(4) < .3282$, $\kappa_2 \leq \zeta_{5,2}^*(d_2) < .3438$, and $\kappa_3 \leq \zeta_{5,3}^*(d_3) < .3021$, whence $\sum \kappa_1 < .9741 < S$. We conclude that $v(x_1^2) \geq 3$, so that $\kappa_1 \leq \zeta_{5,3}(4) < .3$. Without loss, if $d_i = 4$, $i = 2, 3$, then $\kappa_i \leq \zeta_{5,3}^*(4) < .3$. If $d_i > 4$ for some i, then $\kappa_i \leq \zeta_5^*(d_i) < .35$. Since $v(x_2^2) + v(x_3^2) \geq 7$ by Lemma 22, we have either $v(x_2^2) \geq 4$ or $v(x_3^2) \geq 4$, whence $\kappa_i \leq \zeta_{5,4}^*(d_i) < .3$ for some $i > 1$. It follows that $\Sigma < .95 < S$, so we conclude that $d_1 \neq 4$.

We may therefore suppose $d_1 = 3$, so that $\kappa_1 \leq \zeta_5^*(3) \leq .3542$. By Lemma 19, $\kappa_i \leq \zeta_5^*(d_i)$ for $i = 2, 3$. As in the argument when $d_1 = 4$, it follows that $d_2 \leq 6$. By Lemma 14 with $\underline{e} = (1, 1, 1)$, we have $v(x_i) \geq 7$ for two choices of i. If $d_2 = 6$, then $\kappa_1+\kappa_2 \leq \max(\zeta_5^*(3)+\zeta_7^*(6), \zeta_7^*(3)+\zeta_5^*(6)) < .6902$. It follows from inspection of ζ_5^* values that $d_3 = 6$. Since $v(x_2^2)+v(x_3^2) \geq 10$ by Lemma 22.1 we have $\sum \kappa_i \leq \zeta_5^*(3) + \zeta_5^*(6) + \zeta_{5,5}^*(6) < .3542 + .3438 + .2709 < .97 < S$.

If $d_2 = 5$, then $\kappa_2 \leq .225$ and $\kappa_3 \leq \zeta_5^*(d_3) < .344$, so $\Sigma < S$.

If $d_2 = 4$, then $\kappa_1 + \kappa_2 \leq \max(\zeta_5^*(3) + \zeta_7^*(4), \zeta_7(3) + \zeta_5(4)) < .7332$. By Lemma 22.3, $\kappa_3 \leq \zeta_{5,3}^*(d)$. It follows that $\zeta_{5,3}^*(d) > .24$, so $d_3 = 4$ or 6 by inspection. If $d_3 = 4$, then $\kappa_2 \leq \zeta_{5,3}^*(4) < .3$ by the same result, and $\sum \kappa_i < \zeta_5^*(3) + 2\zeta_{5,3}^*(4) < S$. Therefore $d_3 = 6$. If $v(x_2^2) \geq 3$, then $\sum \kappa_i \leq \zeta_5^*(3) + \zeta_{5,3}^*(4) + \zeta_{5,3}^*(6) < S$. If $v(x_2^2) = 2$, then $v(x_3^2) \geq 8$ by Lemma 22 and $\sum \kappa_i < \zeta_5^*(3)+\zeta_{5,2}^*(4)+\zeta_{5,5}^*(6) < S$, so we may assume that $v(x_2^2) = 1$. From Lemma 14 with $\underline{e} = (3, 2, 3)$ we have $4v(x_3^3)+6 \geq 28$ whence $v(x_3^3) \geq 6$ and $\kappa_3 < \zeta_{5,5,6}^*(6) < .2 < S - \kappa_1 - \kappa_2$. This shows that $d_2 \neq 4$.

If $d_2 = 3$ then $\kappa_2 \leq \zeta_7^*(3) \leq .3386$ and, by Lemma 22.2, $\kappa_3 \leq \zeta_{5,5}^*(d_3) \leq .2735$, so $\Sigma \leq .97 < S$.

Step 6 $d_2 \leq 4$

By Lemma 25 and the previous step, $S \geq .9748$. Assume that $d_2 > 4$. By Step 4, $v(x_1) \geq 4$. If $v(x_1) = 4$, then $\kappa_1 \leq \zeta_4(2) < .532$, and $\kappa_i \leq \zeta_{10,6,2}^*(d_i)$ by Lemma 16. By inspection, $\kappa_i < .22$ for $i \geq 2$. This implies that $\Sigma < S$. We conclude that $v(x_1) > 4$.

We have $\kappa_1 \leq \zeta_5(2) < .5157$. If $d_i > 8$ and $d_i \neq 12$, then $\kappa_i \leq \zeta^*(d_i) < .2$. If $d_i = 12$, then $\kappa_i \leq \zeta_7^*(12) < .232$. It follows that either $d_2 \leq 8$ or $d_2 = d_3 = 12$. In the latter case, Lemma 14 with $\underline{e} = (2, 3, 3)$ shows that $v(x_2^3) + v(x_3^3) \geq 7$, so $v(x_i^3) \geq 4$ for some $i > 1$, and $\kappa_i \leq \zeta_{7,1,4}^*(12) < .21$. This implies that $\Sigma < S$. We conclude that $d_2 \leq 8$. From Lemma 14 with $\underline{e} = (2, d_2, 2)$ we have $v(x_3^2) \geq 2 \cdot 14/d_2 > 3$. Consequently, $\kappa_3 \leq \zeta_{7,4}^*(d_3)$. Suppose $d_2 = 8$. Then $\kappa_2 \leq \zeta_7^*(8) < .254$. If $d_2 > 12$, then $\kappa_3 < .19$ by Step 1 and $\Sigma < S$, so $d_2 \leq 12$. By Lemma 14 with $\underline{e} = (2, 2, 12)$, $v(x_2^2) > 2$, so $\kappa_2 \leq \zeta_{7,3}^*(8) < .223$. Since

$\zeta^*_{7,4}(12) < .222$, we conclude that $\kappa_2 + \kappa_3 < .446 < S - \kappa_1$, a contradiction. Therefore $d_2 < 8$. Since $\zeta^*_7(7) < .15$, it is evident that $d_2 \neq 7$.

Suppose $d_2 = 6$. Then $A(\underline{d}) < 2.34$ and $S > .9764$, so $\kappa_2 + \kappa_3 \geq S - \kappa_1 > .4607$. Set $w = v(x_2^2)$. Then w is necessarily even because x_2^2 has order 3. If $w = 2$, then $\kappa_2 \leq \zeta^*_7(6) < .336$. By Lemma 18, $d_3 \geq 14$ and $\kappa_3 \leq \zeta^1(d_3)$. By Step 1 and inspection of the values of $\zeta^1(d)$ for $14 \leq d \leq 30$ we have $\kappa_3 < .08$, so $\Sigma < S$ in this case. If $w = 4$, then $\kappa_2 \leq \zeta^*_{7,4}(6) < .2735$. By Lemma 18, $d_3 \geq 7$ and $\kappa_3 \leq \zeta^2(d_3)$. Observing that $\zeta^2(d) < .1431$ for $7 \leq d \leq 28$, we conclude from Step 1 that $\Sigma < S$ in this case as well. If $w = 6$, then $\kappa_2 \leq \zeta^*_{7,6}(6) < .2579$. We have $d_3 \geq d_2 = 6$, and, by Lemma 18, $\kappa_3 \leq \zeta^3(d_3)$. Since $\zeta^3(d) < .18$ for $6 \leq d \leq 12$ we conclude from Step 1 that $\kappa_3 < .18$, whence, once again, $\Sigma < S$. It follows that $w \geq 8$, so $\kappa_2 \leq \zeta^*_{8,8}(6) < .2527$. From Lemma 14 with $\underline{d} = (2, 6, 2)$ we have $v(x_3^2) \geq 5$, so $\kappa_3 \leq \zeta^*_{7,5}(d_3)$. If $d_3 > 6$ and $d_3 \neq 12$, then $\zeta^*_{7,5}(d_3) < .2$ and $\Sigma < S$. Therefore either $d_3 = 6$ or $d_3 = 12$. Recall that, by Lemma 14 with $\underline{e} = (2, 3, 3)$, $v(x_2^3) + v(x_3^3) \geq 7$. If $v(x_2^3) = 1$, then $\kappa_3 \leq \zeta^*_{7,5,6}(d_3) < .2$, and $\Sigma < S$. Therefore $v(x_2^3) \geq 2$, so $\kappa_2 < \zeta^*_{8,8,2}(6) < .211$. If $d_3 = 6 = d_2$, then we may assume that $\kappa_3 \leq \kappa_2$, whence $\kappa_2 + \kappa_3 < .43$. If $d_3 = 12$, then $\kappa_3 \leq \zeta^*_{7,5}(12) < .217$ and $\kappa_2 + \kappa_3 < .43$. In either case, $\Sigma < S$. Therefore $d_2 \neq 6$.

Suppose $d_2 = 5$. Then $\kappa_2 < .2063$, so $\kappa_3 \geq S - \kappa_1 - \kappa_2 > .25$. We have $\kappa_3 < \zeta^*_{7,6}(d_3)$ by Lemma 14 with $\underline{e} = (2, 5, 2)$. It follows from Step 1 and inspection that $d_3 = 6$. From Lemma 14 with $\underline{e} = (2, 5, 3)$ we have $v(x_3^3) \geq 2$, so $\kappa_3 < \zeta^*_{7,6,2}(d_3) < .22$, a contradiction.

Step 7 *If $d_2 = 4$, then $n = 16$, $\underline{d} = (2, 4, 5)$, $v(x_1) = 4$, $v(x_2) = 12$, and $v(x_3) = 16$.*

Suppose $d_2 = 4$. Then $A < 2.25$ and $S > .9773$. Also, $\kappa_2 \leq \zeta^*_7(4) < .379$.

Assume that $v(x_1) = 4$. then $\kappa_1 \leq \zeta_4(2) < .532$. By Lemma 16, $d_3 > 3$ and $\kappa_2 \leq \zeta^4(d_3)$, so $\kappa_2 < .255$ by inspection and Step 1. From Lemma 14 with $\underline{e} = (1, 4, 4)$ we have $v(x_3^4) \geq 2n - 4 \cdot 4 \geq 12$, so $v(x_3) \geq 12$ and $v(x_3^2) \geq 12$ as well. By Lemma 21 we have $v(x_3^3) \geq 4$. Therefore $\kappa_3 \leq \zeta^*_{12,12,4,12}(d_3)$. If $d_3 > 5$ then $\kappa_3 < .178$ by Step 1 and inspection. Therefore $d_3 = 5$. We have $n \leq d_2 v(x_1) \leq 16$ and $v(x_i) \geq n - 4 \geq 10$, $i = 2, 3$. Since 2 has multiplicative order 4 modulo 5, we also have $4 | v(x_3)$, so $v(x_3) = 12$ or 16. If $v(x_3) = 12$, then $v(x_2) \geq 2n - v(x_1) - v(x_3) = 16$. However, $v(x_2) \leq 3n/4$ because x_2 is an element of order 4 acting in characteristic 2. These inequalities are not compatible with the condition $n \leq 16$. We conclude that $v(x_3) = 16$, $n = 16$, and $v(x_2) = 12$.

We may therefore assume that $v(x_1) > 4$. Then $\kappa_1 \leq \zeta_5(2) < .5157$. Set $w = v(x_2^2)$. Assume that $w \leq 2$. Then, by Lemma 18, $d_3 \geq 14$, and $\kappa_3 \leq \zeta^1(d_3)$. So $\kappa_3 < .08 < S - \kappa_1 - \kappa_2$. Therefore $w > 2$. If $w = 3$ or 4, then $\kappa_2 \leq \zeta^*_{7,3}(4) < .2852$, $d_3 \geq 7$, and $\kappa_3 \leq \zeta^2(d_3)$, so $\kappa_3 < .144$ by inspection and Step 1. Once again, $\Sigma < S$. If $w = 5$ or 6, then $\kappa_2 \leq \zeta^*_{7,5}(4) < .2618$, and $\kappa_3 \leq \zeta^3(d_3)$. If $d_3 \geq 6$, then $\kappa_3 < .19$ and $\Sigma < S$, so $d_3 = 5$. By Lemma 18, $w = 6$. Thus, $\kappa_2 \leq \zeta^*_{7,6}(4) < .2579$ and $\kappa_3 \leq \zeta^3(5) < .2004$, so $\Sigma < S$. We conclude that

$w = v(x_2^2) \geq 7$, so $\kappa_2 \leq \zeta^*_{7,7}(4) < .2559$. By Lemma 21, $\kappa_3 \leq \zeta^*_{7,7,4}(d_3)$. If $d_3 > 5$, then $\kappa_3 < .2$ by Step 1 and inspection, so $\Sigma < S$. If $d_3 = 5$, then $S = .9793$, and $\kappa_3 \leq .2063$, so once again $\Sigma < S$. This completes the argument that $d_3 \neq 4$.

Step 8 *If* $d_2 = 3$ *then* $\underline{d} = (2, 3, 7)$.

It suffices to assume that $d_2 = 3$ and $d_3 > 7$. We have $A(\underline{d}) < 2.17$ and $S > .9781$. Also, $v(x_2)$ is even because x_2 is an element of order 3 acting over $\mathbf{F}_2$. In particular, $v(x_2) \geq 8$ and $\kappa_2 < .33595$. We have $\kappa_3 \leq \zeta^*_{10,10,7,5,3}(d_3)$ by Lemma 20. By inspection, $\kappa_3 \leq .132$. If $v(x_1) \geq 6$, then $\kappa_1 < .50782$ and $\Sigma < S$, so $v(x_1) = 5$ by Lemma 19. We have $\kappa_1 \leq \zeta_5(2) < .5157$.

It follows that $v(x_2) \geq n-5 \geq 9$, whence $v(x_2) \geq 10$, and $\kappa_2 \leq \zeta_{10}(3) < .334$. We have $\kappa_1 + \kappa_2 < .8497$.

