Abstract. Theory of multi-parameter analysis has been a central subject in harmonic analysis and has received substantial progress in the past decades. Motivated by the classical $H^p$ ($p \leq 1$) product theory in the Euclidean spaces and the recent development of $L^p$ ($p > 1$) multi-parameter analysis on groups of stratified type, we build up the $H^p$ ($p \leq 1$) theory of multi-parameter analysis on spaces of homogeneous type. We first establish the Littlewood-Paley theory, Calderon reproducing formulas, and Plancherel-Polya inequalities on product spaces and then introduce and develop the product $H^p$ theory. Atomic decomposition is given on such $H^p$ spaces (see Theorem 4.3) and boundedness of singular integrals on such $H^p$ spaces and from $H^p$ to $L^p$ are established (see Theorems 5.1 and 5.2). A Journe type covering lemma is also proved in the product of two homogeneous spaces which is of its independent interest (see Lemma 4.2). Results in this paper grow out of the product $H^p$ theory of two stratified groups, such as the Heisenberg groups, developed earlier by the first two authors ([HL1], [HL2]).
1 Introduction

Our goal of this article is to develop the product theory on spaces of homogeneous type. The classical theory of Calderón-Zygmund was described by certain singular integral operators which commute with the one parameter dilations on $\mathbb{R}^n$, given by $\rho_\delta(x) = \delta x$ for all $\delta > 0$. The product theory of Fourier analysis on $\mathbb{R}^n$ emphasizes operators built out of product acting on each $\mathbb{R}^1$’s, and which commute with the action of multi-parameter scaling on $\mathbb{R}^n$, given by $\rho_\delta(x) = (\delta_1 x_1, \delta_2 x_2, \ldots, \delta_n x_n)$ with $\delta = (\delta_1, \delta_2, \ldots, \delta_n), \delta_i > 0$ for $1 \leq i \leq n$. In fact, this theory has a long history: beginning with the strong maximal function of Jessen, Marcinkiewicz and Zygmund ([JMZ]), the original form of the Marcinkiewicz multiplier theorem ([S1]). It was then later developed in the setting of product $L^p$ theory of Calderón-Zygmund operators (see R. Fefferman-Stein [FS] and Journe [J1]), and of Hardy space $H^p$ for $p < 1$ and $BMO$ spaces (see for example the works by Chang-R. Fefferman, Gundy-Stein, Journe, Pipher in [CF1], [CF2], [CF3], [F1], [F2], [F4], [Cha], [F6], [J2], [GS], [P]). Recently, the product $L^p$ ($p > 1$) theory plays a crucial role in the study of many questions arising in multi-parameter analysis, such as Marcinkiewicz multipliers and multi-parameter structures on Heisenberg-type groups (see works by Muller-Ricci-Stein in [MRS1, MRS2]), operators on nilpotent Lie groups given by convolution with certain flag singular kernels (see work by Nagel-Ricci-Stein in [NRS]), etc. More recently, to estimate fundamental solutions of $\Box_b$ on certain model domains in several complex variables, Nagel and Stein developed the product theory of singular integrals with non convolution kernels, namely the $L^p$ theory for $1 < p < \infty$ ([NS1, NS2, NS3]).

The main purpose of our paper is to develop a satisfactory product theory for $0 < p \leq 1$ on product of two spaces of homogeneous type, namely, the theory of Hardy spaces (including atomic decomposition) and boundeness of singular operators on such Hardy spaces $H^p$ and from $H^p$ to $L^p$. Results in this paper include the product $H^p$ theory, developed in [HL1] and [HL2], of two stratified groups such as the Heisenberg group as a special case. Our methods are quite different from those given in [NS3] for $1 < p < \infty$ and also in the classical product theory in Euclidean spaces in [CF1, CF2, CF3, F1, F4, F6] because we mainly establish the Hardy space theory using the Calderón reproducing formula and Littlewood-Paley analysis which hold in test function spaces in the product of homogeneous spaces, which are particularly suitable for the $H^p$ theory when $0 < p \leq 1$.

To see how our methods work, let us recall some basic ideas and results of the product theory on $\mathbb{R}^n$. The simplest example of a product-type singular integral on $\mathbb{R}^n$ is the double Hilbert transform $H_1H_2$ on $\mathbb{R}^2$ defined by

$$H_1H_2(f) = f * \frac{1}{x_1 x_2}.$$ 

For such tensor products the $L^p$-boundedness for $1 < p < \infty$ is trivial consequence of Fubini’s theorem. But for operators defined by $T(f) = f * K$ where $K$ is defined on $\mathbb{R}^n \times \mathbb{R}^m$ and satisfies all the analogous estimates to those satisfied by $\frac{1}{x_1 x_2}$, but cannot
be written in the tensor product form $K_1(x_1)K_2(x_2)$, then the arguments which deal with $H_1H_2$ fail. Fefferman and Stein developed a new method to deal with these more general product-type operators with convolutional kernels. The basic idea they used is to develop the product-type Littlewood-Paley theory for $L^p(\mathbb{R}^n \times \mathbb{R}^m)$, $1 < p < \infty$. This follows from the original vector-valued Littlewood-Paley theory on $\mathbb{R}^n$ and an iteration argument. Their methods work for product-type convolutional operators very well, see [FS] for more details. Furthermore, Journé considered general product-type operators with non-convolutional kernels. He proved the $T1$ theorem in the product setting. A new idea Journé used is the vector-valued $T1$ theorem on $\mathbb{R}^n$ and a basic result he proved is the so-called Journé's covering lemma. We also refer to the work of Pipher [P] for Journé type covering lemmas of more parameters.

The product $H^p$ theory has an extensive history. A counterexample given by Carleson showed that the product $H^p$ theory cannot be obtained by a routine matter of iterating one dimensional methods ([Car]). Gundy and Stein established in [GS] the product $H^p$ theory by using the non-tangential maximal function and the area integral or their probabilistic analogues resulted by introducing two-time Brownian motion, i.e., the martingale maximal function, and the corresponding square function. Chang and Fefferman obtained atomic decomposition of the product $H^p$ spaces ([CF1]). The key tool they used is Calderón reproducing formula on product $\mathbb{R}^n$, which follows from using the Fourier transform. Since the support of each atom is an open set in the product $\mathbb{R}^n$, one cannot use atomic decomposition to get the $H^p - L^p$ boundedness of product Calderón-Zygmund operators while this worked very well on the single space $\mathbb{R}^n$. Nevertheless, R. Fefferman found that it would suffice to check the action of operators on atoms whose supports are rectangles in the product space to show the boundedness of operators on the product $H^p$ spaces ([F4]). As a consequence, R. Fefferman proved the $H^p - L^p$ boundedness for a class of operators introduced by Journé who proved the $L^\infty - BMO$ boundedness. The Journé’s covering lemma plays a crucial role in the proofs of these results ([J1, J2]).