By inspection, if $d > 7$ and $d \neq 8$ or 12, then $\zeta^*_{10,10,7,5,3}(d) < .114$. It follows that $d_3 = 8$ or 12. If $d_3 = 8$, then $A(\underline{d}) < 2.05$, so $S > .9793$ and $\Sigma \leq \zeta_5(2) + \zeta_{10}(3) + \zeta^*_{10,10,7,5}(8) < .9793 < S$. We conclude that $d_3 = 12$, whence $A(\underline{d}) < 2.09$ and $S > .9789$. Since x_3^4 has order 3, $v(x_3^4)$ must be even, and $v(x_3^4) \geq 6$. If $v(x_2) = 10$, then $v(x_3) \geq 2n - v(x_1) - v(x_2) \geq 13$, so $\kappa_3 \leq \zeta^*_{13,10,7,6}(12)$, and $\sum \kappa_i \leq \zeta_5(2) + \zeta_{10}(3) + \zeta^*_{13,10,7,6}(12)$. If $v(x_2) > 10$, then $v(x_2) \geq 12$ and $\sum \kappa_i \leq \zeta_5(2) + \zeta_{10}(3) + \zeta^*_{10,10,7,6}(12)$. In either case, $\Sigma < S$, a contradiction.

Step 9 *If* $\underline{d} = (2, 3, 7)$ *then* $n \leq 21$.

Otherwise $v(x_1) \geq 8$, $v(x_2) \geq 11$, and $v(x_3) \geq 11$, so $\sum \kappa_i \leq \zeta_8(2) + \zeta_{11}(3) + \zeta_{11}(7) < .9795 < S$. □

3 Proof of Theorem 2

Retaining the notation of 2.1, assume that Ω is a primitive point action for G with $|\Omega| \geq 10^4$ and that $x \in G$.

3.1 Linear and Symplectic Groups

Proposition 33 *If* Ω *consists of all points in the* L *action or* Sp *action, then* $f(x) - q^{-v(x)} < 1/100$.

Proof. We have $N = (q^n - 1)/(q - 1)$, so $q^{n-1} < N < 2q^{n-1} \leq q^n$.

Suppose x is a linear transformation. Then the fixed points of x are contained in the union of its eigenspaces, the largest of which has dimension $n - v$. We claim $f(x) - q^{-v(x)} < q^{-n/2} < 1/100$. It suffices to establish the first inequality.

If $v \leq n/2$, then the fixed points of x lying outside the largest eigenspace are contained in a space of dimension $n - v$. This implies that $f(x) \leq \dfrac{q^{n-v} - 1}{q - 1} +$

$\dfrac{q^v-1}{q-1}$, so

$$\begin{aligned}\frac{F(x)}{N} - q^{-v} &= \frac{q^{n-v}-1}{q^n-1} + \frac{q^v-1}{q^n-1} - q^{-v} \\ &< q^{-(n-v)} \le q^{-n/2}.\end{aligned}$$

If $v = \frac{n+1}{2}$, then the fixed points of x lying outside the largest eigenspace are contained in the union of two nontrivial spaces having total dimension $n - v = (n+1)/2$. For fixed m, the largest value of $q^a + q^{m-a}$ for a in $\{1, 2, \ldots, m-1\}$ is $q^{m-1} + q$. Therefore $F(x) \le \dfrac{q^{n-v}-1}{q^n-1} + \dfrac{q^{(n-1)/2}-1}{q-1} + 1$, so

$$\frac{F(x)}{N} - q^{-v} < \frac{q^{(n-1)/2}-1}{q^n-1} + \frac{q-1}{q^n-1} < q^{-n/2}.$$

If $v \ge n/2 + 1$, then $F(x) \le (q-1)\dfrac{q^{n/2-1}-1}{q-1}$ and

$$F(x)/N \le (q-1)\left(\frac{q^{n/2-1}-1}{q^n-1}\right) < q^{-n/2}.$$

This completes the analysis for x a linear transformation.

Now suppose x is not a linear transformation. Then x induces a field automorphism of order, say, d. Then $F(x) \le \dfrac{q^{n/d}-1}{q^{1/d}-1}$, so $f(x) > .01$ implies that

$$q^{n(d-1)/d} < \frac{q^n-1}{q^{n/d}-1} < 100\frac{q-1}{q^{1/d}-1} = 100q^{(d-1)/d}\left(\frac{1-q^{-1}}{1-q^{-1/d}}\right).$$

Since $q^{-1/d} \le 1/2$, we have $q^{n(d-1)/d} < 200q^{(d-1)/d}$. It follows that $q^{(n-1)(d-1)/d} < 200$.

By the first line of this argument, $2q^{n-1} > N > 10000$. Therefore $q^{n-1} > 5000 > 200^{3/2}$, whence $\frac{d-1}{d}\frac{3}{2} < 1$, so $d = 2$.

If x is not a standard field automorphism, then $F(x) \le \dfrac{q^{n/2-1}-1}{q^{1/2}-1} + 1$, so

$$\begin{aligned}.01 < f(x) &\le (q^{1/2}+1)\left(\tfrac{q^{n/2-1}-1}{q^n-1}\right) + \tfrac{1}{N} \\ &< \tfrac{3}{2}q^{1/2} \cdot q^{-(n/2+1)} + .0001\end{aligned}$$

This implies that $q^{n+1} < \left(\dfrac{1}{.0066}\right)^2 < 160^2$.

On the other hand, we have $F(x) > .01N > 100$, so $\dfrac{q^{n/2-1}-1}{q^{1/2}-1} + 1 > 100$. It follows from this that $q^{n-2} > 99^2$, whence $q^3 < (160/99)^2$, which is impossible. Therefore x must be a standard field automorphism.

We have $f(x) = \frac{q^{1/2}+1}{q^{n/2}+1}$ and $v_q(x) = n/2$. If $f(x) - q^{-v_q(x)} > .01$, then $q^{-(n-1)/2} > .01$, whence $q^{n-1} < 10000$. On the other hand, $q^{n-1} \cdot \dfrac{q}{q-1} > \dfrac{q^n - 1}{q-1} = N > 10000$. That is,

$$q^{n-1} < 10000 < \frac{q^n}{q-1}.$$

Since $n > 2$, the first inequality implies that $q < 100$. Since q is both a perfect square and a prime power, it is an easy inspection that these two inequalities cannot both hold. □

Proposition 34 *If Ω consists of hyperplanes of type δ in the Sp action, then $f(x) < q^{-v(x)} + 1/100$.*

Proof. We have $N = \frac{1}{2}(q^n + \delta q^{n/2})$. Since q is a power of 2 and $2^{14}+2^7 < 20000$, we have $q^n \geq 2^{16}$.

If x is a field automorphism, then $F(x) \leq q^{n/2}$ in either action, so $f(x) \leq 2(q^{n/2} - 1)^{-1} < .01$.

If x is in InnDiag, then $F(x) \leq \frac{1}{2}(q^{n-v} + q^{n/2})$, so $F(x) - q^{-v(x)}N < \frac{1}{2}q^{n/2}$, and $f(x) - q^{-v} < .01$, as before. □

3.2 Unitary and Orthogonal Groups

To complete the proof of Theorem 2 we assume that V admits a nondegenerate orthogonal or unitary form, and that the action of G is on the points of type t in V.

For a subspace W of V, let $\pi(W) = \pi_t(W)$ be the number of points of type t in W.

35 *Define α and b as in Table 3.*

1. *$N = \pi(V) \geq \alpha(q^n - bq^{n/2})$.*
2. *Let W be an m-dimensional subspace of V with radical of dimension r. Then*

$$\pi(W) \leq \alpha(q^m + bq^{(m+r)/2}).$$

   In particular,

   (a) *$\pi(W) \leq \alpha q^m(b+1)$ and*

   (b) *$\pi(W) \leq \alpha(q^m + bq^{n/2})$.*

3. *If W is an m-dimensional subspace of V, then $\pi(W) \leq q^m/(q-1)$.*

Table 3: Values of α, b

action	restriction on q	α^{-1}	b
$W, \mathbf{s}$	none	$q^{1/2}(q-1)$	$q^{1/2}$
$W, \mathbf{n}$	none	$q^{1/2}(q^{1/2}+1)$	1
$O^{\cdot}, \mathbf{s}$	none	$q(q-1)$	q
$O^{\cdot}, \mathbf{n}$ or δ	q odd	$2q$	$q^{1/2}$
$O^{\cdot}, \mathbf{n}$	q even	q	1

Proof. Statement 1 and the case $r = 0$ of statement 2 follow from **3.7** of [FM00]. Statement 3 holds because W has $(q^m - 1)/(q-1)$ one-dimensional subspaces. The special cases of statement 2 follow from it because $\dim \operatorname{rad} W \leq \max(\dim W, n - \dim W)$.

Setting W_R to be the radical of W, we have $\pi(W) = |W_R|\pi(W/W_r) + \pi(W_R)$. If $t = \mathbf{n}$ or δ, then $\pi(W_R) = 0$, and the general case of statement 2, follows from the special case $r = 0$. If $t = \mathbf{s}$, then the non-degenerate inequality is stronger, namely, $\pi(W) \leq \alpha\left(q^m + b(q^{m/2} - 1)\right)$, and the general case follows from this because $\pi(W_R) < \alpha b q^r$. □

36 *If* $\dim W = m$*, then* $q^{m-2} < \pi(W) < 2q^{m-1}$.

Proof. The upper bound follows easily, as in the argument for Proposition 33. The lower bound follows from the previous result. □

37 *Let x be a linear element of G with $v(x) = v$.*

1. *If* $v < n/2$*, then* $F(x) \leq \alpha(q^{n-v} + bq^{n/2}) + \alpha q^v(b+1)$.
2. *If* $v = n/2$*, then* $F(x) \leq 2\alpha q^{n/2}(b+1)$.
3. *If* $v = (n+1)/2$*, then* $F(x) \leq 2\alpha q^{(n-1)/2}(b+1) + 1$.
4. *If* $v > (n+1)/2$*, then* $F(x) \leq q^{n/2}$.

Proof. The first three statements follow immediately from Lemma 35. The last statement holds because x has at most $q-1$ eigenspaces each of which contains fewer than $(q^{n/2-1})/(q-1)$ points. □

38 *Let x be a field automorphism of G of order d modulo InnDiag(G).*

1. *If* $d > 2$*, then* $F(x) < q^{n/3}/(q^{1/3} - 1)$.
2. *If* $d = 2$*, then* $F(x) < q^{n/2}/(q^{1/2} - 1)$.
3. *If* $d = 2$ *and* $v_q > n/2$*, then* $F(x) < q^{(n-1)/2}/(q^{1/2} - 1) + 1$.

Proof. The first two statements come from counting the points in an n-dimensional vector space over $\mathbf{F}_{q^{1/d}}$.

The last statement holds because the fixed points of x are contained in the union of two disjoint proper subspaces of an n-dimensional space over $\mathbf{F}_{q^{1/d}}$. □

39 *If x is in InnDiag(G) and $F(x) - q^{-v}N > N/100$, then $n < 2\log_q(401) + 2\log_q(b)$, where b is listed in Table 3.*

Proof. Set $v = v(x)$.

Suppose first that $v < n/2$. Then

$$\begin{array}{rcl} F(x) - q^{-v}N & \leq & \alpha\left(q^{n-v} + bq^{n/2} + (b+1)q^{v}\right) - \alpha\left(q^{n-v} - bq^{n/2-v}\right) \\ & = & \alpha b q^{n/2}\left(1 + \frac{b+1}{b} q^{-(n/2-v)} + q^{-v}\right). \end{array}$$

Since $(b+1)/b \leq 2$ and $q^{-v} < 1$, we have $F(x) - q^{-v}N < 4\alpha b q^{n/2}$.

Now suppose $v = n/2$. Then

$$\begin{array}{rcl} F(x) - q^{-v}N & \leq & \alpha q^{n/2}(2b+2) - \alpha q^{n/2}(1 - bq^{-n/2}) \\ & = & \alpha q^{n/2}(2b + 1 + bq^{-n/2}) \\ & < & 4\alpha b q^{n/2} \end{array}$$

because $1 \leq b$ and $q^{-n/2} < 1$.

Suppose $v = (n+1)/2$. Then

$$\begin{array}{rcl} F(x) - q^{-v}N & \leq & \alpha q^{(n-1)/2}(2b+2) - \alpha q^{(n-1)/2}(1 - bq^{-n/2}) + 1 \\ & = & \alpha q^{(n-1)/2}(2b + 1 + bq^{-n/2}) + 1 \\ & < & 4\alpha b q^{(n-1)/2} + 1. \end{array}$$

Finally, suppose $v > (n+1)/2$. Then

$$F(x) < (q-1) \cdot q^{n-v}/(q-1) = q^{n-v} \leq q^{n/2}q^{-1} < 4\alpha b q^{n/2}$$

because $4\alpha b > q^{-1}$. Note that in case $U, \mathbf{n}$, $\alpha b = \dfrac{1}{q(1 + q^{-1/2})} > \dfrac{1}{q} \cdot \dfrac{1}{2}$.

In all cases, $F(x) - q^{-v}N < 4\alpha q^{n}\gamma$ where $\gamma = bq^{-n/2}$. Therefore $.01 < (F(x) - q^{-v}N)/N < 4\gamma/(1-\gamma)$. This implies that $\gamma > 1/401$, whence $q^{n/2} < 401b$, and $n < 2\log_q(401) + 2\log_q(b)$. □

The obvious computation establishes the following result.

40 *If x is in InnDiag(G) and $f(x) - q^{-v} > 1/100$, then n is at most the value listed in Table 4.*

Proposition 41 *If x is in InnDiag(G), then either $f(x) \leq q^{-v(x)} + 1/100$ or x has two complementary eigenspaces of dimension $n/2$ each consisting of totally singular points. In the latter case, one of the following is true.*

1. *In the $(U, \mathbf{s})$ action, $n = 8$ and $q = 4$. [$N = 10965$, $v_q = 4$, $F = 170$]*

2. *In the* $(O^+, \mathbf{s})$ *action,* (n, q) *is one of the following*

   (a) $(6, 11)$ *[*$N = 16226$, $v_q = 3$, $F = 266$*]*

   (b) $(6, 13)$ *[*$N = 31110$, $v_q = 3$, $F = 340$*]*

   (c) $(8, 5)$ *[*$N = 19656$, $v_q = 4$, $F = 312$*]*

Proof. It follows from an examination of the residual cases that x must have two complementary eigenspaces of dimension $n/2$ when $f(x) - q^{-v} > .01$. In the unitary case, $N = (q^{n/2} - 1)(q^{(n-1)/2} + 1)/(q - 1)$ and $F(x) = 2(q^{n/2} - 1)/(q - 1)$. In the orthogonal case, the action must be of type $(O^+, \mathbf{s})$, $N = (q^{n/2} - 1)(q^{n/2-1} + 1)/(q - 1)$ and $F(x) = 2(q^{n/2} - 1)/(q - 1)$. The result follows from a routine computation. Note that $q > 2$ because x has two distinct eigenvalues. □

Lemma 42 *Assume that* x *has order* d *modulo InnDiag and* $f(x) > .01$*. Then* $d < 4$*. If* $d = 2$*, then* n *is bounded by the valued listed in Table 5. If* $d = 3$*, then* n *is bounded by the valued listed in Table 6.*

Proof. The fixed points of x are contained in a space of dimension at most n over the subfield. By Lemma 36, $F(x) < 2q^{(n-1)/2}$ and $N > q^{n-2}$. Therefore $F/N < 2q^{-(n-3)/2}$. It follows that $q^{(n-3)/2} < 200$, so $n < 2\log_q(200) + 3$. If $d = 2$, then n is bounded as in Table 5.

If $d > 2$, then $F(x) < 2q^{(n-1)/d}$. Since $F(x)/N > .01$ and $N > 10000$, we have $F(x)^2 > N$. Therefore $4q^{(2n-2)/d} > q^{n-2}$. If $d = 3$, then $q^{(n-4)/3} < 4$, so $n < 3\log_q(4) + 4$, and one of the possibilities listed in Table 6 must hold. If $d > 3$, then $q^{n-2} < 4q^{(2n-2)/d} \leq 4q^{(n-1)/2}$, so $q^{n-3} < 16$. Since $q \geq 2^d \geq 16$ and $n \geq 4$, this is impossible. □

The following propositions follow from routine computations in the cases listed in Tables 5 and 6.

Proposition 43 *If* G *is unitary and* x *is not in InnDiag(G), then either* $f(x) \leq p^{-v_p(x)} + 1/100$ *or* x *fixes a space of dimension* n *over* $\mathbf{F}_{q^{1/2}}$ *consisting of singular points. In the latter case,* (n_q, q) *is one of the following:*

1. $(4, 49)$ *[*$N = 17200$, $v_q = 2$, $F = 400.$*]*
2. $(4, 64)$ *[*$N = 33345$, $v_q = 2$, $F = 585.$*]*
3. $(4, 81)$ *[*$N = 59860$, $v_q = 2$, $F = 820.$*]*
4. $(6, 9)$ *[*$N = 22204$, $v_q = 3$, $F = 364.$*]*
5. $(8, 4)$ *[*$N = 10965$, $v_q = 4$, $F = 255.$*]*

Proposition 44 *If the fixed points of* x *are the points of an orthogonal space of dimension* n *over a subfield of order* $q^{1/2}$ *and* $f(x) > p^{-v_p(x)} + 1/100$ *then one of the following is true.*

1. $n_q = 6$ *and* $q = 16$ *[*$N = 70161$, $v_q = 3$, $F = 1385$*]*
2. $n_q = 10$ *and* $q = 4$ *[*$N = 87637$, $v_q = 5$, $F = 1023$*]*

Table 4: Maximum values of n

q	$U,\mathbf{s}$	$U,\mathbf{n}$	$O,\mathbf{s}$	$O,\mathbf{n}$
2			18	16
3			12	11
4	9	8	10	8
5			9	8
7			8	7
8			6	4
9	6	5	7	6
11			6	5
13			6	5
16	5	4	6	4
17			6	5
19			6	5
23			5	4
25	4	3	5	4

Table 5: Maximum values of n, field automorphisms of order 2

q	n
4	10
9	7
16	6
25	6
> 25	≤ 5

Table 6: Maximum values of n, field automorphisms of order 3

q	n
8	5
27	5
≥ 64	≤ 4

4 Proof of Theorem 3

We assume here that $\underline{x}$ and V satisfy one of the conditions listed in Table 2. Suppose Ω is a primitive G-set of [projective] points in V with $|\Omega| \geq 10000$.

That is, one of the following is true where $n_p = \dim_{\mathbf{F}_p}(V)$.

1. $\underline{x}$ has signature $(2, 3, 7)$ and one of the following holds.
   (a) $p = 11$ and $n_p = 5$ or 6.
   (b) $p = 7$ and $n_p = 6$.
   (c) $p = 5$ and $n_p = 7, 8,$ or 9.
   (d) $p = 3$ and $n_p = 12$.
   (e) $p = 2$ and $14 \leq n_p \leq 21$.
2. $\underline{x}$ has signature $(2, 3, 8)$, $p = 3$, and $n_p = 10$.
3. $\underline{x}$ has signature $(2, 4, 5)$, $p = 2$, and $n_p = 16$ Furthermore $v_p(x_1) = 4$, $v_p(x_2) = 12$, and $v_p(x_3) = 16$.

Then V is an n_q-dimensional $\mathbf{F}_q$-module where $q^{n_q} = p^n$. and n_q and q satisfy the conditions listed for point actions.

45 *The number $CP(X, n, q, t)$ of t-points in a classical n-space of type X over $GF(q)$ is given in Table 7.*

Proof. See [FM00]. □

We calculate a lower bound for $g(\underline{x})$ in each of the cases using the following lemma.

Lemma 46 *1. If $\underline{d} = (2, 3, 7)$, then $v(x_1) \geq n/3$, $v(x_2) \geq n/2$, and $v(x_3) \geq n/2$.*

2. If $\underline{d} = (2, 3, 8)$ then $v(x_1) \geq n/3$, $v(x_3) \geq n/2$, $v(x_2) \geq n/2$, $v(x_2^2) \geq n/2$, and $v(x_2^4) \geq n/5$.

3. If $\underline{d} = (2, 4, 5)$, then $v(x_1) \geq n/4$, $v(x_2) \geq n/2$, $v(x_2^2) \geq n/4$, and $v(x_3) \geq n/2$.

4. The number of t-points in an n-space with radical of dimension r of type X over $\mathbf{F}_q$ is $(q^r - 1)/(q - 1) + q^r CP(X, n - r, q, t)$ for singular points and $q^r CP(X, n - r, q, t)$ for non-singular points.

5. Assume that q is even. Let $G = O(2m+1, q) \cong Sp(2m, q)$ act on the $2m+1$-dimensional orthogonal space V, where V has a 1-dimensional radical R. If x is a linear transformation in G then x fixes at most $q^m(q^{m-v(x)}+1)/2$ complements to R of each type.

6. If W is a space of codimension v in the non-degenerate space V then $\dim \operatorname{rad} W \leq v$.

7. *Let* $\mathrm{Fix}_2(x)$ *be the number of fixed points of* x *lying outside its principal eigenspace. Set* $v = v(x)$*. Then*

   (a) *If* $(o(x), q-1) = 1$ *then* $\mathrm{Fix}_2(x) = 0$*.*

   (b) $\mathrm{Fix}_2(x) = 0$ *in case of type* S*.*

   (c) *If* $2v \leq n$ *then* $\mathrm{Fix}_2(x)$ *is bounded by the number of type* t *points in some* v*-dimensional space.*

   (d) *If* $(o(x), q-1) = d_0$ *and every* $n-v$*-dimensional space contains at most* M *points then* $\mathrm{Fix}_2(x) \leq (d_0 - 1)M$*.*

8. *If* $\mathrm{Fix}(x^j) \leq F_j$ *for all positive powers of* x*, then*

$$\mathrm{Ind}(x) \geq \frac{d-1}{d}N - \frac{1}{d}\left(\sum_{k|d, k<d} \phi(\frac{d}{k})F_k\right)$$

9. *If* $\mathrm{Ind}x_i \geq H_i$ *for all* i *then* $g(\underline{x}) \geq \frac{1}{2}\sum H_i - N + 1$*.*

Proof. The first three statements follow from Lemma 14.

The fourth statement is a straightforward count of points in $R \oplus W$ where R is totally singular of dimension r and W is non-degenerate.

Statement 5 still requires proof.

The next statement is clear because $\mathrm{rad}\, W \subseteq W^{\perp}$.

To prove 7, note that the principal eigenspace of x has dimension $n - v$, and every fixed point of x lying outside the principal eigenspace must lie in an eigenspace of dimension at most $n - v$.

All eigenvalues of x must have order dividing both $o(x)$ and $q - 1$, so there are at most $d_0 = (o(x), q-1)$ eigenvalues in toto. Statements 7a and 7d now follow immediately.

In type S only the eigenvalue $\lambda = 1$ corresponds to fixed points, so statement 7b holds.

The total dimension of all secondary eigenspaces is at most v, and all secondary fixed points of x lie in the direct sum of such subspaces. Statement 7c follows.

Statements 8 and 9 follow easily from the Cauchy-Frobenius and Riemmann-Hurwitz Formulas, respectively. □

In all cases except $L_{14}(2)$ acting on the points in its natural module and $U_8(2^2)$ acting on singular points the lower bound is larger than 2.

However, in those cases, we use the following additional facts:

1. If x has order 7 and acts as a linear transformation over $\mathbf{F}_2$ or $\mathbf{F}_4$ then x has a single eigenspace and $3|v(x)$.

2. If x has order 3 and acts as a linear transformation over $\mathbf{F}_2$ then x has a single eigenspace and $2|v(x)$.

Table 7: Number of t-points in classical n-space of type X over $\mathbf{F}_q$.

X	condition	t	$CP(X,n,q,t)$
L			$\dfrac{q^n-1}{q-1}$
O^ϵ	$n=2m$	singular	$\dfrac{(q^m-\epsilon 1)(q^{m-1}+\epsilon 1)}{q-1}$
O^ϵ	$n=2m$	δ	$\dfrac{(2,q)}{2}(q^m-\epsilon 1)q^{m-1}$
O^ϵ	$n=2m+1$	singular	$\dfrac{q^{2m}-1}{q-1}$
O^ϵ	$n=2m+1$	δ	$\dfrac{q^m(q^m-\epsilon\delta)}{2}$
U	$q=q_0^2$	singular	$\dfrac{(q_0^n-(-1)^n)(q_0^{n-1}+(-1)^n)}{q-1}$
U	$q=q_0^2$	non-singular	$\dfrac{(q_0^n-(-1)^n)q_0^{n-1}}{q_0+1}$
S	$n=2m, q$ even	ϵ hyperplane	$\dfrac{q^m(q^m+\epsilon 1)}{2}$

Using these additional facts, it is easy to establish the following lemma and complete the proof of Theorem 3.

Lemma 47 *If* $\underline{d} = (2,3,7)$ *and the action is either* $L_{14}(2)$ *on points or* $U_8(2^2)$ *on singular points, then the genus is at least* 20*.*

Proof. Suppose $G = L_{14}(2)$. Then x_i has only one eigenspace for $i = 1,2,3$, $2|v(x_2)$, and $3|v(x_3)$. It follows that $v_1 \geq 5$, $v_2 \geq 8$, and $v_3 \geq 9$. Furthermore, $\mathrm{Ind}(x_1) \geq \frac{1}{2}(2^{14} - 2^9) = 7936$, $\mathrm{Ind}(x_2) \geq \frac{2}{3}(2^{14} - 2^6) = 10880$, and $\mathrm{Ind}(x_3) \geq \frac{6}{7}(2^{14} - 2^5) = 14016$. This implies that $g(\underline{x}) > 30$.

Suppose $G = U_8(2^2)$. Then x_1 and x_3 have at most one eigenspace, and $3|v(x_3)$. We have $v_1 \geq 3$, $v_2 \geq 4$, and $v_3 = 6$, and it follows that $g(\underline{x}) > 2$. □

Note: GAP4 code confirming the calculations here is contained in Appendix B.

References

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[FGM02b] Daniel Frohardt, Robert Guralnick, and Kay Magaard. Incidence matrices, permutation characters, and the minimal genus of a permutation group. *Journal of Combinatorial Theory, Series A*, 98:87–105, 2002.

[FM00] Daniel Frohardt and Kay Magaard. Grassmannian fixed point ratios. *Geometriae Dedicata*, 82:21–104, 2000.

[FM01] Daniel Frohardt and Kay Magaard. Composition factors of monodromy groups. *Annals of Mathematics*, 154:327–345, 2001.

[GT90] Robert Guralnick and John Thompson. Finite groups of genus zero. *Journal of Algebra*, 131:303–341, 1990.

[Sco77] Leonard Scott. Matrices and cohomology. *Annals of Mathematics*, pages 473–492, 1977.

A GAP computations for the proof of Theorem 1

Table 8: Conversion of TEXto gap code

TEX	gap code
$\mu_*(d,p)$	`mustar(d,p)`
$\zeta(d,p)$	`zeta(d,p)`
$\zeta_{s1,s2,\ldots,sk}(d,p)$	`zetal([s1,s2,...,sk],d,p)`
$\zeta_s(d,p)$	`zetal1(s,d,p)`
$\zeta^s(d,p)$	`zetau(s,d,p)`
$\zeta^*(d,p)$	`zetastar(d,p)`
$\zeta^*_{s1,s2,\ldots,sk}(d,p)$	`zetastarsub([s1,s2,...,sk],d,p)`