In the recent paper ([NS3]), Nagel and Stein considered the product space $M = M_1 \times M_2 \times \cdots \times M_n$ where each factor $M_i$, $1 \leq i \leq n$, is either a compact connected smooth manifold, or arises as the boundary of a model polynomial domain in $\mathbb{C}^2$. Both are associated with the real vector fields which are of finite-type. They proved the $L^p$ boundedness of certain operators on these product spaces. The key idea of the proof is to use the product-type Littlewood-Paley theory on $L^p$, $1 < p < \infty$. A crucial role to develop this theory is a reproducing formula which is constructed by use of heat kernel on each $M_i$ and then product-type reproducing formula is a tensor product. Since their reproducing formula holds only on $L^2$, so it is not sufficient to use this reproducing formula to develop the product $H^p(M)$ theory for $0 < p \leq 1$ in this setting.

To develop the product theory on spaces of homogeneous type, we first consider that $X = X_1 \times X_2$, where $(X_i, \rho_i, \mu_i)_{d_i, \theta_i}$, $i = 1, 2$, are spaces of homogeneous type in the sense of Coifman and Weiss ([CW1, CW2]; see also definition below). It was proved that if
\{S_{k_i}\}_{k_i \in \mathbb{Z}}, \ i = 1, 2, \text{is an approximation to the identity on } X_i, \text{ and set } D_{k_i} = S_{k_i} - S_{k_i-1} \text{for } k_i \in \mathbb{Z}, \text{then the following Calderón reproducing formulae hold}

\begin{align*}
(1.1) \quad f(x) &= \sum_{k_i = -\infty}^{\infty} \tilde{D}_{k_i} D_{k_i}(f)(x) = \sum_{k_i = -\infty}^{\infty} D_{k_i} \overline{D}_{k_i}(f)(x),
\end{align*}

where the series converge both in $L^p(X_i)$ with $1 < p < \infty$ and in some test function spaces and its dual spaces on $X_i$, $i = 1, 2$, see [HS, H1] for more details.

The main tool to develop the product theory on $X$ is the following product-type Calderón reproducing formula

\begin{align*}
(1.2) \quad f(x_1, x_2) &= \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2}(f)(x_1, x_2) \\
&= \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} D_{k_1} D_{k_2} \overline{D}_{k_1} \overline{D}_{k_2}(f)(x_1, x_2),
\end{align*}

where the series converge in $L^p(X)$, $1 < p < \infty$, in product test function spaces and its dual spaces on $X$.

This formula together with the vector-valued Littlewood-Paley theory and an iteration argument yields the product Littlewood-Paley theory for $L^p(X)$, $1 < p < \infty$. To establish $H^p(X)$ space, we formally introduce the product Littlewood-Paley-Stein $S$ function and define the product $H^p(X)$ norm for distribution space mentioned above. As in the product $\mathbb{R}^n$ case, the formula in (1.2) is a key tool to obtain atomic decomposition of $H^p(X)$ space. To see the Littlewood-Paley-Stein $S$ function and $g$ function are equivalent on $H^p(X)$, one needs the discrete product-type Calderón reproducing formula. In fact, the following discrete Calderón reproducing formula on each $(X_i, \rho_i, \mu_i)_{d_i, \theta_i}, \ i = 1, 2$, was proved in [H3]:

\begin{align*}
(1.3) \quad f(x) &= \sum_{k = -\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu = 1}^{N(k, \tau)} \mu(Q_{\nu}^{k, \nu}) \tilde{D}_{k}(x, y_{\tau}^{k, \nu}) D_{k}(f)(y_{\tau}^{k, \nu}) \\
&= \sum_{k = -\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu = 1}^{N(k, \tau)} \mu(Q_{\nu}^{k, \nu}) D_{k}(x, y_{\tau}^{k, \nu}) \overline{D}_{k}(f)(y_{\tau}^{k, \nu}),
\end{align*}

where $\{Q_{\nu}^{k, \nu}\}_{k \in \mathbb{Z}, \tau \in I_k, \nu = 1, \ldots, N(k, \tau)}$ is the collection of dyadic cubes in the sense of Christ ([Chr1, Chr2]), Sawyer-Wheeden ([SW]) and the series converge in $L^p(X_i)$ with $1 < p < \infty$, in some test function spaces and its dual spaces on $X_i$ and $i = 1, 2$, see [H2] for more details.
Similarly, the discrete product-type Calderón reproducing formula is given by

\[(1.4) \quad f(x_1, x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{N(k_1, \tau_1)} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{N(k_2, \tau_2)} \mu_1(Q_{\tau_1}^{k_1, \nu_1}) \mu_2(Q_{\tau_2}^{k_2, \nu_2})
\times \widetilde{D}_{k_1}(x_1, y_{\tau_1}^{k_1, \nu_1}) \widetilde{D}_{k_2}(x_2, y_{\tau_2}^{k_2, \nu_2}) D_{k_1} D_{k_2}(f)(y_{\tau_1}^{k_1, \nu_1}, y_{\tau_2}^{k_2, \nu_2})
\times D_{k_1}(x_1, y_{\tau_1}^{k_1, \nu_1}) D_{k_2}(x_2, y_{\tau_2}^{k_2, \nu_2}) \mathcal{T}_{k_1} \mathcal{T}_{k_2}(f)(y_{\tau_1}^{k_1, \nu_1}, y_{\tau_2}^{k_2, \nu_2}),\]

where the series converge in $L^p(X)$, $1 < p < \infty$, in product test function spaces and its dual spaces on $X$.

The formula of (1.4) implies the so-called Plancherel-Pólya inequalities. As a simple consequence, we show the above $H^p(X)$ space can be characterized by the product Littlewood-Paley-Stein $g$ function. Using Littlewood-Paley-Stein $g$ function, we will also prove the product $T1$ theorem of the Calderón-Zygmund operators.

To explain how our results include the product $H^p$ theory on two stratified groups such as the Heisenberg group, we give some preliminary introduction here.

We begin with some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to the books [FS] and [VSCC] for analysis on stratified groups. Let $G$ be a finite-dimensional, stratified, nilpotent Lie algebra. Assume that

$G = \oplus_{i=1}^{s} V_i$,

with $[V_i, V_j] \subset V_{i+j}$ for $i + j \leq s$ and $[V_i, V_j] = 0$ for $i + j > s$. Let $X_1, \cdots, X_l$ be a basis for $V_1$ and suppose that $X_1, \cdots, X_l$ generate $G$ as a Lie algebra. Then for $2 \leq j \leq s$, we can choose a basis $\{X_{i_j}\}$, $1 \leq i \leq k_j$, for $V_j$ consisting of commutators of length $j$. We set $X_{i_1} = X_i$, $i = 1, \cdots, l$ and $k_1 = l$, and we call $X_{i_1}$ a commutator of length 1.

If $G$ is the simply connected Lie group associated with $\mathcal{G}$, then the exponential mapping is a global diffeomorphism from $\mathcal{G}$ to $G$. Thus, for each $g \in G$, there is $x = (x_{ij}) \in R^N$ for $1 \leq i \leq k_j, 1 \leq j \leq s$ and $N = \sum_{j=1}^{s} k_j$ such that

$g = \exp(\sum x_{ij} X_{ij})$.

A homogeneous norm function $| \cdot |$ on $G$ is defined by

$|g| = (\sum |x_{ij}|^{2s/j})^{1/2s}$,

and $Q = \sum_{j=1}^{s} jk_j$ is said to be the homogeneous dimension of $G$. The dilation $\delta_r$ on $G$ is defined by

$\delta_r(g) = \exp(\sum r^j x_{ij} X_{ij})$ if $g = \exp(\sum x_{ij} X_{ij})$. 