File mustar

```
\input ~/gap/scripts/mustar

#  file mustar
#  gap code setting up functions used in
#  verifying calculations in Primitive Monodromy
#  Groups of Genus at Most Two
#

Divisors := function(n)
     return Filtered([1..n], d -> n/d in Integers); end;

zeta := function(d,p)
#  zeta(d,p)  = [in TeX] \zeta(d,p)
     local sum;
     sum := Sum( List( Filtered( Divisors(d), m -> m > 1), k -> Phi(k)/p ) );
     return (1+sum)/d; end;

zetal := function(list, d, p)
#  zetal([s1,s2,...,sk]d,p)  = [in TeX] \zeta_{s1,s2,\ldots,sk}(d,p)
     local l, sum1, sum2;
     l := Length(list);
     sum1 := Sum( List( Filtered([1..l], i -> i in Divisors(d) ),
             j -> Phi(d/j) *p^(-list[j]) ) );
     sum2 := Sum( List( Filtered( Divisors(d), m -> 1 < m and m < d/l ),
             j -> Phi(j) * p^(-1) ) ) ;
     return (1 + sum1 + sum2)/d ; end;

zetal1 := function(k, d, p)
#  zetal1(k,d,p)  = [in TeX] \zeta_{k}(d,p)
     return zetal([k],d,p); end;
```

```
zetau := function(t, d, p, n)
#  zetalu(k,d,p,n)  = [in TeX] \zeta_{k}(d)
#  the p is implicit and the n should be the minimum
#  dimension assoicated with p
     local sum;
     sum := Sum( List( Filtered( Divisors(d), i -> i < d ),
            m -> Phi(d/m) * p^(-Maximum(1,n-m*t) ) ) );
            return (1 + sum)/d; end;

Relativelyprime := function(a,b)
#  Relativelyprime(a,b) is true iff a and b are relatively prime
     return Gcd(a,b) = 1; end;

ppart := function(d,p)
#  ppart(d,p) is the largest power of p dividing d
     if Relativelyprime(d,p) then return 1; fi;
     return p*ppart(d/p,p); end;

modorder   := function(x,y)
#  multiplcative order of x mod y
if not Relativelyprime(x,y) then return "ERROR"; fi;
return Minimum(Filtered([1..y-1], k -> EuclideanRemainder(x^k,y) = 1 ) );
end;

mustar := function(d,p)
#  mustar(d,p)  = [in TeX] \mu_\ast(d,p)
local dp, dq, x, others;
if d = 1 then return 0; fi;
dp := ppart(d,p);
dq := d/dp;
if dq = 1 then return d/p; fi;
if dp > 1 then return mustar(dp,p) + mustar(dq,p); fi;
x := modorder(p,d);
others := List(Filtered(Divisors(d), a -> 1 < a and a < d and
               Relativelyprime(a,d/a) ),
               b -> mustar(b,p) + mustar(d/b,p) );
return Minimum( Concatenation(others,[x]) ); end;

zetastar := function(d,p)
#  zetastar(d,p)  = [in TeX] \zeta^\ast(d,p)
   local sum;
   sum := Sum( List( Filtered( Divisors(d), k -> k > 1 ),
                    m -> Phi(m) * p^(-mustar(m,p)) ) );
   return ( 1 + sum ) / d; end;

zetastarsub := function(list,d,p)
```

```
#  zetastar([s1,s2,...,sk],d,p)  = [in TeX] \zeta^\ast_{s1,s2,\ldots,sk}(d,p)
   local alpha, sum;
   alpha := function(i)
       if i > Length(list) then return mustar(d/i,p); fi;
       return Maximum(list[i],mustar(d/i,p)); end;
   sum := Sum(List( Filtered(Divisors(d), k -> k > 1),
                m -> Phi(m) *p^(- alpha(d/m)) ) );
   return ( 1 + sum ) / d ; end;

A := function(tuple)
#  the function A defined in Primitive Monodromy Groups ...
   return Sum( List( tuple, d -> (d-1)/d ) ); end;

C := function(i,tuple)
#  C(i,[e1,e2,...,ek]) = [in TeX] \uC_i(\ue)
     return 2/(tuple[i]*(2-A(tuple)) ); end;

S := function(tuple)
# The lower bound for \sum \kappa_i in Primitive Mondodromy Groups
      return Length(tuple) - 20002/10000 - A(tuple)/100; end;
```

File Schecks

```
\input ~/gap/scripts/Schecks

#  gap script to check inequalities involving S
#
checklistr := [
Maximum( Filtered( [3..8] , r -> r < 40004 * 11 / ( ( 98 * 11 - 100 ) * 100 ) ) ) = 4 ,
Maximum( Filtered( [3..8] , r -> r < 40004 * 7 / ( ( 98 * 7 - 100 ) * 100 ) ) ) = 4 ,
Maximum( Filtered( [3..8] , r -> r < 40004 * 5 / ( ( 98 * 5 - 100 ) * 100 ) ) ) = 5 ,
Maximum( Filtered( [3..8] , r -> r < 40004 * 3 / ( ( 98 * 3 - 100 ) * 100 ) ) ) = 6 ,
Maximum( Filtered( [3..8] , r -> r < 40004 * 2 / ( ( 98 * 2 - 100 ) * 100 ) ) ) = 8
];
checklistS := [
1 -   20002 * 7  / ( ( 99 * 7 - 100 ) * 100 ) * (1/100) - 2/10000  > 9761/10000 ,
1 -   20002 * 5  / ( ( 99 * 5 - 100 ) * 100 ) * (1/100) - 2/10000  > 9744/10000 ,
1 -   20002 * 3  / ( ( 99 * 3 - 100 ) * 100 ) * (1/100) - 2/10000  > 9693/10000 ,
1 -   20002 * 2  / ( ( 99 * 2 - 100 ) * 100 ) * (1/100) - 2/10000  > 9589/10000 ,
S([3,4,4]) >= 9781/10000 ,
#   Note that A([3,4,4]) = 1/2 + 2/3 + 1 is an upper bound for A([2,3,d])
S([2,3,7]) >= 9795/10000
];
```

File large

```
\input ~/gap/scripts/large


#  gap script to check the inequalities in the p>7
#  analysis in "Primitive Monodromy Groups of Genus
#  at most Two"
p := 23;
checklist23 := [
zetal1(2,2,29) + zetal1(2,3,29) + zetal1(2,7,29) < 9787/10000 ,
zetal1(2,2,p) + zetal1(2,3,p) + zetal1(2,8,p) < 9778/10000 ,
zetal1(2,2,29) + zetal1(2,4,29) + zetal1(2,5,29) < 9787/10000 ,
zetal1(2,2,p) + zetal1(2,4,p) + zetal1(2,6,p) < 9787/10000 ,
zetal1(2,2,p) + zetal1(2,4,p) + zetal1(2,7,p) < 9787/10000 ,
zetal1(2,2,p) + zetal1(2,5,p) + zetal1(2,5,p) < 9787/10000 ,
zetal1(2,3,p) + zetal1(2,3,p) + zetal1(2,4,p) < 9787/10000 ,
zetal1(2,2,p) + zetal1(2,4,p) + zetal1(4,5,p) < 9628/10000 ,
zeta(2,p) + zetal([3,2],4,p) + zetal1(4,5,p) < 974/1000 ,
zetal1(2,2,p) + zetal1(2,3,p) + zetal1(3,7,p) < 9786/10000
];

p := 11;
checklist11 := [
2*zeta(2,p) + zetal1(2,2,p) + zetal1(2,3,p) < 195/100 ,
zeta(2,p) + 2*zetal1(2,2,p) + zetal1(2,3,p) < 195/100,
zeta(4,p) < 9775/30000 ,
zeta(5,p) < (9775/10000 - zeta(3,p))/2 ,
Maximum( List([11,13,17,19],
      pp -> zetal1(2,4,pp) - zetal1(2,3,pp) ) ) < 0 ,
Maximum( List([11,13,17,19],
      pp -> zeta(5,pp) - zetal1(2,4,pp) ) ) < 0 ,
2*zetal1(2,3,p) + zetal1(2,4,p) < 9775/10000 ,
zeta(8,p) < (9775/10000 - zeta(2,p) )/2 ,
zetal1(2,5,p) - zetal1(2,6,p) < 0 ,
zetal1(2,7,p) - zetal1(2,6,p) < 0 ,
zeta(8,p) - zetal1(2,6,p) < 0 ,
zeta(2,p) + 2*zetal1(2,6,p) < 9775/10000 ,
zetal1(2,2,p) + zetal1(3,4,p) < 9775/10000 - 2002/10000 ,
Maximum( List( [7..20], d -> zetal1(3,d,p) ) ) < 2002/10000 ,
zetal1(2,2,p) + zetal1(3,4,p) + zetal1(3,5,p) < 9781/10000 ,
zetal1(2,2,p) + zetal([3,2],4,p) + zetal1(3,6,p) < 97/100 ,
zetal1(2,2,p) + zetal1(3,4,p) + zetal([3,2],6,p) < 97/100 ,
zetal1(2,2,p) + zetal1(3,3,p) < 8426/10000 ,
zeta(21,p) < 135/1000 ,
zetal1(3,9,p) < 137/1000 ,
```

```
Maximum( List( [11..20], d -> zetal1(3,d,p) ) ) < 137/1000 ,
zetal1(2,2,p) + zetal1(3,3,p) +
     Maximum( List( [8,10], d -> zetal([3,3],d,p) ) ) < 9781/10000 ,
zetal1(2,2,17) + zetal1(3,3,17) + zetal1(3,7,17) < 9795/10000 ,
zetal1(2,2,13) + zetal1(3,3,13) + zetal1(4,7,13) < 9795/10000 ,
zetal1(3,2,p) + zetal1(4,3,p) + zetal1(3,7,p) < 9795/10000
];
```

File seven

```
\input ~/gap/scripts/seven


#  gap script to check the inequalities in the p=7
#  analysis in "Primitive Monodromy Groups of Genus
#  at most Two"
p := 7;
n := 6;
checklist7 := [
2*zeta(2,p) + 2*zetau(2,3,p,n)  < 19/10 ,
zeta(2,p) + 2*zetal1(2,2,p) + zetal1(2,3,p) < 1976/1000 ,
3* zetal1(2,2,p) + zeta(3,p) < 1976/1000 ,
Maximum( List( [6..24], d -> zetau(1,d,p,n) ) ) <  168/1000 ,
zeta(2,p) < 572/1000 ,
Maximum(List([4..7], d -> zetal1(2,d,p))) < 3/10 ,
3 * Maximum( List([4..12], d -> zetal1(2,d,p) ) ) < 9761/10000 ,
2 * zetal1(2,3,p) + Maximum( List([5..12], d -> zetal1(2,d,p) ) ) < 9761/10000 ,
zetal1(2,3,p) + zetal1(4,3,p) + zetal1(4,4,p) < 97/100 ,
2*zetal1(3,3,p) + zetal1(2,4,p) < 97/100 ,
zetal1(2,2,p) + 465/1000 <= 9761/10000 ,
Maximum(List([7..20], d -> zetal1(3,d,p))) < 2/10 ,
Maximum(List([5..12], d -> zetau(2,d,p,n) ) ) < 21/100 ,
zetal1(2,2,p) + zetau(2,4,p,n) + zetau(2,5,p,n) < 97/100 ,
zetal1(2,2,p) + 4678/10000 < 9781/10000 ,
zetau(2,3,p,n) + 1341/10000 < 4678/10000 ,
Maximum(List([8,9,12], d -> zetal([4,3,2,2],d,p))) < 1341/10000 ,
zetal1(3,2,p) < 5015/10000 ,
zetal1(3,6,p) + Maximum( List([7..40], d -> zetal1(3,d,p) ) ) < 4747/10000 ,
zetal1(3,6,p) + zetal([3,2],6,p) < 4747/10000 ,
2*zetal1(3,5,p) < 4747/10000 ,
zetal1(3,4,p) < 2872/10000 ,
Maximum(List( [7..20], d -> zetal1(3,d,p) ) ) < 1875/10000 ,
zetal([3,2],4,p) < 257/1000 ,
zetal([3,2],6,p) < 2/10 ,
zetal1(3,2,p) + zetal1(3,3,p) < 8368/10000 ,
```

```
Maximum( List([19..30], d -> zetal1(3,d,p) ) ) < 1413/10000 ,
zetal1(3,2,p) + zetal1(4,3,p) + zetal1(4,7,p) < 9781/1000
];
```

File five

```
\input ~/gap/scripts/five


#  gap script to check the arithmetic involving zetas
#  in the p=5 analysis in "Primitive Monodromy Groups
#  of Genus at most Two"
p := 5;
n := 7;
checklist5 := [
3*zeta(2,p) + 2*zetal1(2,2,p) < 29744/10000 ,
Maximum( List( [3..10], d -> zetau(3,3,p,n) ) ) <= 3344/10000 ,
2*zeta(2,p) + 2* 3344/10000 < 19744/10000 ,
zeta(2,p) + 2*zetal1(3,2,p) + zetal1(3,3,p) < 195/100 ,
3*zetal1(2,2,p) + zetal1(2,3,p) = 192/100 ,
2*zetal1(3,2,p) + 2*zeta(3,p) < 195/100 ,
zeta(3,p) < zetal1(3,2,p) ,
3*zetal1(3,2,p) +
   Maximum( List( [3..10], d -> zetal1(3,d,p) ) )  < 19/10 ,
zeta(2,p) +
   2*Maximum( List( [3..30], d -> zetau(1,7,p,n) ) ) < 9/10 ,
zetal1(3,2,p) = 504/1000 ,

Maximum( List([4..11], d-> zetal1(2,d,p) ) ) <= 32/100 ,
zetal1(2,3,p) = 36/100 ,
Maximum( List( [4..11], d -> zetal1(3,d,p) ) ) = 304/1000 ,
2*zetal1(3,3,p) < 6774/10000 ,
Maximum( List([5..14], d-> zetal1(3,d,p) ) ) <= 27/100 ,
zetal([3,2],4,p) = 264/1000 ,
zetal1(2,2,p) = 52/100 ,
Maximum( List([5..19], d-> zetau(2,d,p,n) ) ) < 203/1000 ,
zetau(2,4,p,n) < 253/1000 ,
zetal1(2,2,p) + zetau(2,4,p,n) + zetau(2,5,p,n) < 973/1000 ,
zetal1(2,2,p) + zetau(2,4,p,n) +
    Maximum( List( [7..20], d -> zetau(2,d,p,n) ) ) < 973/1000 ,
zetal1(2,2,p) + zetau(2,4,p,n) + zetau(2,6,p,n) < 976/1000 ,
zetal1(3,2,p) = 504/1000 ,
Maximum( List([7..19], d-> zetal1(4,d,p) ) ) < 21/100 ,
zetastar(6,p) <= 268/1000 ,
Maximum( List([7..24], d-> Minimum( zetastar(d,p), zetal([4,2],d,p) ) ))
```

```
       < 174/1000 ,
zetal([4,2],6,p) < 214/1000 ,
zetal1(4,5,p) + zetal([4,2],6,p) < 47/100 ,
zetal1(4,4,p) = 3008/10000 ,
Maximum( List([7..23], d-> zetal([4,3],d,p) ) ) <= 214/1000 ,
zetal([4,2],4,p) = 2608/10000 ,
zetal([4,3],6,p) <= 2032/10000 ,
zetal1(3,2,p) + zetal1(4,3,p) = 8384/10000 ,
Maximum( List([8..30], d-> zetal([4,4,3,2],d,p) ) ) < 1397/10000 ,
zetal1(4,2,p) = 5008/10000 ,
zetal1(4,2,p) + zetal1(4,3,p) + zetal1(4,7,p) <  S([2,3,7])

];
```