We call a curve $\gamma : [a, b] \rightarrow \mathbb{G}$ "a horizontal curve" connecting two points $x, y \in \mathbb{G}$ if $\gamma(a) = x$, $\gamma(b) = y$ and $\gamma'(t) \in V_1$ for all $t$. Then the Carnot-Caratheodory distance between $x, y$ is defined as

$$d_{cc}(x, y) = \inf_{\gamma} \int_a^b < \gamma'(t), \gamma'(t) >^{\frac{1}{2}} dt,$$

where the infimum is taken over all horizontal curves $\gamma$ connecting $x$ and $y$. It is known that any two points $x, y$ on $\mathbb{G}$ can be joined by a horizontal curve of finite length and then $d_{cc}$ is a left invariant metric on $\mathbb{G}$. We can define the metric ball centered at $x$ and with radius $r$ associated with this metric by

$$B_{cc}(x, r) = \{ y : d_{cc}(x, y) < r \}.$$

We must notice that this metric $d_{cc}$ is equivalent to the pseudo-metric $\rho(x, y) = |x^{-1}y|$ defined by the homogeneous norm $| \cdot |$ in the following sense (see [FS])

$$C \rho(x, y) \leq d_{cc}(x, y) \leq C \rho(x, y).$$

We denote the metric ball associated with $\rho$ as $D(x, r) = \{ y \in \mathbb{G} : \rho(x, y) < r \}$. An important feature of both of these distance functions is that these distances and thus the associated metric balls are left invariant, namely,

$$d_{cc}(xz, zy) = d(x, y), B_{cc}(x, r) = xB_{cc}(0, r)$$

and

$$\rho(zx, zy) = \rho(x, y), D(x, r) = xD(0, r).$$

For simplicity, we will use the left invariant metric $d_{cc}$ to study the product theory of two stratified groups. An important property of the metric ball is that

$$\mu(B_{cc}(x, r)) = c_Q r^Q$$

for all $x \in \mathbb{G}$ and $r > 0$, where $\mu$ is the Lebesgue measure on $\mathbb{G}$ and $Q$ is the homogeneous dimension. Therefore, the space $(\mathbb{G}, d_{cc}, \mu)$ is a space of homogeneous type.

If we consider two stratified groups $(\mathbb{G}_1, d^1_{cc}, \mu)$ and $(\mathbb{G}_2, d^2_{cc}, \mu)$, the product $H^p$ theory developed in this paper includes the case of product theory on $G_1 \times G_2$ as a special case. Of particular interests are the case $H^p(G_1 \times G_2)$ when $G_1$ or $G_2$ is the renowned Heisenberg group. Such product $H^p$ theory was developed earlier by the first two authors in ([HL1], [HL2]). It is this work which motivated the generalization to the $H^p$ product theory of two homogeneous spaces in the current paper.

The following final remarks are in order. First of all, as we pointed out at the beginning of the introduction, the methods employed in this paper are different from those classical product Hardy space theory in several ways. We develop a discrete Calderon
reproducing formula and establish a Plancherel-Polya inequalities in product spaces. These tools are used for the first time in product spaces. Indeed, we adapt successfully to the case of product spaces the methods of the Littlewood-Paley theory, Calderon reproducing formula, Plancherel-Pólya inequalities in the single homogeneous spaces developed by the first author over the past decade (see e.g., [H1], [H2], [H3]). Second, we would like to point out that this paper is a substantial extension and expansion of the earlier unpublished manuscripts by the first two authors in the case of the product theory of two stratified groups about ten years ago (see [HL1], [HL2]).\(^1\) We were motivated then by the important development of multi-parameter analysis on \(R^n\) by R. Fefferman and Pipher ([FP1] and [FP2]), Ricci and Stein ([RS]), and Nagel-Ricci-Stein ([NRS]), and Nagel and Stein ([NS1], [NS2], [NS3], [NS4]), on Heisenberg-type groups of Muller-Ricci-Stein ([MRS1], [MRS2]). Third, on the one hand for the convenience of the reader, we have made every effort to make our presentation in this paper self-contained. On the other hand, we have strived to simplify the exposition so that similar estimates which appear in one part of this paper will be very brief in other parts of the paper. Therefore, we must apologize for the lengthy computations and estimates given in some parts of this paper due to the very complicated nature of product theory itself, and also some somewhat concise exposition in other parts. We hope that an interested reader will be patient enough while reading this paper.

A brief description of the contents of this paper follows. In Section 2 we first introduce a class of test functions and its dual space (distribution space), and then establish a special product-type Calderón reproducing formula. In Section 3, we develop the product Littlewood-Paley-Stein theory for \(L^p\), \(1 < p < \infty\). The product \(H^p\) space is established in Section 4, and we prove the Journé’s covering lemma for product spaces of homogeneous type (Lemma 4.2). Atomic decomposition for \(H^p\) spaces is proved in this section (See Theorem 4.3). The boundedness of Calderón-Zygmund operators on \(H^p\) space and from \(H^p\) to \(L^p\) are derived in Section 5 (See Theorems 5.1 and 5.2).

\(^1\)Those works in that framework have been presented by the second author in the invited talks at the international conference in harmonic analysis in Kiel, Germany in 1998 and also in the AMS special session of harmonic analysis in Chicago in 1999.
2 Special Calderón reproducing formulae

We begin with recalling some necessary definitions and notation on spaces of homogeneous type.

A quasi-metric \( \rho \) on a set \( X \) is a function \( \rho : X \times X \to [0, \infty) \) satisfying that

(i) \( \rho(x, y) = 0 \) if and only if \( x = y \);

(ii) \( \rho(x, y) = \rho(y, x) \) for all \( x, y \in X \);

(iii) There exists a constant \( A \in [1, \infty) \) such that for all \( x, y, z \in X \),

\[
\rho(x, y) \leq A[\rho(x, z) + \rho(z, y)].
\]

Any quasi-metric defines a topology, for which the balls

\[
B(x, r) = \{y \in X : \rho(y, x) < r\}
\]

for all \( x \in X \) and all \( r > 0 \) form a basis.

In what follows, we set \( \operatorname{diam} X = \sup\{\rho(x, y) : x, y \in X\} \) and \( \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \). We also make the following conventions. We denote by \( f \sim g \) that there is a constant \( C > 0 \) independent of the main parameters such that \( C^{-1}g < f < Cg \). Throughout the paper, we denote by \( C \) a positive constant which is independent of the main parameters, but it may vary from line to line. Constants with subscripts, such as \( C_1 \), do not change in different occurrences. For any \( q \in [1, \infty] \), we denote by \( q' \) its conjugate index, namely, \( 1/q + 1/q' = 1 \). Let \( A \) be a set and we will denote by \( \chi_A \) the characteristic function of \( A \).

**Definition 2.1** Let \( d > 0 \) and \( \theta \in (0, 1] \). A space of homogeneous type, \((X, \rho, \mu)_{d, \theta}\), is a set \( X \) together with a quasi-metric \( \rho \) and a nonnegative Borel regular measure \( \mu \) on \( X \), and there exists a constant \( C_0 > 0 \) such that for all \( 0 < r < \operatorname{diam} X \) and all \( x, x', y \in X \),

\[
\mu(B(x, r)) \sim r^d
\]

and

\[
|\rho(x, y) - \rho(x', y)| \leq C_0\rho(x, x')^\theta[\rho(x, y) + \rho(x', y)]^{1-\theta}.
\]

The space of homogeneous type was first introduced by Coifman and Weiss [CW1] and its theory has developed significantly in the past three decades. For a variant of the space of homogeneous type as given in the above definition, we refer to [MS1]. In [MS1], Macias and Segovia have proved that one can replace the quasi-metric \( \rho \) of the space of homogeneous type in the sense of Coifman and Weiss by another quasi-metric \( \bar{\rho} \) which yields the same topology on \( X \) as \( \rho \) such that \((X, \bar{\rho}, \mu)\) is the space defined by Definition 2.1 with \( d = 1 \).

Throughout this section to Section 6, we will always assume that \( \mu(X) = \infty \).

Let us now recall the definition of the space of test functions on spaces of homogeneous type.
**Definition 2.2** ([H1]) Let $X$ be a space of homogeneous type as in Definition 2.1. Fix $\gamma > 0$ and $\beta > 0$. A function $f$ defined on $X$ is said to be a test function of type $(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$, if $f$ satisfies the following conditions:

1. $|f(x)| \leq C \frac{r^\gamma}{(r + \rho(x, x_0))^{d+\gamma}}$,
2. $|f(x) - f(y)| \leq C \left( \frac{\rho(x, y)}{r + \rho(x, x_0)} \right)^\beta \frac{r^\gamma}{(r + \rho(x, x_0))^{d+\gamma}}$
   
   for $\rho(x, y) \leq \frac{1}{2A} [r + \rho(x, x_0)]$;

3. $\int_X f(x) \, d\mu(x) = 0$.

If $f$ is a test function of type $(x_0, r, \beta, \gamma)$, we write $f \in G(x_0, r, \beta, \gamma)$, and the norm of $f$ in $G(x_0, r, \beta, \gamma)$ is defined by

$$
\|f\|_{G(x_0, r, \beta, \gamma)} = \inf\{C : (i) \text{ and } (ii) \text{ hold}\}.
$$

Now fix $x_0 \in X$ and let $G(\beta, \gamma) = G(x_0, 1, \beta, \gamma)$. It is easy to see that

$$
G(x_1, r, \beta, \gamma) = G(\beta, \gamma)
$$

with an equivalent norm for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $G(\beta, \gamma)$ is a Banach space with respect to the norm in $G(\beta, \gamma)$. Also, let the dual space $(G(\beta, \gamma))'$ be all linear functionals $L$ from $G(\beta, \gamma)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in G(\beta, \gamma)$,

$$
|L(f)| \leq C \|f\|_{G(\beta, \gamma)}.
$$

We denote by $\langle h, f \rangle$ the natural pairing of elements $h \in (G(\beta, \gamma))'$ and $f \in G(\beta, \gamma)$. Clearly, for all $h \in (G(\beta, \gamma))'$, $\langle h, f \rangle$ is well defined for all $f \in G(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$.

It is well-known that even when $X = \mathbb{R}^n$, $G(\beta, \gamma)$ is not dense in $G(\beta_2, \gamma)$ if $\beta_1 > \beta_2$, which will bring us some inconvenience. To overcome this defect, in what follows, for a given $\epsilon \in (0, \theta]$, we let $\hat{G}(\beta, \gamma)$ be the completion of the space $G(\epsilon, \epsilon)$ in $G(\beta, \gamma)$ when $0 < \beta, \gamma < \epsilon$.

**Definition 2.3** ([H1]) Let $X$ be a space of homogeneous type as in Definition 2.1. A sequence $\{S_k\}_{k \in \mathbb{Z}}$ of linear operators is said to be an approximation to the identity of order $\epsilon \in (0, \theta]$ if there exists $C_1 > 0$ such that for all $k \in \mathbb{Z}$ and all $x, x', y$ and $y' \in X$, $S_k(x, y)$, the kernel of $S_k$ is a function from $X \times X$ into $\mathbb{C}$ satisfying

1. $|S_k(x, y)| \leq C_1 \frac{2^{-k \epsilon}}{(2^{-k} + \rho(x, y))^{d+\epsilon}}$.
By Coifman’s construction in [DJS], one can construct an approximation
Let
\[ S \]
that for all
\[ X \]
×
Lemma 2.1
Remark 2.1
(10)
(5)
(4)
(8)
(7)
\[ S_k(x, y) = 0 \]
if \( \rho(x, y) \geq C_22^{-k} \) and \( \|S_k\|_{L^\infty(X \times X)} \leq C_32^{kd} \);
\[ |S_k(x, y) - S_k(x', y)| \leq C_32^{k(d+\epsilon)}\rho(x, x')\epsilon; \]
\[ |S_k(x, y) - S_k(x, y')| \leq C_32^{k(d+\epsilon)}\rho(y, y')\epsilon; \]
(10) \[ ||S_k(x, y) - S_k(x', y')|| - |S_k(x', y) - S_k(x', y')|| \leq C_32^{k(d+2\epsilon)}\rho(x, x')\epsilon\rho(y, y')\epsilon. \]

**Remark 2.1** By Coifman’s construction in [DJS], one can construct an approximation to the identity of order \( \theta \) having compact support satisfying the above Definition 2.3.

We now recall the continuous Calderón reproducing formulae on spaces of homogeneous type in [HS, H1].

**Lemma 2.1** Let \( X \) be a space of homogeneous type as in Definition 2.1, \( \epsilon \in (0, \theta] \), \( \{S_k\}_{k \in \mathbb{Z}} \) be an approximation to the identity of order \( \epsilon \) and \( D_k = S_k - S_{k-1} \) for \( k \in \mathbb{Z} \). Then there
are families of linear operators \( \{ \tilde{D}_k \}_{k \in \mathbb{Z}} \) and \( \{ D_k \}_{k \in \mathbb{Z}} \) such that for all \( f \in \mathcal{G}(\beta, \gamma) \) with \( \beta, \gamma \in (0, \epsilon) \),

\[
(2.4) \quad f = \sum_{k=-\infty}^{\infty} \tilde{D}_k D_k(f) = \sum_{k=-\infty}^{\infty} D_k \tilde{D}_k(f),
\]

where the series converge in the norm of both the space \( \mathcal{G}(\beta', \gamma') \) with \( 0 < \beta' < \beta \) and \( 0 < \gamma' < \gamma \) and the space \( L^p(X) \) with \( p \in (1, \infty) \). Moreover, \( \tilde{D}_k(x, y) \), the kernel of \( \tilde{D}_k \) for all \( k \in \mathbb{Z} \) satisfies the conditions (i) and (ii) of Definition 2.3 with \( \epsilon \) replaced by any \( \epsilon' \in (0, \epsilon) \), and

\[
(2.5) \quad \int_X \tilde{D}_k(x, y) \, d\mu(y) = 0 = \int_X \tilde{D}_k(x, y) \, d\mu(x);
\]

\( D_k(x, y) \), the kernel of \( D_k \) satisfies the conditions (i) and (iii) of Definition 2.3 with \( \epsilon \) replaced by any \( \epsilon' \in (0, \epsilon) \) and (2.5).