File three

```
\input ~/gap/scripts/three


#  gap script to check arithmetic inequalities
#  in the p=3 analysis of "Primitive Monodromy
#  Groups of Genus at most Two"
p := 3;
n := 10;
checklist3 := [
Maximum( List( [25..42], d -> zetastar(d,p) ) )  < 11/100 ,
Maximum( List( [13..24], d -> zetastar(d,p) ) )  < 2/10 ,
Maximum( List( [3..12], d -> zeta(d,p) ) ) < zeta(2,p) ,
Maximum( List( [3..12], d -> zetal1(2,d,p) ) ) < zetal1(2,2,p) ,
Maximum( List( [3..12], d -> zetal1(3,d,p) ) ) < zetal1(3,2,p) ,
Maximum( List( [3..12], d -> zetal1(4,d,p) ) ) < zetal1(4,2,p) ,
4*zeta(2,p) + 2*zetal1(2,2,p) < 38/10 ,
Maximum( List( [4..12], d -> zetau(3,d,p,n) ) ) < 3/10 ,
Maximum( List( [3..12], d -> zetau(4,d,p,n) ) ) < 35/100 ,
3*zeta(2,p) = 2 ,
2*zetau(4,3,p,n) < 7/10 ,
2*zeta(2,p) + 3*zetal1(3,2,p) < 29/10 ,
zeta(2,p) + 4*zetal1(2,2,p) < 29/10 ,
zetal1(1,2,p) + zetal1(1,2,p) < 13334/10000 ,
zetal1(1,2,p) + zetal1(2,2,p) < 12223/10000 ,
zetal1(2,2,p) + zetal1(2,2,p) < 11852/10000 ,
zetal1(1,2,p) + zetal1(3,2,p) < 11852/10000 ,
2* Maximum( List( [5..12], d -> zetau(2,d,p,n) ) ) < 401/1000 ,
2* Maximum( List( [4..12], d -> zetau(3,d,p,n) ) ) < 511/1000 ,
```

```
2* Maximum( List( [3..12], d -> zetau(4,d,p,n) ) ) < 67/100 ,
zeta(2,p) < 67/100 ,
zetal1(3,2,p) < 52/100 ,
Maximum( List( [3..12],d -> zeta(d,p) ) )  < 56/100 ,
Maximum( List( [3..12],d -> zetal1(3,d,p) ) )  < 36/100 ,
zeta(2,p) + 2*zetal1(4,2,p) < 168/100 ,
Maximum( List( [5..12], d-> zetau(2,d,p,n) ) ) < 21/100 ,
zetal1(2,2,p) + 2*zetal1(3,2,p) < 16/10 ,
Maximum( List( [4..12], d-> zetau(4,d,p,n) ) ) < 28/100 ,
3*Maximum( List( [3..12], d -> zetal1(3,2,p) ) ) < 156/100 ,
Maximum( List( [7..12], d -> zetal1(4,d,p) ) ) < 35/100 ,
zeta(2,p) < 67/100 ,
Maximum( List( [10..42], d -> zetau(1,d,p,n) ) ) < 11/100 ,
zetal1(2,2,p) < 556/1000 ,
Maximum( List( [5..12], d -> zetau(2,d,p,n) ) ) < 201/1000  ,
Maximum( List( [4..12], d -> zetal1(3,d,p) ) ) <= zetal1(3,4,p)  ,
zetal1(3,4,p) < 3519/10000 ,
zetal1(3,4,p) < 9714/10000 - 619/1000 ,
Maximum( List( [4..12], d -> zetau(3,d,p,n) ) ) < 254/1000 ,
Maximum( List( [4..12], d -> zetal1(4,d,p) ) ) < 34/100 ,
Maximum( List( [4..12], d -> zetau(4,d,p,n) ) ) < 28/100 ,
Maximum( List( [4..12], d -> zetal1(5,d,p) ) ) <= zetal1(5,4,p)  ,
zetal1(5,4,p) < 336/1000 ,
Maximum( List( [7..24], d -> zetastar(d,p) ) )  <= 25/100 ,
zetastar(5,p) < 25/100 ,
zetal([5,2],4,p) < 28/100 ,
zetal([5,2],6,p) < 28/100 ,
zetal1(5,4,p) < 34/100 ,
zetal1(5,6,p) < 34/100 ,
zetal1(3,3,p) < 36/100 ,
Maximum( List( [4..12], d -> zetau(3,d,p,n) ) ) < 26/100 ,
zetal1(4,3,p) < 342/1000 ,
Maximum( List( [5,7,9,11,13,15,17,19], d -> zetastar(d,p) ) )  < 21/100 ,
Maximum( List( [4..20], d -> zetal1(4,d,p) ) )  <= zetal1(4,3,p) ,
Maximum( List( [3..6], d -> zetal([4,2],2*d,p) ) )  <= zetal([4,2],4,p)  ,
Maximum( List( [3..6], d -> zetal1(4,2*d,p) ) )  <= zetal1(4,4,p)  ,
zetal([4,2],4,p) + zetal1(4,4,p) < 6295/10000 ,
2*zetal1(4,3,p) + zetal([4,2],4,p) < 9714/10000 ,
#
#   end of d1 > 2 argument
#
zetal1(3,2,p) < 5186/10000 ,
Maximum( List( [9..12], d -> zetastar(d,p) ) ) < 22/100 ,
Maximum( List( [5,7,9,11], d -> zetal1(5,d,p) ) ) < 204/1000 ,
Maximum( List( [6..12], d -> zetal1(5,d,p) ) ) < 336/1000 ,
Maximum( List( [11..50], d -> zetau(1,d,p,n) ) ) < 11/100 ,
```

```
Maximum( List( [6..12], d -> zetal([5,2],d,p) ) ) < 281/1000 ,
Maximum( List( [6..24], d -> zetau(2,d,p,n) ) ) < 17/100 ,
zetal([5,3],6,p) < 236/1000 ,
Maximum( List( [8..20], d -> zetastarsub([5,3],d,p) ) ) < 204/1000 ,
zetal([5,3],6,p) + zetal([5,3,2],6,p) < 435/1000 ,
zetal1(5,4,p) < 336/1000 ,
zetal([5,2],4,p) < 28/100 ,
zetau(2,6,p,n) < 17/100 ,
zetal([5,3],4,p) < 2614/10000 ,
zetastar(7,p) < 15/100 ,
zetastar(10,p) < 15/100 ,
zetastar(11,p) < 15/100 ,
Maximum( List( [13..20],d -> zetastar(d,p) ) ) < 15/100 ,
Maximum( List( [6,8,9,12] , d -> zetal([5,4,2],d,p) ) ) < 191/1000 ,
zetal1(3,2,p) + zetau(3,4,p,n) + zetau(3,5,p,n) < 975/1000 ,
zetal1(4,2,p) < 507/1000 ,
zetal([5,3],4,p) < 262/1000 ,
zetal1(5,5,p) < 204/1000 ,
zetal1(4,2,p) + zetal1(6,3,p) < 8405/10000 ,
zetal1(5,2,p) + zetal1(5,3,p) < 8405/10000 ,
Maximum( List( [9..24] , d -> zetal([5,5,4,2],d,p) ) ) < 137/1000 ,
zetal1(5,2,p) + zetal1(7,3,p) + zetal1(7,7,p) < 9795/10000

];
```

File two

```
\input ~/gap/scripts/two


#  gap script to check arithmetic inequalities
#  (and some equalities) in the p=2 analysis of
#  "Primitive Monodromy Groups of Genus at most Two"
p := 2;
n := 14;
checklist2 := [
zeta(2,p) = 75/100 ,
Maximum( List( [3,4], d -> zetastar(d,p) ) )  <= 5/10 ,
Maximum( List( [5,6], d -> zetastar(d,p) ) )  <= 375/1000 ,
Maximum( List( [7,8], d -> zetastar(d,p) ) )  < 282/1000 ,
Maximum( List( [9..12], d -> zetastar(d,p) ) )  <= 25/100 ,
Maximum( List( [13,14], d -> zetastar(d,p) ) )  < 19/100 ,
Maximum( List( [15..30], d -> zetastar(d,p) ) )  <= 15/100 ,
Maximum( List( [31..42], d -> zetastar(d,p) ) )  <= 94/1000 ,
Maximum( List( [43..100], d -> zetastar(d,p) ) )  < 8/100 ,
```

```
#
6*zeta(2,p) + 2*zetal1(2,2,p) = 575/100 ,
5*zeta(2,p) + 2*zetal1(3,2,p) <= 4875/1000 ,
Maximum( List( [3..6],d -> zetau(6,d,p,n) ) )  < 34/100 ,
4*zeta(2,p) + 2*34/100 < 37/10 ,
3*zeta(2,p) + 3*zetal1(4,2,p) < 39/10 ,
2*zeta(2,p) + zetal1(2,2,p) + 3*zetal1(3,2,p) < 39/10 ,
zeta(2,p) + 5*zetal1(2,2,p) < 39/10 ,
Maximum( List( [3..6],d -> zetau(6,d,p,n) ) )  < 34/100 ,
zetal1(5,2,p) < 52/100 ,
zetal1(5,3,p) < 4/10 ,
zetal1(5,4,p) < 4/10 ,
zeta(2,p) + zetal1(2,2,p) < 14/10 ,
zetal1(4,2,p) < 54/100 ,
zetal1(4,3,p) < 41/100 ,
zetal1(4,4,p) < 41/100 ,
2*zetal1(2,2,p) < zeta(2,p) + zetal1(3,2,p) ,
zeta(2,p) + zetal1(3,2,p) < 132/100 ,
zetal1(3,2,p) < 57/100 ,
zetal1(3,3,p) < 44/100 ,
zetal1(3,4,p) < 44/100 ,
zeta(2,p) + 3*zetal1(4,2,p) < 24/10 ,
zetal1(2,2,p) + 3*zetal1(3,2,p) < 24/10 ,
Maximum( List( [1..3], a -> zetal1(a,2,p) + 4*zetal1(7-a,2,p) ))  < 28/10 ,
#
#   end of Step 2 checks
#
Maximum( List( [5..12], d -> zetau(3,d,p,n) ) ) < 201/1000 ,
zeta(2,p) + zetal1(3,2,p) < 132/100 ,
Maximum( List( [4..8], d -> zetau(4,d,p,n) ) ) < 26/100 ,
zeta(2,p) + zetal1(4,2,p) < 1282/1000 ,
Maximum( List( [3..6], d -> zetau(5,d,p,n) ) ) < 335/1000 ,
zeta(2,p) + zetal1(5,2,p) < 1266/1000 ,
Maximum( List( [3..6], d -> zetau(6,d,p,n) ) ) < 336/1000 ,
375/1000 < zetal1(5,4,p) ,
zetal1(5,4,p) < 5/10 ,
Filtered([1..14],k -> zetastarsub([k],3,p) > zetastarsub([k],4,p) ) = [] ,
zetal1(2,2,p) + zetal1(5,2,p) + 2*zetastarsub([5],4,p) < 193/100 ,
zetal1(3,2,p) + zetal1(4,2,p) + 2*zetastarsub([4],4,p) < 191/100 ,
2*zetal1(4,2,p) + zetastarsub([3],4,p) + zetastarsub([4],4,p) < 191/100 ,
2*zetal1(5,2,p) + zetastarsub([2],4,p) + zetastarsub([5],4,p) < 193/100 ,
2*zetal1(6,2,p) < 102/100 ,
Maximum( List([1..3],
       t -> zetastarsub([t],4,p) + zetastarsub([7-t],4,p))) <9/10 ,
zeta(2,p) + zetal1(6,2,p) < 126/100 ,
zetastarsub([6],5,p) < 34/100 ,
```

```
zetastarsub([6],6,p) < 34/100 ,
zetastarsub([6],3,p) < 39/100 ,
zetastarsub([6],4,p) < 39/100 ,
zetastarsub([6,5],4,p) < 3/10 ,
zetastarsub([6,5],5,p) < 3/10 ,
zetastarsub([6,5],6,p) < 3/10 ,
zetal1(6,3,p) < 35/100 ,
zeta(2,p) + 2*zetal1(6,2,p) < 1766/1000 ,
Maximum( List( [7..30], d -> zetau(2,7,p,n) ) )  < 144/1000 ,
zetal1(2,2,p) + 2*zetal1(5,2,p) < 1657/1000 ,
Maximum( List( [4..8], d  -> zetau(4,d,p,n) ) ) < 255/1000 ,
zetal1(3,2,p) + 2*zetal1(4,2,p) = 1625/1000 ,
Maximum( List( [3..6], d -> zetau(6,d,p,n) ) )  < 336/1000 ,
3* zetal1(4,2,p) < 1594/1000 ,
Maximum( List([3..4], d -> zetal1(7,d,p) ) ) < 379/1000 ,
#
#  end of Step 3 checks
#
zetal1(1,2,p) <= 75/100 ,
zetal1(2,2,p) <= 625/1000 ,
zetal1(3,2,p) <= 563/1000 ,
Maximum( List( [14..42], d -> zetau(1,d,p,n) ) )  < 8/100 ,
Maximum( List( [7..42], d -> zetau(2,d,p,n) ) )  < 15/100 ,
Maximum( List( [5..42], d -> zetau(3,d,p,n) ) )  < 201/1000 ,
#
#  end of Step 4 checks
#
zetastarsub([4],3,p) < 41/100 ,
zetastarsub([4],4,p) < 41/100 ,
Maximum( List( [4..8], d -> zetau(4,d,p,n) ) ) < 255/1000 ,
zetastarsub([5],3,p) < 3907/10000 ,
zetastarsub([5],4,p) < 3907/10000 ,
zetastarsub([5],6,p) < 35/100 ,
Maximum( List( [5,7,8], d -> zetastarsub([5],d,p) )) < 27/100 ,
zetastarsub([5],6,p) + 2*zetastarsub([5,3],6,p) < 95/100 ,
zetastarsub([5],4,p) + 2* 27/100 < 9714/10000 ,
zetastarsub([5],4,p) + zetastarsub([5],6,p) < 7345/10000 ,
Maximum( List( [9..12], d -> zetastarsub([5],d,p) )) < 24/100 ,
zetastarsub([5],4,p) + zetastarsub([5,2],4,p) < 7188/10000 ,
Maximum( List( [4..6], d -> zetastarsub([5,3],d,p) )) < 3021/10000 ,
Maximum( List( [4..6], d -> zetastarsub([5,4],d,p) )) < 2813/10000 ,
zetastarsub([5,2],4,p) < 3282/10000 ,
Maximum( List( [4..6], d -> zetastarsub([5,2],d,p) )) < 3438/10000 ,
Maximum( List( [4..6], d -> zetastarsub([5,3],d,p) )) < 3021/10000 ,
zetastarsub([5,3],4,p) < 3/10 ,
Maximum( List( [5..6], d -> zetastarsub([5],d,p) )) < 35/100 ,
```

```
zetastarsub([5],3,p) < 3542/10000 ,
zetastarsub([5],3,p) + zetastarsub([7],6,p) < 6902/10000 ,
zetastarsub([7],3,p) + zetastarsub([5],6,p) < 6902/10000 ,
zetastarsub([5],6,p) < 3438/10000 ,
zetastarsub([5,5],6,p) < 2709/10000 ,
zetastarsub([5],3,p) + zetastarsub([5],6,p)
    + zetastarsub([5,5],6,p) < 97/100 ,
zetastarsub([5],5,p) <= 225/1000 ,
Maximum( List( [5..8], d -> zetastarsub([5],d,p) ) )  < 344/1000 ,
zetastarsub([5],3,p) + zetastarsub([7],4,p)  < 7332/10000 ,
zetal1(7,3,p) + zetal1(5,4,p)  < 7332/10000 ,
Maximum( List( [5,7,8], d -> zetastarsub([5,3],d,p) )) < 24/100 ,
zetastarsub([5],3,p) + 2*zetastarsub([5,3],4,p) < 97/100 ,
zetastarsub([5],3,p) + zetastarsub([5,3],4,p)
        + zetastarsub([5,3],6,p) < 97/100 ,
zetastarsub([5],3,p) + zetastarsub([5,2],4,p)
        + zetastarsub([5,5],6,p) < 97/100 ,
zetastarsub([5,5,6],6,p) < 2/10 ,
zetastarsub([7],3,p) < 3386/10000 ,
Maximum( List( [4..8], d -> zetastarsub([5,5],d,p) ) ) < 2735/10000 ,
#
#  end of Step 5 checks
#
zetal1(4,2,p) < 532/1000 ,
Maximum( List([5..12], d -> zetastarsub([10,6,2],d,p) ) ) < 22/100 ,
zetal1(5,2,p) < 5157/10000 ,
Maximum( List([9..11], d -> zetastarsub([7],d,p) ) ) < 2/10 ,
zetastarsub([7],12,p) < 232/1000 ,
zetastarsub([7,1,4],12,p) < 21/100 ,
zetastarsub([7],8,p) < 254/1000 ,
zetastarsub([7,3],8,p) < 223/1000 ,
zetastarsub([7,4],12,p) < 222/1000 ,
zetastarsub([7],7,p) < 15/100 ,
zetal1(6,2,p) < 9764/10000 - 4608/10000 ,
zetastarsub([7],6,p) < 336/1000 ,
Maximum( List([14..30], d -> zetau(1,d,p,n) ) ) < 8/100 ,
zetastarsub([7,4],6,p) < 2735/10000 ,
Maximum( List([7..28], d -> zetau(2,d,p,n) ) ) < 1431/10000 ,
zetastarsub([7,6],6,p) < 2579/10000 ,
Maximum( List([6..12], d -> zetau(3,d,p,n) ) ) < 18/100 ,
zetastarsub([8,8],6,p) < 2527/10000 ,
Maximum( List([7..11], d -> zetastarsub([7,5],d,p) ) ) < 2/10 ,
zetastarsub([7,5,6],6,p) < 2/10 ,
zetastarsub([8,8,2],6,p) < 211/1000 ,
zetastarsub([7,5],12,p) < 217/1000 ,
zetastarsub([7],5,p)  < 2063/10000 ,
```

```
Maximum( List([5,7,8], d -> zetastarsub([7,6],d,p) ) ) < 25/100 ,
zetastarsub([7,6,2],6,p) < 22/100 ,
#
# end of Step 6 checks
#
zetal1(5,2,p) < 9764/10000 - 4607/10000 ,
zetau(3,6,p,n) < 18/100 ,
zetastarsub([7],4,p) < 379/1000 ,
zetal1(4,2,p) < 532/1000 ,
Maximum( List( [4..8], d -> zetau(4,d,p,n) ) ) < 255/1000 ,
Maximum( List( [6..14], d -> zetastarsub([12,12,4,12],d,p) ) ) < 178/1000 ,
zetal1(5,2,p) < 5157/10000 ,
Maximum( List( [14..100], d -> zetau(1,d,p,n) ) ) < 8/100 ,
zetastarsub([7,3],4,p) < 2852/10000 ,
Maximum( List( [7..30], d -> zetau(2,d,p,n) ) ) < 144/1000 ,
zetastarsub([7,5],4,p) < 2618/10000 ,
Maximum( List( [6..30], d -> zetau(3,d,p,n) ) ) < 19/100 ,
zetastarsub([7,6],4,p) < 2579/10000 ,
zetau(3,5,p,n) < 2004/10000 ,
zetastarsub([7,7],4,p) < 2559/10000 ,
Maximum( List( [6..30], d -> zetastarsub([7,7,4],d,p) ) ) < 2/10 ,
zetal1(7,5,p) < 2073/10000 ,
#
# end of Step 7 checks
#
zetal1(8,3,p) < 33595/100000 ,
Maximum( List( [8..30], d -> zetastarsub([10,10,7,5,3],d,p) ) ) < 132/1000 ,
zetal1(6,2,p) < 50782/100000 ,
zetal1(10,3,p) < 334/1000 ,
zetal1(5,2,p) < 5157/10000 ,
Maximum( List( [9..11], d -> zetastarsub([10,10,7,5,3],d,p) ) ) < 114/1000 ,
Maximum( List( [13..30], d -> zetastarsub([10,10,7,5,3],d,p) ) ) < 114/1000 ,
zetal1(5,2,p) + zetal1(10,3,p) + zetastarsub([10,10,7,5],8,p) < 9793/10000 ,
zetal1(5,2,p) + zetal1(12,3,p) + zetastarsub([10,10,7,6],12,p) < 9789/10000 ,
zetal1(5,2,p) + zetal1(10,3,p) + zetastarsub([13,10,7,6],12,p) < 9789/10000 ,
zetal1(8,2,p) + zetal1(11,3,p) + zetal1(11,7,p) < 9795/10000

];
```

File Aug9.log

```
\input ~/gap/logs/Aug9.log


gap> Read("../scripts/mustar");
gap> Read("../scripts/Schecks");
```

```
gap> Read("../scripts/large");
gap> Read("../scripts/seven");
gap> Read("../scripts/five");
gap> Read("../scripts/three");
gap> Read("../scripts/two");
gap> checklistr;
[ true, true, true, true, true ]
gap> checklistS;
[ true, true, true, true, true, true ]
gap> checklist23;
[ true, true, true, true, true, true, true, true, true, true ]
gap> checklist11;
[ true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true ]
gap> checklist7;
[ true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true ]
gap> checklist5;
[ true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true ]
gap> checklist3;
[ true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true ]
gap> checklist2;
[ true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
```

```
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true,
  true, true, true, true, true, true, true, true, true, true ]
gap> LogTo();
```

B GAP computations for the proof of Theorem 3

File leftovers

```
\input ~/gap/jun07/leftovers

#  file leftovers
#  gap code setting up functions used in
#  verifying calculations for proving Theorem 3
#  in Primitive Monodromy Groups of Genus at Most Two
#

iseven := function(n)
# true if n is an even integer
     return Gcd(n,2) = 2; end;

squareroots := function(q)
#  returns list of positive integers that
#  square to q
#  not good for large values of q
     return Filtered([1..q], x -> x^2 = q ) ;
     end;

issquare := function(q)
#  true if q is a perfect square
#  not good for large values of q
     return squareroots(q) <> [];
     end;

sqrt := function(q)
#  if q is a perfect integer square returns its square root
#  otherwise returns "ERROR"
#  not practical for large values of q
     local sqrq;
     sqrq := squareroots(q);
     if sqrq = [] then return "ERROR sqrt"; fi;
     return sqrq[1];
     end;

kfactor := function(q)
#  coefficient for counting non-singular points of orthogonal space
     return Gcd(q,2)/2;
     end;

epsterm := function(type)
```

```
#  converts "+" to 1 and "-" to -1
     if type = "+" then return 1; fi;
     if type = "-" then return -1; fi;
     return "ERROR epsterm";
     end;

compatible := function( tl, n, q )
#  tl is a typelist ( [L], [O,+], [O,-], [U], or [S] )
#  n is a dimension
#  q is the order of a finite field
#  returns true if there is a relevant group of that
#  type for the given dimension and order
#  consider symplectic groups only in even characteristic
     if tl[1] = "L" then return true; fi;
     if tl[1] = "O" and iseven(n) then return true; fi;
     if tl[1] = "O" and not iseven(q) then return true; fi;
     if tl[1] = "U" and issquare(q) then return true; fi;
     if tl[1] = "S" and iseven(n) and iseven(q) then return true; fi;
     return false;
     end;

classicalpoints := function(tlist,n,q,type)
#  tlist is ["L"] or ["O","+"] or ["O","-"] or ["U"] or ["S"]
#  type is the type of point +, -, or 0
#  the number of points of
     local k, e, m, q0;
     if n = 0 then return 0; fi;
     if n < 0 then return "ERROR classicalpoints n too small"; fi;
     if tlist[1] = "L" then return (q^n-1)/(q-1); fi;
     if tlist[1] = "O" then
           if iseven(n) then
                e := epsterm(tlist[2]);
                m := n/2;
                if type = "0" then
                      return (q^m-e)*(q^(m-1)+e) / (q-1); fi;
                k := kfactor(q);
                return k * (q^m-e) * q^(m-1);
                fi;
           if iseven(q) then return "ERROR1 classicalpoints"; fi;
           m := (n-1)/2;
           if type = "0" then return (q^(2*m)-1) / (q-1) ; fi;
           e := epsterm(tlist[2])*epsterm(type);
           return q^m * ( q^m - e ) / 2;
           fi;
     if tlist[1] = "U" then
           q0 := sqrt(q);
```

```
          e := (-1)^n;
          if type = "0" then return (q0^n - e ) * (q0 ^ ( n-1 ) +  e ) / (q-1) ; fi;
          return (q0^n-e)*q0^(n-1)/(q0+1);
          fi;
    if tlist[1] = "S" then
          e := epsterm(type);
          m := n/2;
          return q^m*(q^m-e)/2;
          fi;
    end;

numberofpoints := function(tlist,n,r,q,type)
#  tlist, n, q, type  as above
#  r is the dimension of the radical
     local addterm;
     if r > n then return 0; fi;
     if not compatible(tlist, n-r , q) then return 0; fi;
     addterm := 0;
     if tlist[1] = "L" then return classicalpoints(["L"],n,q,type); fi;
     if type = "0" then addterm := classicalpoints(["L"],r,q,type); fi;
     return q^r * classicalpoints( tlist, n-r , q , type ) + addterm;
     end;

maxpts := function(tlist, m, r, q, type)
#  tlist, q as above
#  maximum number of points of the given type
#  in an m-dimensional space with r-dimensional radical
     if tlist[1] in ["L","U"] or type = "0" then
          return numberofpoints(tlist,m,r,q,type); fi;
     if tlist[1] = "O" and type in ["+","-"] then
          return Maximum(List(["+","-"], t -> numberofpoints(tlist,m,r,q,t)));
          fi;
     if tlist[1] = "S" and type in ["+", "-"] then
          return "Symplectic not finished"; fi;
     return "ERROR maxpts";
     end;

maximumespacepoints := function(tlist,n,q,type,v)
#  tlist, n, q, type as above
#  for linear, orthogonal, or unitary spaces
#     returns maximum number of points in the principal eigenspace
#     of an element with commutator subspace of codimension v
#  for symplectic spaces returns the number of hyperplanes
#     of the given type complementary to the radical
     local tpossible;
     if tlist[1] = "S" then return q^(n/2)*(q^((n/2)-v)+1)/2; fi;
```

```
     tpossible := Filtered([0..v],x -> 2*x <= n);
     if iseven(q) then
         tpossible := Filtered(tpossible, x -> iseven(n-v-x)); fi;
     if tpossible = [] then return 0; fi;
     return Maximum(List(tpossible, t -> maxpts( tlist, n-v, t, q, type) ) );
     end;

allowablens := function(p)
# based on  results of Primitive Monodromy
# Groups of Genus at Most Two
     if p = 11 then return [5,6]; fi;
     if p = 7 then return [6]; fi;
     if p = 5 then return [7,8,9]; fi;
     if p = 3 then return [10,12]; fi;
     if p = 2 then return [14..21]; fi;
     end;

tlist := [ [ "L" ] , [ "O" , "+" ] , [ "O" , "-" ] , [ "U" ] , [ "S" ]  ];

IsDivisor := function(a,b)
# true if b is a divisor of a
     return b in DivisorsInt(a); end;

ProperDivisors := function(n)
# proper divisors of n
     return Filtered(DivisorsInt(n), d -> d < n ) ;
     end;

typestocheck := function(tl,m,q)
#  given the typelist, dimension, and order
#  returns a list of the types of point* to check
#  (*hyperplane in the symplectic case)
     if tl[1] = "L" then return ["0"]; fi;
     if tl[1] = "U" then return ["0","+"]; fi;
     if tl[1] = "O" then
          if iseven(q) or iseven(m) then return ["0","+"]; fi;
             return ["0","+","-"];
          fi;
     if tl[1] = "S" then
          if iseven(q) and iseven(m) then return ["+","-"]; fi;
          fi;
     end;

maximumpoints := function(tlist,m,q,type)
#  maximum number of points in a space of dimension m
return (q^m-1)/(q-1); end;
```

```
secondaryfps := function(tl, n , q , t, d, v)
#  gives the largest possible number of points outside
#  the largest eigenspace
#  fixed by an element with commutator of codimension v
#  tl is typelist
#  q is fieldsize
#  n is dimension of the big space
#  t tis the type of point
#  d is the order of the element
#  v is the codimension of the largest eigenspace
     local d0;
     d0 := Gcd(d,q-1);
     if d0 = 1 then return 0; fi;
     if tl = ["S"] then return 0; fi;
     if 2*v <= n then return maximumpoints(tl,v,q,t); fi;
     return (d0-1)* maximumpoints(tl,n-v,q,t);
     end;

maxfps := function(tl, n , q , t , d , v )
#  gives the largest possible number of fixed points
#  tl is typelist
#  q is fieldsize
#  n is dimension of the big space
#  t tis the type of point
#  d is the order of the element
#  v is the codimension of the largest eigenspace
     local sfps;
     sfps := 0;
          sfps := secondaryfps(tl, n, q, t, d, v); # fi;
     return maximumespacepoints(tl, n, q, t, v) + sfps;
     end;

minindex := function(tl, n, q, t, d, vlist)
#  tl, n, q as above
#  t is point type
#  returns the smallest possible permutation index
#  for an element x of order d satisfying
#  v(x^j) >= vlist[j]
     local pd;
     pd := ProperDivisors(d);
     if Maximum(pd) > Length(vlist) then
         return "ERROR minindex:  inadequate vlist"; fi;
     return
     classicalpoints(tl, n, q, t)*(d-1)/d
         - Sum(pd, h -> Phi(d/h)*maxfps(tl,n,q, t, d, vlist[h])) / d;
```

```
     end;

Floor := function(rational)
#  standard floor function
     local r2, flnegr;
     if rational < 0 then
        flnegr := Floor(-1*rational);
        if flnegr + rational = 0 then return -1*flnegr;
               else return -1*flnegr -1; fi;
        fi;
     if rational < 1 then return 0; fi;
     if rational < 2 then return 1; fi;
     r2 := Floor(rational/2);
     return 2*r2+ Floor(rational - 2*r2);
     end;

Ceiling := function(rational)
#  standard integer ceiling function
#  Example:
#  gap> Ceiling(7/3);
#  gap> 3;
     return -1*Floor(-1*rational);
     end;

genusofsystem := function(tl, n, q, t, dtuple, vtuple)
#  the genus of generators with signature dtuple and
#  vs bounded below by vtuple will be at least as
#  large as what this function returns
    local degree, indexsum, minin;
    degree := classicalpoints(tl,n,q,t);
    minin := List([1..Length(dtuple)],
                  i -> minindex(tl, n, q, t, dtuple[i],vtuple[i]) );
    indexsum := Sum(minin);
    Print(dtuple,minin,"\n");
    return Ceiling(indexsum/2 - degree + 1);
    end;

minimalv := function(n, dtuple)
#  crude lower bounds for v
     if dtuple = [2,3,7] then
     return [ [Ceiling(n/3)], [Ceiling(n/2)], [Ceiling(n/2) ] ];
     fi;
     if dtuple = [2,3,8] then
     return [ [Ceiling(n/3)], [Ceiling(n/2)],
     [Ceiling(n/2) , Ceiling(n/2), 1, Ceiling(n/5) ] ];
     fi;
```

```
     ##  if dtuple = [ 2, 4, 5 ] then
     ##  return [ [Ceiling(n/4)], [Ceiling(n/2), Ceiling(n/4)],
     ##  [Ceiling(n/2) ] ];
     ##  fi;
     return "ERROR minimalv dtuple not programmed";
     end;

genuscheck := function(tl, n, q, t, dtuple)
#  a lower bound for the genus of a system of type dtuple
#  in the classical group with parameters tl, n, q, t
return genusofsystem(tl,n,q,t,dtuple,minimalv(n,dtuple));
end;

minv := function(n, dtuple, q)
#  lower bounds for v using results from obtaining list1
#  use for (2,4,5) case
     local lint;
     if dtuple = [2,3,7] then
     return [ [Ceiling(n/3)], [Ceiling(n/2)], [Ceiling(n/2) ] ];
     fi;
     if dtuple = [2,3,8] then
     return [ [Ceiling(n/3)], [Ceiling(n/2)],
     [Ceiling(n/2) , Ceiling(n/2), 1, Ceiling(n/5) ] ];
     fi;
     if dtuple = [ 2, 4, 5 ] then
     lint := LogInt(q,2);
     if lint in [1,2,4] then
     return [ [ 4/lint ],  [12/lint,Ceiling(7/lint)],
     [16/lint] ];fi;
     return [ [n], [n,n], [n]];
     fi;
     return "ERROR minimalv dtuple not programmed";
     end;

genuscheckspecial := function(tl, n, q, t, dtuple)
#  a lower bound for the genus of a system of type dtuple
#  in the classical group with parameters tl, n, q, t
#  use for (2,4,5) case only
     if not q in [2,4,16] then return "q= 256 NOT RELEVANT"; fi;
     return genusofsystem(tl,n,q,t,dtuple,minv(n,dtuple,q));
     end;

###


troublemakers := [];
```

```
for p in [2,3,5,7,11] do
  for n in allowablens(p) do
    for d in ProperDivisors(n) do
      nn := n/d;
      q := p^d;
      for tl in tlist do
        if compatible(tl,nn,q) then
          for tt in typestocheck(tl,nn,q) do
            clpts := classicalpoints(tl,nn,q,tt);
            if clpts > 10000 then
              Print([nn, q, tl, tt, clpts],"\n");
              gc := genuscheck(tl,nn,q,tt,[2,3,7]);
              Print("(2,3,7)-genus is at least ",gc,"\n\n");
              if gc < 10 then
                Append(troublemakers, [ [ [ tl,nn,q,tt], gc ] ] );fi;
               fi;
            od;
          fi;
        od;
      od;
    od;
  od;

troublemakers2 := [];
p:=2;
n := 16;
for d in ProperDivisors(n) do
  nn := n/d;
  q := p^d;
  for tl in tlist do
    if compatible(tl,nn,q) then
      for tt in typestocheck(tl,nn,q) do
        clpts := classicalpoints(tl,nn,q,tt);
        if clpts > 10000 then
          Print([nn, q, tl, tt, clpts],"\n");
          gc := genuscheckspecial(tl,nn,q,tt,[2,4,5]);
          Print("(2,4,5)-genus is at least ",gc,"\n\n");
          if gc < 10 then
            Append(troublemakers2, [ [ [ tl,nn,q,tt], gc ] ] );fi;
          fi;
        od;
      fi;
    od;
  od;

troublemakers3 := [];
```

```
p:=3;
n := 10;
for d in ProperDivisors(n) do
  nn := n/d;
  q := p^d;
  for tl in tlist do
    if compatible(tl,nn,q) then
      for tt in typestocheck(tl,nn,q) do
        clpts := classicalpoints(tl,nn,q,tt);
        if clpts > 10000 then
          Print([nn, q, tl, tt, clpts],"\n");
          gc := genuscheck(tl,nn,q,tt,[2,3,8]);
          Print("(2,3,8)-genus is at least ",gc,"\n");
          if gc < 10 then
            Append(troublemakers3, [ [ [ tl,nn,q,tt], gc ] ] );fi;
          fi;
        od;
      fi;
    od;
  od;

Print("\n","Need further checking for (2,3,7) systems:  \n",troublemakers,"\n");
Print("\n","Need further checking for (2,4,5) systems:  ",troublemakers2,"\n");
Print("\n","Need further checking for (2,3,8) systems:  ",troublemakers3,"\n");
```