By an argument of duality, Han and Sawyer in [HS, H1] also establish the following continuous Calderón reproducing formulae on spaces of distributions, \( \left( \tilde{\mathcal{G}}(\beta, \gamma) \right)' \) with \( \beta, \gamma \in (0, \epsilon) \).

**Lemma 2.2** With all the notation as in Lemma 2.1, then for all \( f \in \left( \tilde{\mathcal{G}}(\beta, \gamma) \right)' \) with \( \beta, \gamma \in (0, \epsilon) \), (2.4) holds in \( \left( \tilde{\mathcal{G}}(\beta', \gamma') \right)' \) with \( \beta < \beta' < \epsilon \) and \( \gamma < \gamma' < \epsilon \).

Let now \( (X_i, \rho_i, \mu_i)_{d_i, \theta_i} \) for \( i = 1, 2 \) be two spaces of homogeneous type as in Definition 2.1 and \( \rho_i \) satisfies (2.1) with \( A \) replaced by \( A_i \) for \( i = 1, 2 \). We now introduce the space of test functions on the product space \( X_1 \times X_2 \) of spaces of homogeneous type.

**Definition 2.4** For \( i = 1, 2 \), fix \( \gamma_i > 0 \) and \( \beta_i > 0 \). A function \( f \) defined on \( X_1 \times X_2 \) is said to be a test function of type \((\beta_1,\beta_2,\gamma_1,\gamma_2)\) centered at \((x_0,y_0)\in X_1 \times X_2\) with width \( r_1, r_2 > 0 \) if \( f \) satisfies the following conditions:

(i) \( |f(x,y)| \leq C \frac{r_1^{\gamma_1}}{(r_1 + \rho_1(x,x_0))^{d_1+\gamma_1}} \frac{r_2^{\gamma_2}}{(r_2 + \rho_2(y,y_0))^{d_2+\gamma_2}} \);

(ii) \( |f(x,y) - f(x',y)| \leq C \frac{\rho_1(x,x')}{(r_1 + \rho_1(x,x_0))^{d_1+\gamma_1}} \frac{r_1^{\gamma_1}}{(r_1 + \rho_1(x,x_0))^{d_1+\gamma_1}} \frac{r_2^{\gamma_2}}{(r_2 + \rho_2(y,y_0))^{d_2+\gamma_2}} \)

for \( \rho_1(x,x') \leq \frac{1}{2A_1} |r_1 + \rho_1(x,x_0)| \);

(iii) \( |f(x,y) - f(x,y')| \leq C \frac{r_1^{\gamma_1}}{(r_1 + \rho_1(x,x_0))^{d_1+\gamma_1}} \frac{\rho_2(y,y')}{(r_2 + \rho_2(y,y_0))^{d_2+\gamma_2}} \frac{r_2^{\gamma_2}}{(r_2 + \rho_2(y,y_0))^{d_2+\gamma_2}} \)

for \( \rho_2(y,y') \leq \frac{1}{2A_2} |r_2 + \rho_2(y,y_0)| \);
In the sequel, if which is the set of all linear functionals \( G \)

**Remark 2.2**

We denote by \( h, f \in G \) with an equivalent norm for all \( (\beta_1,\beta_2;\gamma_1,\gamma_2) \) centered at \( (x_0,y_0) \in X_1 \times X_2 \) with width \( r_1, r_2 > 0 \), we write \( f \in G(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2) \) and we define the norm of \( f \) by

\[
\|f\|_{G(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2)} = \inf\{C: (i), (ii), (iii) and (iv) hold\}.
\]

**Remark 2.2** In the sequel, if \( \beta_1 = \beta_2 = \beta \) and \( \gamma_1 = \gamma_2 = \gamma \), we will then simply write

\[
f \in G(x_0,y_0;r_1,r_2;\beta;\gamma)
\]

and similar for any other parameter.

We now denote by \( G(\beta_1,\beta_2;\gamma_1,\gamma_2) \) the class of \( G(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2) \) with \( r_1 = r_2 = 1 \) for fixed \( (x_0,y_0) \in X_1 \times X_2 \). It is easy to see that

\[
G(x_1,y_1;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2) = G(\beta_1,\beta_2;\gamma_1,\gamma_2)
\]

with an equivalent norm for all \( (x_1,y_1) \in X_1 \times X_2 \). We can easily check that the space \( G(\beta_1,\beta_2;\gamma_1,\gamma_2) \) is a Banach space. Also, we denote by \( (G(\beta_1,\beta_2;\gamma_1,\gamma_2))^\prime \) its dual space which is the set of all linear functionals \( \mathcal{L} \) from \( G(\beta_1,\beta_2;\gamma_1,\gamma_2) \) to \( \mathbb{C} \) with the property that there exists \( C \geq 0 \) such that for all \( f \in G(\beta_1,\beta_2;\gamma_1,\gamma_2) \),

\[
|\mathcal{L}(f)| \leq C\|f\|_{G(\beta_1,\beta_2;\gamma_1,\gamma_2)}.
\]

We denote by \( \langle h,f \rangle \) the natural pairing of elements \( h \in (G(\beta_1,\beta_2;\gamma_1,\gamma_2))^\prime \) and \( f \in G(\beta_1,\beta_2;\gamma_1,\gamma_2) \). Clearly, for all \( h \in (G(\beta_1,\beta_2;\gamma_1,\gamma_2))^\prime \), \( \langle h,f \rangle \) is well defined for all \( f \in G(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2) \) with \( (x_0,y_0) \in X_1 \times X_2 \), \( r_1 > 0 \) and \( r_2 > 0 \). By the same reason as the case of non product spaces, we denote by \( \hat{G}(\beta_1,\beta_2;\gamma_1,\gamma_2) \) the completion of the space \( G(\epsilon_1,\epsilon_2) \) in \( G(\beta_1,\beta_2;\gamma_1,\gamma_2) \) when \( 0 < \beta_1, \gamma_1 < \epsilon_1 \) and \( 0 < \beta_2, \gamma_2 < \epsilon_2 \).
Lemma 2.3 Let \((x_1, x_2) \in X_1 \times X_2, r_i > 0, \epsilon_i \in (0, \theta_i]\) and \(0 < \beta_i, \gamma_i < \epsilon_i\) for \(i = 1, 2\). If the linear operators \(T_1\) and \(T_2\) are respectively bounded on the spaces \(G(x_1, r_1, \beta_1, \gamma_1)\) and \(G(x_2, r_2, \beta_2, \gamma_2)\) with operator norms \(C_{4.1}\) and \(C_{4.2}\), then the operator \(T_1 T_2\) is bounded on \(G(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)\) with an operator norm \(C_{4.1} C_{4.2}\).