File jun7.log

```
\input ~/gap/jun07/jun8.log

gap> Read("~/gap/jun07/leftovers");
[ 14, 2, [ "L" ], "O", 16383 ]
[ 2, 3, 7 ][ 7936, 32512/3, 97536/7 ]
(2,3,7)-genus is at least -28

[ 15, 2, [ "L" ], "O", 32767 ]
[ 2, 3, 7 ][ 15872, 21760, 195840/7 ]
(2,3,7)-genus is at least 39

[ 16, 2, [ "L" ], "O", 65535 ]
[ 2, 3, 7 ][ 32256, 43520, 391680/7 ]
(2,3,7)-genus is at least 332

[ 16, 2, [ "O", "+" ], "O", 32895 ]
[ 2, 3, 7 ][ 16128, 21760, 195840/7 ]
(2,3,7)-genus is at least 39
```

```
[ 16, 2, [ "O", "+" ], "+", 32640 ]
[ 2, 3, 7 ][ 16072, 21680, 195120/7 ]
(2,3,7)-genus is at least 175

[ 16, 2, [ "O", "-" ], "O", 32639 ]
[ 2, 3, 7 ][ 16072, 64768/3, 194304/7 ]
(2,3,7)-genus is at least 72

[ 16, 2, [ "O", "-" ], "+", 32896 ]
[ 2, 3, 7 ][ 16128, 65408/3, 28032 ]
(2,3,7)-genus is at least 87

[ 16, 2, [ "S" ], "+", 32640 ]
[ 2, 3, 7 ][ 16000, 64768/3, 194304/7 ]
(2,3,7)-genus is at least 35

[ 16, 2, [ "S" ], "-", 32896 ]
[ 2, 3, 7 ][ 16128, 21760, 195840/7 ]
(2,3,7)-genus is at least 38

[ 8, 4, [ "L" ], "O", 21845 ]
[ 2, 3, 7 ][ 10752, 14450, 130560/7 ]
(2,3,7)-genus is at least 83

[ 8, 4, [ "O", "+" ], "+", 16320 ]
[ 2, 3, 7 ][ 8040, 32350/3, 97560/7 ]
(2,3,7)-genus is at least 62

[ 8, 4, [ "O", "-" ], "+", 16448 ]
[ 2, 3, 7 ][ 8064, 32566/3, 98208/7 ]
(2,3,7)-genus is at least 28

[ 8, 4, [ "U" ], "O", 10965 ]
[ 2, 3, 7 ][ 5376, 21590/3, 65280/7 ]
(2,3,7)-genus is at least -14

[ 8, 4, [ "U" ], "+", 10880 ]
[ 2, 3, 7 ][ 5360, 7170, 65040/7 ]
(2,3,7)-genus is at least 32

[ 8, 4, [ "S" ], "+", 32640 ]
[ 2, 3, 7 ][ 16000, 64768/3, 194304/7 ]
(2,3,7)-genus is at least 35

[ 8, 4, [ "S" ], "-", 32896 ]
[ 2, 3, 7 ][ 16128, 21760, 195840/7 ]
```

```
(2,3,7)-genus is at least 38

[ 4, 16, [ "S" ], "+", 32640 ]
[ 2, 3, 7 ][ 16192, 64768/3, 194304/7 ]
(2,3,7)-genus is at least 131

[ 4, 16, [ "S" ], "-", 32896 ]
[ 2, 3, 7 ][ 16320, 21760, 195840/7 ]
(2,3,7)-genus is at least 134

[ 2, 256, [ "S" ], "+", 32640 ]
[ 2, 3, 7 ][ 16192, 64768/3, 194304/7 ]
(2,3,7)-genus is at least 131

[ 2, 256, [ "S" ], "-", 32896 ]
[ 2, 3, 7 ][ 16320, 21760, 195840/7 ]
(2,3,7)-genus is at least 134

[ 17, 2, [ "L" ], "0", 131071 ]
[ 2, 3, 7 ][ 64512, 261632/3, 112128 ]
(2,3,7)-genus is at least 856

[ 18, 2, [ "L" ], "0", 262143 ]
[ 2, 3, 7 ][ 129024, 523264/3, 224256 ]
(2,3,7)-genus is at least 1709

[ 18, 2, [ "O", "+" ], "0", 131327 ]
[ 2, 3, 7 ][ 64512, 261632/3, 112128 ]
(2,3,7)-genus is at least 600

[ 18, 2, [ "O", "+" ], "+", 130816 ]
[ 2, 3, 7 ][ 64400, 261152/3, 783456/7 ]
(2,3,7)-genus is at least 872

[ 18, 2, [ "O", "-" ], "0", 130815 ]
[ 2, 3, 7 ][ 64400, 260608/3, 781824/7 ]
(2,3,7)-genus is at least 666

[ 18, 2, [ "O", "-" ], "+", 131328 ]
[ 2, 3, 7 ][ 64512, 87296, 785664/7 ]
(2,3,7)-genus is at least 696

[ 18, 2, [ "S" ], "+", 130816 ]
[ 2, 3, 7 ][ 64256, 260608/3, 781824/7 ]
(2,3,7)-genus is at least 593
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[ 18, 2, [ "S" ], "-", 131328 ]
[ 2, 3, 7 ][ 64512, 261632/3, 112128 ]
(2,3,7)-genus is at least 599

[ 9, 4, [ "L" ], "O", 87381 ]
[ 2, 3, 7 ][ 43008, 58084, 523776/7 ]
(2,3,7)-genus is at least 579

[ 9, 4, [ "U" ], "O", 43605 ]
[ 2, 3, 7 ][ 21440, 28900, 261120/7 ]
(2,3,7)-genus is at least 218

[ 9, 4, [ "U" ], "+", 43776 ]
[ 2, 3, 7 ][ 21552, 29044, 37488 ]
(2,3,7)-genus is at least 267

[ 6, 8, [ "L" ], "O", 37449 ]
[ 2, 3, 7 ][ 18432, 74752/3, 31974 ]
(2,3,7)-genus is at least 214

[ 6, 8, [ "O", "+" ], "+", 32704 ]
[ 2, 3, 7 ][ 16100, 65296/3, 195450/7 ]
(2,3,7)-genus is at least 191

[ 6, 8, [ "O", "-" ], "+", 32832 ]
[ 2, 3, 7 ][ 16128, 21840, 196122/7 ]
(2,3,7)-genus is at least 162

[ 6, 8, [ "S" ], "+", 130816 ]
[ 2, 3, 7 ][ 64256, 260608/3, 781824/7 ]
(2,3,7)-genus is at least 593

[ 6, 8, [ "S" ], "-", 131328 ]
[ 2, 3, 7 ][ 64512, 261632/3, 112128 ]
(2,3,7)-genus is at least 599

[ 2, 512, [ "S" ], "+", 130816 ]
[ 2, 3, 7 ][ 65152, 260608/3, 781824/7 ]
(2,3,7)-genus is at least 1041

[ 2, 512, [ "S" ], "-", 131328 ]
[ 2, 3, 7 ][ 65408, 261632/3, 112128 ]
(2,3,7)-genus is at least 1047

[ 19, 2, [ "L" ], "O", 524287 ]
[ 2, 3, 7 ][ 260096, 349184, 3142656/7 ]
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(2,3,7)-genus is at least 4830

[ 20, 2, [ "L" ], "0", 1048575 ]
[ 2, 3, 7 ][ 520192, 698368, 6285312/7 ]
(2,3,7)-genus is at least 9657

[ 20, 2, [ "O", "+" ], "0", 524799 ]
[ 2, 3, 7 ][ 260096, 349184, 3142656/7 ]
(2,3,7)-genus is at least 4318

[ 20, 2, [ "O", "+" ], "+", 523776 ]
[ 2, 3, 7 ][ 259872, 1046560/3, 3139680/7 ]
(2,3,7)-genus is at least 4851

[ 20, 2, [ "O", "-" ], "0", 523775 ]
[ 2, 3, 7 ][ 259872, 1045504/3, 3136512/7 ]
(2,3,7)-genus is at least 4450

[ 20, 2, [ "O", "-" ], "+", 524800 ]
[ 2, 3, 7 ][ 260096, 1048064/3, 3144192/7 ]
(2,3,7)-genus is at least 4512

[ 20, 2, [ "S" ], "+", 523776 ]
[ 2, 3, 7 ][ 259584, 1045504/3, 3136512/7 ]
(2,3,7)-genus is at least 4305

[ 20, 2, [ "S" ], "-", 524800 ]
[ 2, 3, 7 ][ 260096, 349184, 3142656/7 ]
(2,3,7)-genus is at least 4317

[ 10, 4, [ "L" ], "0", 349525 ]
[ 2, 3, 7 ][ 174080, 232562, 2095104/7 ]
(2,3,7)-genus is at least 3448

[ 10, 4, [ "O", "+" ], "0", 87637 ]
[ 2, 3, 7 ][ 43520, 57970, 523776/7 ]
(2,3,7)-genus is at least 522

[ 10, 4, [ "O", "+" ], "+", 261888 ]
[ 2, 3, 7 ][ 130440, 522614/3, 1569888/7 ]
(2,3,7)-genus is at least 2571

[ 10, 4, [ "O", "-" ], "0", 87125 ]
[ 2, 3, 7 ][ 43400, 172886/3, 520704/7 ]
(2,3,7)-genus is at least 584
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[ 10, 4, [ "O", "-" ], "+", 262400 ]
[ 2, 3, 7 ][ 130560, 523478/3, 224640 ]
(2,3,7)-genus is at least 2448

[ 10, 4, [ "U" ], "O", 174933 ]
[ 2, 3, 7 ][ 87040, 348502/3, 1047552/7 ]
(2,3,7)-genus is at least 1497

[ 10, 4, [ "U" ], "+", 174592 ]
[ 2, 3, 7 ][ 86960, 348182/3, 1046592/7 ]
(2,3,7)-genus is at least 1676

[ 10, 4, [ "S" ], "+", 523776 ]
[ 2, 3, 7 ][ 260608, 1045504/3, 3136512/7 ]
(2,3,7)-genus is at least 4817

[ 10, 4, [ "S" ], "-", 524800 ]
[ 2, 3, 7 ][ 261120, 349184, 3142656/7 ]
(2,3,7)-genus is at least 4829

[ 5, 16, [ "L" ], "O", 69905 ]
[ 2, 3, 7 ][ 34816, 139708/3, 59904 ]
(2,3,7)-genus is at least 741

[ 5, 16, [ "U" ], "O", 17425 ]
[ 2, 3, 7 ][ 8672, 34748/3, 104448/7 ]
(2,3,7)-genus is at least 164

[ 5, 16, [ "U" ], "+", 52480 ]
[ 2, 3, 7 ][ 26144, 34956, 314808/7 ]
(2,3,7)-genus is at least 558

[ 4, 32, [ "L" ], "O", 33825 ]
[ 2, 3, 7 ][ 16896, 22528, 202752/7 ]
(2,3,7)-genus is at least 371

[ 4, 32, [ "O", "+" ], "+", 32736 ]
[ 2, 3, 7 ][ 32705/2, 65410/3, 196230/7 ]
(2,3,7)-genus is at least 360

[ 4, 32, [ "O", "-" ], "+", 32800 ]
[ 2, 3, 7 ][ 32767/2, 65534/3, 28086 ]
(2,3,7)-genus is at least 359

[ 4, 32, [ "S" ], "+", 523776 ]
[ 2, 3, 7 ][ 261376, 1045504/3, 3136512/7 ]
```