Proof. Let \(f \in G(x_1, x_2; r_1, r_2, \beta_1, \beta_2, \gamma_1, \gamma_2)\). By Definition 2.4, we see that for any fixed \(x \in X_1, f(x, \cdot) \in G(x_2, r_2, \beta_2, \gamma_2)\) and

\[
\|f(x, \cdot)\|_{G(x_2, r_2, \beta_2, \gamma_2)} \leq \|f\|_{G(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} \frac{r_1^\gamma_1}{(r_1 + \rho_1(x, x_1))^{d_1 + \gamma_1}};
\]

and for any \(x, x' \in X_1\) with \(\rho(x, x') \leq \frac{1}{2A_1}[r_1 + \rho_1(x, x_1)]\), \(f(x, \cdot) - f(x', \cdot) \in G(x_2, r_2, \beta_2, \gamma_2)\)

and

\[
\|f(x, \cdot) - f(x', \cdot)\|_{G(x_2, r_2, \beta_2, \gamma_2)} \leq \|f\|_{G(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} \frac{r_1^\gamma_1}{(r_1 + \rho_1(x, x_1))^{d_1 + \gamma_1}}.
\]

The assumption on \(T_2\), (2.6) and (2.7) yield that for any \(x \in X_1, T_2 f(x, \cdot) \in G(x_2, r_2, \beta_2, \gamma_2)\) and

\[
\|T_2 f(x, \cdot)\|_{G(x_2, r_2, \beta_2, \gamma_2)} \leq C_{4.2} \|f\|_{G(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} \frac{r_1^\gamma_1}{(r_1 + \rho_1(x, x_1))^{d_1 + \gamma_1}};
\]

and for any \(x, x' \in X_1\) with \(\rho(x, x') \leq \frac{1}{2A_1}[r_1 + \rho_1(x, x_1)]\), \(T_2 f(x, \cdot) - T_2 f(x', \cdot) \in G(x_2, r_2, \beta_2, \gamma_2)\)

and

\[
\|T_2 f(x, \cdot) - T_2 f(x', \cdot)\|_{G(x_2, r_2, \beta_2, \gamma_2)} \leq C_{4.2} \|f\|_{G(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} \frac{r_1^\gamma_1}{(r_1 + \rho_1(x, x_1))^{d_1 + \gamma_1}}.
\]

The estimates (2.8) and (2.9) imply that for any \(y \in X_2, T_2 f(\cdot, y) \in G(x_1, r_1, \beta_1, \gamma_1)\) and

\[
\|T_2 f(\cdot, y)\|_{G(x_1, r_1, \beta_1, \gamma_1)} \leq C_{4.2} \|f\|_{G(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} \frac{r_2^\gamma_2}{(r_2 + \rho_2(y, x_2))^{d_2 + \gamma_2}};
\]

and for any \(y, y' \in X_2\) with \(\rho(y, y') \leq \frac{1}{2A_2}[r_2 + \rho_2(y, x_2)]\), \(T_2 f(\cdot, y) - T_2 f(\cdot, y') \in G(x_1, r_1, \beta_1, \gamma_1)\)

and

\[
\|T_2 f(\cdot, y) - T_2 f(\cdot, y')\|_{G(x_1, r_1, \beta_1, \gamma_1)} \leq C_{4.2} \|f\|_{G(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} \frac{r_2^\gamma_2}{(r_2 + \rho_2(y, x_2))^{d_2 + \gamma_2}}.
\]
From the assumption on $T_1$, (2.10) and (2.11), it follows that for any $y \in X_2$, $T_1 T_2 f(\cdot, y) \in \mathcal{G}(x_1, r_1, \beta_1, \gamma_1)$ and

\[(2.12) \quad \|T_1 T_2 f(\cdot, y)\|_{\mathcal{G}(x_1, r_1, \beta_1, \gamma_1)} \leq C_{4,1} C_{4,2} \|f\|_{\mathcal{G}(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} \frac{r_2^{\gamma_2}}{(r_2 + \rho_2(y, x_2))^{d_2 + \gamma_2}};\]

and for any $y, y' \in X_2$ with $\rho(y, y') \leq \frac{1}{2^{N_1}} [r_2 + \rho_2(y, x_2)]$, $T_1 T_2 f(\cdot, y) - T_1 T_2 f(\cdot, y') \in \mathcal{G}(x_1, r_1, \beta_1, \gamma_1)$ and

\[(2.13) \quad \|T_1 T_2 f(\cdot, y) - T_1 T_2 f(\cdot, y')\|_{\mathcal{G}(x_1, r_1, \beta_1, \gamma_1)} \leq C_{4,1} C_{4,2} \|f\|_{\mathcal{G}(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} \times \left( \frac{\rho_2(y, y')}{r_2 + \rho_2(y, x_2)} \right)^{\beta_2} \frac{r_2^{\gamma_2}}{(r_2 + \rho_2(y, x_2))^{d_2 + \gamma_2}}.\]

The estimates (2.12) and (2.13) actually tell us that $T_1 T_2 f \in \mathcal{G}(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ and

\[
\|T_1 T_2 f\|_{\mathcal{G}(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} \leq C_{4,1} C_{4,2} \|f\|_{\mathcal{G}(x_1, x_2; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)};
\]

which completes the proof of Lemma 2.3.

To establish special continuous Calderón reproducing formulae on the product spaces $X_1 \times X_2$, we first need to recall some details of the proof of the same formulae for the non-product space case in [H1], namely Lemma 2.1. One of the keys for establishing these formulae is Coifman’s idea in [DJS]. Let $X$ be a space of homogeneous type as in Definition 2.1, $\{S_k\}_{k \in \mathbb{Z}}$ be an approximation to the identity of order $\epsilon \in (0, \theta]$ on $X$ as in Definition 2.3 and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$. Then, it is easy to see that

\[(2.14) \quad I = \sum_{k=-\infty}^{\infty} D_k \quad \text{in} \quad L^2(X)\]

Let $N \in \mathbb{N}$. Coifman’s idea is to rewrite (2.14) into

\[(2.15) \quad I = \left( \sum_{k=-\infty}^{\infty} D_k \right) \left( \sum_{j=-\infty}^{\infty} D_j \right) = \sum_{|j|>N} \sum_{k=-\infty}^{\infty} D_{k+j} D_k + \sum_{k=-\infty}^{\infty} \sum_{|j|\leq N} D_{k+j} D_k = R_N + T_N,\]
where

\[(2.16) \quad R_N = \sum_{|j|>N} \sum_{k=-\infty}^{\infty} D_{k+j} D_k\]

and

\[(2.17) \quad T_N = \sum_{k=-\infty}^{\infty} D_k^N D_k\]

with

\[D_k^N = \sum_{|j|\leq N} D_{k+j}.\]

It was proved in [H1] that there are constants \(C_4 > 0\) and \(\delta > 0\) independent of \(N \in \mathbb{N}\) such that for all \(f \in \mathcal{G}(x_1, r, \beta, \gamma)\) with \(x_1 \in X, r > 0\) and \(0 < \beta, \gamma < \epsilon,\)

\[(2.18) \quad \|R_N f\|_{\mathcal{G}(x_1, r, \beta, \gamma)} \leq C_4 2^{-N\delta} \|f\|_{\mathcal{G}(x_1, r, \beta, \gamma)}.\]

Thus, if we choose \(N \in \mathbb{N}\) such that

\[(2.19) \quad C_4 2^{-N\delta} < 1,\]

then \(T_N\) in (2.17) is invertible in the space \(\mathcal{G}(x_1, r, \beta, \gamma)\), namely, \(T_N^{-1}\) exists in the space \(\mathcal{G}(x_1, r, \beta, \gamma)\) and there is a constant \(C > 0\) such that for all \(f \in \mathcal{G}(x_1, r, \beta, \gamma),\)

\[(2.20) \quad \|T_N^{-1} f\|_{\mathcal{G}(x_1, r, \beta, \gamma)} \leq C \|f\|_{\mathcal{G}(x_1, r, \beta, \gamma)}.\]

For such chosen \(N \in \mathbb{N}\), letting

\[(2.21) \quad \tilde{D}_k = T_N^{-1} D_k^N,\]

we then obtain the first formula in (2.4) by (2.17). The proof of the second formula in (2.4) is similar.

Using this idea, we can obtain the following continuous Calderón reproducing formula of separable variable type on product spaces of homogeneous-type spaces, which is also the main theorem of this section.

**Theorem 2.1** Let \(i = 1, 2, \epsilon_i \in (0, \theta_i], \{S_{k_i}\}_{k_i \in \mathbb{Z}}\) be an approximation to the identity of order \(\epsilon_i\) on space of homogeneous type, \(X_i,\) and \(D_{k_i} = S_{k_i} - S_{k_i-1}\) for all \(k_i \in \mathbb{Z}.\) Then there are families of linear operators \(\{\tilde{D}_{k_i}\}_{k_i \in \mathbb{Z}}\) on \(X_i\) such that for all \(f \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)\) with \(\beta_i, \gamma_i \in (0, \epsilon_i)\) for \(i = 1, 2,\)

\[f = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2} (f),\]
where the series converge in the norm of both the space \( \mathcal{G}(\beta_1', \beta_2'; \gamma_1', \gamma_2') \) with \( \beta_i' \in (0, \beta_i) \) and \( \gamma_i' \in (0, \gamma_i) \) for \( i = 1, 2 \), and \( L^p(X_1 \times X_2) \) with \( p \in (1, \infty) \). Moreover, \( \tilde{D}_{k_i}(x_i, y_i) \), the kernel of \( \tilde{D}_{k_i} \) for \( x_i, y_i \in X_i \), and all \( k_i \in \mathbb{Z} \) satisfies the conditions (1) and (2) of Definition 2.3 with \( \epsilon_i \) replaced by any \( \epsilon_i' \in (0, \epsilon_i) \), and

\[
\int_{X_i} \tilde{D}_{k_i}(x_i, y_i) \, d\mu_i(y_i) = 0 = \int_{X_i} \tilde{D}_{k_i}(x_i, y_i) \, d\mu_i(x_i),
\]

where \( i = 1, 2 \).

**Proof.** We prove (2.21) by taking advantage of its nature of separation of variables. For \( i = 1, 2 \), let \( I_i \) be the identity operator on \( L^2(X_i) \). We rewrite (2.15) into

\[
(2.22) \quad I_i = R_{N_i} + T_{N_i},
\]

where \( R_{N_i} \) and \( T_{N_i} \) are defined by (2.16) and (2.17) instead of \( D_k \) and \( N \) there respectively by \( D_{k_i} \) and \( N_i \in \mathbb{N} \). By (2.18), we know that there are constants \( C_{5,i} > 0 \) and \( \delta_i > 0 \) independent of \( N_i \in \mathbb{N} \) such that for all \( f \in \mathcal{G}(x_i, r_i, \beta_i, \gamma_i) \) with \( x_i \in X_i, r_i > 0, \beta_i' \in (0, \beta_i] \) and \( \gamma_i' \in (0, \gamma_i] \),

\[
\| R_{N_i} f \| \mathcal{G}(x_i, r_i, \beta_i', \gamma_i') \leq C_{5,i} 2^{-N_i \delta_i} \| f \| \mathcal{G}(x_i, r_i, \beta_i', \gamma_i').
\]

We now choose \( N_i \in \mathbb{N} \) large enough such that

\[
(2.24) \quad C_{5,i} 2^{-N_i \delta_i} < 1,
\]

and thus \( T_{N_i} \) is invertible in the space \( \mathcal{G}(x_i, r_i, \beta_i', \gamma_i') \), namely, \( T_{N_i}^{-1} \) exists in \( \mathcal{G}(x_i, r_i, \beta_i', \gamma_i') \) and there is a constant \( C > 0 \) such that for all \( f \in \mathcal{G}(x_i, r_i, \beta_i', \gamma_i') \),

\[
(2.25) \quad \| T_{N_i}^{-1} f \| \mathcal{G}(x_i, r_i, \beta_i', \gamma_i') \leq C \| f \| \mathcal{G}(x_i, r_i, \beta_i', \gamma_i').
\]

Similarly to (2.20), we now define

\[
(2.26) \quad \tilde{D}_{k_i} = T_{N_i}^{-1} D_{k_i}^{N_i}
\]

for \( k_i \in \mathbb{Z} \). With these \( \tilde{D}_{k_i} \) as in (2.26), we now verify (2.21) holds in \( \mathcal{G}(\beta_1', \beta_2'; \gamma_1', \gamma_2') \). Let \( f \in \mathcal{G}(\beta_1', \beta_2'; \gamma_1', \gamma_2') \) and \( L_i \in \mathbb{N} \). We wish to show that

\[
(2.27) \quad \lim_{L_1, L_2 \to \infty} \left\| f - \sum_{|k_1|<L_1} \sum_{|k_2|<L_2} \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2} (f) \right\| \mathcal{G}(\beta_1', \beta_2'; \gamma_1', \gamma_2') = 0.
\]
By (2.26) and Lemma 2.3, we can write

\[
\begin{align*}
  &f - \sum_{|k_1|<L_1} \sum_{|k_2|<L_2} \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2}(f) \\
  &= f - \sum_{|k_1|<L_1} \sum_{|k_2|<L_2} T_{N_1}^{-1} D_{k_1}^N T_{N_2}^{-1} D_{k_2}^N D_{k_1} D_{k_2}(f) \\
  &= f - T_{N_1}^{-1} T_{N_2}^{-1} \left\{ \sum_{|k_1|<L_1} \sum_{|k_2|<L_2} D_{k_1}^N D_{k_2}^N D_{k_1} D_{k_2}(f) \right\} \\
  &= f - T_{N_1}^{-1} T_{N_2}^{-1} \left[ T_{N_1} - \sum_{|k_1|\geq L_1} D_{k_1}^N D_{k_1} \right] \left[ T_{N_2} - \sum_{|k_2|\geq L_2} D_{k_2}^N D_{k_2} \right] (f) \\
  &= \left[ f - T_{N_1}^{-1} T_{N_2}^{-1} T_{N_1} T_{N_2}(f) \right] + T_{N_1}^{-1} T_{N_2}^{-1} T_{N_1} \sum_{|k_2|\geq L_2} D_{k_2}^N D_{k_2}(f) \\
  &\quad + T_{N_1}^{-1} T_{N_2}^{-1} \sum_{|k_1|\geq L_1} D_{k_1}^N D_{k_1} T_{N_2,2}(f) \\
  &\quad - T_{N_1}^{-1} T_{N_2}^{-1} \sum_{|k_1|\geq L_1} D_{k_1}^N D_{k_1} \sum_{|k_2|\geq L_2} D_{k_2}^N D_{k_2}(f) \\
  &= F_1 + F_2 + F_3 - F_4.
\end{align*}
\]