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(2,3,7)-genus is at least 5201

[ 4, 32, [ "S" ], "-", 524800 ]
[ 2, 3, 7 ][ 261888, 349184, 3142656/7 ]
(2,3,7)-genus is at least 5213

[ 2, 1024, [ "S" ], "+", 523776 ]
[ 2, 3, 7 ][ 261376, 1045504/3, 3136512/7 ]
(2,3,7)-genus is at least 5201

[ 2, 1024, [ "S" ], "-", 524800 ]
[ 2, 3, 7 ][ 261888, 349184, 3142656/7 ]
(2,3,7)-genus is at least 5213

[ 21, 2, [ "L" ], "O", 2097151 ]
[ 2, 3, 7 ][ 1040384, 4192256/3, 12576768/7 ]
(2,3,7)-genus is at least 20092

[ 7, 8, [ "L" ], "O", 299593 ]
[ 2, 3, 7 ][ 149504, 199680, 256356 ]
(2,3,7)-genus is at least 3178

[ 3, 128, [ "L" ], "O", 16513 ]
[ 2, 3, 7 ][ 8192, 11008, 99072/7 ]
(2,3,7)-genus is at least 165

[ 10, 3, [ "L" ], "O", 29524 ]
[ 2, 3, 7 ][ 14560, 19602, 176418/7 ]
(2,3,7)-genus is at least 160

[ 12, 3, [ "L" ], "O", 265720 ]
[ 2, 3, 7 ][ 131200, 176904, 227448 ]
(2,3,7)-genus is at least 2057

[ 12, 3, [ "O", "+" ], "O", 88816 ]
[ 2, 3, 7 ][ 43720, 58968, 75816 ]
(2,3,7)-genus is at least 437

[ 12, 3, [ "O", "+" ], "+", 88452 ]
[ 2, 3, 7 ][ 87197/2, 58806, 529254/7 ]
(2,3,7)-genus is at least 556

[ 12, 3, [ "O", "-" ], "O", 88330 ]
[ 2, 3, 7 ][ 87197/2, 58644, 527796/7 ]
(2,3,7)-genus is at least 492
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[ 12, 3, [ "O", "-" ], "+", 88695 ]
[ 2, 3, 7 ][ 43720, 58968, 75816 ]
(2,3,7)-genus is at least 558

[ 6, 9, [ "L" ], "O", 66430 ]
[ 2, 3, 7 ][ 32800, 44226, 56862 ]
(2,3,7)-genus is at least 515

[ 6, 9, [ "O", "+" ], "+", 29484 ]
[ 2, 3, 7 ][ 29069/2, 19602, 176418/7 ]
(2,3,7)-genus is at least 187

[ 6, 9, [ "O", "-" ], "+", 29565 ]
[ 2, 3, 7 ][ 14575, 19656, 25272 ]
(2,3,7)-genus is at least 188

[ 6, 9, [ "U" ], "O", 22204 ]
[ 2, 3, 7 ][ 10930, 14742, 18954 ]
(2,3,7)-genus is at least 110

[ 6, 9, [ "U" ], "+", 44226 ]
[ 2, 3, 7 ][ 43649/2, 29430, 264870/7 ]
(2,3,7)-genus is at least 322

[ 4, 27, [ "L" ], "O", 20440 ]
[ 2, 3, 7 ][ 10192, 13608, 17496 ]
(2,3,7)-genus is at least 209

[ 7, 5, [ "L" ], "O", 19531 ]
[ 2, 3, 7 ][ 9672, 13000, 117000/7 ]
(2,3,7)-genus is at least 164

[ 8, 5, [ "L" ], "O", 97656 ]
[ 2, 3, 7 ][ 48422, 65000, 585000/7 ]
(2,3,7)-genus is at least 842

[ 8, 5, [ "O", "+" ], "O", 19656 ]
[ 2, 3, 7 ][ 9672, 13000, 117000/7 ]
(2,3,7)-genus is at least 39

[ 8, 5, [ "O", "+" ], "+", 39000 ]
[ 2, 3, 7 ][ 19297, 77750/3, 233250/7 ]
(2,3,7)-genus is at least 269

[ 8, 5, [ "O", "-" ], "O", 19406 ]
[ 2, 3, 7 ][ 19219/2, 38500/3, 16500 ]
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(2,3,7)-genus is at least 67

[ 8, 5, [ "O", "-" ], "+", 39125 ]
[ 2, 3, 7 ][ 38719/2, 26000, 234000/7 ]
(2,3,7)-genus is at least 271

[ 4, 25, [ "L" ], "O", 16276 ]
[ 2, 3, 7 ][ 8112, 10816, 97500/7 ]
(2,3,7)-genus is at least 154

[ 4, 25, [ "U" ], "+", 13000 ]
[ 2, 3, 7 ][ 12949/2, 25898/3, 77850/7 ]
(2,3,7)-genus is at least 116

[ 9, 5, [ "L" ], "O", 488281 ]
[ 2, 3, 7 ][ 242172, 976250/3, 2928750/7 ]
(2,3,7)-genus is at least 4711

[ 9, 5, [ "O", "+" ], "O", 97656 ]
[ 2, 3, 7 ][ 96719/2, 65000, 585000/7 ]
(2,3,7)-genus is at least 811

[ 9, 5, [ "O", "+" ], "+", 195000 ]
[ 2, 3, 7 ][ 96547, 389750/3, 1169250/7 ]
(2,3,7)-genus is at least 1751

[ 9, 5, [ "O", "+" ], "-", 195625 ]
[ 2, 3, 7 ][ 193719/2, 391000/3, 1173000/7 ]
(2,3,7)-genus is at least 1759

[ 9, 5, [ "O", "-" ], "O", 97656 ]
[ 2, 3, 7 ][ 48422, 65000, 585000/7 ]
(2,3,7)-genus is at least 842

[ 9, 5, [ "O", "-" ], "+", 195625 ]
[ 2, 3, 7 ][ 193719/2, 391000/3, 1173000/7 ]
(2,3,7)-genus is at least 1759

[ 9, 5, [ "O", "-" ], "-", 195000 ]
[ 2, 3, 7 ][ 96547, 389750/3, 1169250/7 ]
(2,3,7)-genus is at least 1751

[ 3, 125, [ "L" ], "O", 15751 ]
[ 2, 3, 7 ][ 7812, 10500, 13500 ]
(2,3,7)-genus is at least 156
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[ 6, 7, [ "L" ], "O", 19608 ]
[ 2, 3, 7 ][ 9600, 12996, 16758 ]
(2,3,7)-genus is at least 70

[ 5, 11, [ "L" ], "O", 16105 ]
[ 2, 3, 7 ][ 7980, 32186/3, 13794 ]
(2,3,7)-genus is at least 148

[ 6, 11, [ "L" ], "O", 177156 ]
[ 2, 3, 7 ][ 87840, 354046/3, 151734 ]
(2,3,7)-genus is at least 1640

[ 6, 11, [ "O", "+" ], "O", 16226 ]
[ 2, 3, 7 ][ 7980, 32186/3, 13794 ]
(2,3,7)-genus is at least 27

[ 6, 11, [ "O", "+" ], "+", 80465 ]
[ 2, 3, 7 ][ 79727/2, 160688/3, 482064/7 ]
(2,3,7)-genus is at least 683

[ 6, 11, [ "O", "-" ], "O", 15984 ]
[ 2, 3, 7 ][ 15839/2, 31702/3, 95106/7 ]
(2,3,7)-genus is at least 54

[ 6, 11, [ "O", "-" ], "+", 80586 ]
[ 2, 3, 7 ][ 39924, 160930/3, 68970 ]
(2,3,7)-genus is at least 684

[ 3, 121, [ "L" ], "O", 14763 ]
[ 2, 3, 7 ][ 7320, 9840, 88572/7 ]
(2,3,7)-genus is at least 145

[ 3, 121, [ "U" ], "+", 13431 ]
[ 2, 3, 7 ][ 13309/2, 8952, 80580/7 ]
(2,3,7)-genus is at least 129

[ 16, 2, [ "L" ], "O", 65535 ]
[ 2, 4, 5 ][ 30720, 49016, 52428 ]
(2,4,5)-genus is at least 548

[ 16, 2, [ "O", "+" ], "O", 32895 ]
[ 2, 4, 5 ][ 15360, 24568, 26316 ]
(2,4,5)-genus is at least 228

[ 16, 2, [ "O", "+" ], "+", 32640 ]
[ 2, 4, 5 ][ 15312, 24417, 26112 ]
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(2,4,5)-genus is at least 282

[ 16, 2, [ "O", "-" ], "O", 32639 ]
[ 2, 4, 5 ][ 15312, 24412, 130556/5 ]
(2,4,5)-genus is at least 280

[ 16, 2, [ "O", "-" ], "+", 32896 ]
[ 2, 4, 5 ][ 15360, 24570, 131584/5 ]
(2,4,5)-genus is at least 229

[ 16, 2, [ "S" ], "+", 32640 ]
[ 2, 4, 5 ][ 15232, 24316, 130046/5 ]
(2,4,5)-genus is at least 140

[ 16, 2, [ "S" ], "-", 32896 ]
[ 2, 4, 5 ][ 15360, 24508, 26214 ]
(2,4,5)-genus is at least 146

[ 8, 4, [ "L" ], "O", 21845 ]
[ 2, 4, 5 ][ 10240, 16360, 17476 ]
(2,4,5)-genus is at least 194

[ 8, 4, [ "O", "+" ], "+", 16320 ]
[ 2, 4, 5 ][ 7656, 24447/2, 13056 ]
(2,4,5)-genus is at least 149

[ 8, 4, [ "O", "-" ], "+", 16448 ]
[ 2, 4, 5 ][ 7680, 24627/2, 65792/5 ]
(2,4,5)-genus is at least 129

[ 8, 4, [ "U" ], "O", 10965 ]
[ 2, 4, 5 ][ 5120, 8200, 8772 ]
(2,4,5)-genus is at least 82

[ 8, 4, [ "U" ], "+", 10880 ]
[ 2, 4, 5 ][ 5104, 8149, 8704 ]
(2,4,5)-genus is at least 100

[ 8, 4, [ "S" ], "+", 32640 ]
[ 2, 4, 5 ][ 15232, 24348, 130046/5 ]
(2,4,5)-genus is at least 156

[ 8, 4, [ "S" ], "-", 32896 ]
[ 2, 4, 5 ][ 15360, 24540, 26214 ]
(2,4,5)-genus is at least 162
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[ 4, 16, [ "S" ], "+", 32640 ]
[ 2, 4, 5 ][ 15232, 24348, 130046/5 ]
(2,4,5)-genus is at least 156

[ 4, 16, [ "S" ], "-", 32896 ]
[ 2, 4, 5 ][ 15360, 24540, 26214 ]
(2,4,5)-genus is at least 162

[ 2, 256, [ "S" ], "+", 32640 ]
(2,4,5)-genus is at least q= 256 NOT RELEVANT

[ 2, 256, [ "S" ], "-", 32896 ]
(2,4,5)-genus is at least q= 256 NOT RELEVANT

[ 10, 3, [ "L" ], "O", 29524 ]
[ 2, 3, 8 ][ 14560, 19602, 50483/2 ]
(2,3,8)-genus is at least 179

Need further checking for (2,3,7) systems:  [ [ [ [ "L" ], 14, 2, "O" ], -28 ], [ [ [ "U" ],

Need further checking for (2,4,5) systems:  [  ]

Need further checking for (2,3,8) systems:  [  ]
gap> LogTo();
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