We will estimate each term separately. We first estimate \(F_1\). By (2.22), we can write

\[
F_1 = f - T_{N_1}^{-1} T_{N_2}^{-1} T_{N_1} T_{N_2}(f) \\
= f - T_{N_1}^{-1} (I_1 - R_{N_1}) T_{N_2}^{-1} (I_2 - R_{N_2})(f) \\
= f - \left\{ \lim_{m_1 \to \infty} \sum_{l_1=0}^{m_1-1} R_{N_1}^{l_1} (I_1 - R_{N_1}) \right\} \left\{ \lim_{m_2 \to \infty} \sum_{l_2=0}^{m_2-1} R_{N_2}^{l_2} (I_2 - R_{N_2}) \right\} (f) \\
= f - \left( I_1 - \lim_{m_1 \to \infty} R_{N_1}^{m_1} \right) \left( I_2 - \lim_{m_2 \to \infty} R_{N_2}^{m_2} \right) (f) \\
= \lim_{m_1 \to \infty} R_{N_1}^{m_1} I_2(f) + \lim_{m_2 \to \infty} I_1 R_{N_2}^{m_2}(f) - \lim_{m_1 \to \infty} R_{N_1}^{m_1} R_{N_2}^{m_2}(f).
\]

The estimate (2.23) and Lemma 2.3 tell us that

\[
\begin{align*}
  \| R_{N_1}^{m_1} I_2(f) \|_{G(\beta_1',\beta_2';\gamma_1',\gamma_2')} &\leq \left( C_{5,1} 2^{-N_1\delta_1} \right)^{m_1} \| f \|_{G(\beta_1',\beta_2';\gamma_1',\gamma_2')} \\
  \| I_1 R_{N_2}^{m_2}(f) \|_{G(\beta_1',\beta_2';\gamma_1',\gamma_2')} &\leq \left( C_{5,2} 2^{-N_2\delta_2} \right)^{m_2} \| f \|_{G(\beta_1',\beta_2';\gamma_1',\gamma_2')}.
\end{align*}
\]
and
\[
\| R_{N_1}^m R_{N_2}^{m_2}(f) \|_{\mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)} \leq \left( C_{5,1} 2^{-N_1 \delta} \right)^{m_1} \left( C_{5,2} 2^{-N_2 \delta_2} \right)^{m_2} \| f \|_{\mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)}.
\]

Thus, the assumption (2.24) leads us that
\[
\begin{align*}
(2.28) \quad \| F_1 \|_{\mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)} & \leq \lim_{m_1 \to \infty} \| R_{N_1}^m I_2(f) \|_{\mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)} \\
& + \lim_{m_2 \to \infty} \| I_1 R_{N_2}^{m_2}(f) \|_{\mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)} \\
& + \lim_{m_1, m_2 \to \infty} \| R_{N_1}^m R_{N_2}^{m_2}(f) \|_{\mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)} \\
& = 0.
\end{align*}
\]

We now assume $\beta'_i \in (0, \beta_i)$ and $\gamma'_i \in (0, \gamma_i)$ for $i = 1, 2$ to be the same as in the theorem. To estimate $F_2$, $F_3$ and $F_4$, we first recall that there exist constants $\sigma_i > 0$ and $C_{6,i} > 0$ independent of $f_i$ and $L_i$ such that for all $f_i \in \mathcal{G}(\beta_i, \gamma_i)$,
\[
(2.29) \quad \| \sum_{|k_i| \geq L_i} D_{k_i}^N D_{k_i}(f) \|_{\mathcal{G}(\beta'_i, \gamma'_i)} \leq C_{6,i} 2^{-L_i \sigma_i} \| f \|_{\mathcal{G}(\beta_i, \gamma_i)},
\]
where $i = 1, 2$; see [H1, pp. 72-76] for a proof of this fact.

We now estimate $F_2$. By (2.22), we can write
\[
F_2 = T_{N_1}^{-1}(I_1 - R_{N_1}) T_{N_2}^{-1} \sum_{|k_2| \geq L_2} D_{k_2}^{N_2} D_{k_2}(f) \\
= \lim_{m_1 \to \infty} \sum_{j=0}^{m_1-1} R_{N_1}^{j}(I_1 - R_{N_1}) T_{N_2}^{-1} \sum_{|k_2| \geq L_2} D_{k_2}^{N_2} D_{k_2}(f) \\
= \left( I_1 - \lim_{m_1 \to \infty} R_{N_1}^m \right) T_{N_2}^{-1} \sum_{|k_2| \geq L_2} D_{k_2}^{N_2} D_{k_2}(f).
\]

The estimates (2.29), (2.23) and (2.25), and Lemma 2.3 yield
\[
\| F_2 \|_{\mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)} \leq C C_{5,2} 2^{-L_2 \sigma_2} \left\{ 1 + \lim_{m_1 \to \infty} \left( C_{5,1} 2^{-N_1 \delta} \right)^{m_1} \right\} \| f \|_{\mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)}.
\]

Then the assumption (2.24) further implies that
\[
(2.30) \quad \lim_{L_2 \to \infty} \| F_2 \|_{\mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)} = 0.
\]

The estimate for $F_3$ is similar to that for $F_2$ by symmetry.
Finally, the estimates (2.29) and (2.25), and Lemma 2.3 lead us that

\[(2.31) \lim_{L_1, L_2 \to \infty} \|F_4\|_{G(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)} \leq \lim_{L_1, L_2 \to \infty} C^2 C_{6, 1} C_{6, 2} 2^{-L_1 \sigma_1} 2^{-L_2 \sigma_2} \|f\|_{G(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)} = 0.\]

The estimates (2.28), (2.30) and (2.31) yield (2.27) and we have verified that (2.21) holds in \(G(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)\) with \(\beta'_i \in (0, \beta_i)\) and \(\gamma'_i \in (0, \gamma_i)\) for \(i = 1, 2\).

We now verify (2.21) also holds in \(L^p(X_1 \times X_2)\) for \(p \in (1, \infty)\). Instead of (2.27), we need to show that

\[(2.32) \lim_{L_1, L_2 \to \infty} \left\| f - \sum_{|k_1| < L_1} \sum_{|k_2| < L_2} \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2}(f) \right\|_{L^p(X_1 \times X_2)} = 0.\]

To see (2.32) is true, we only need to note that the following facts are true:

(i) If \(T_i\) is bounded in \(L^p(X_i)\) for \(p \in (1, \infty)\) with an operator norm \(C_{7, i}\) for \(i = 1, 2\), then \(T_1 T_2\) is also bounded in \(L^p(X_1 \times X_2)\) with an operator norm \(C_{7, 1} C_{7, 2}\).

(ii) Let \(i = 1, 2\). The operator \(R_{N_i}\) in (2.22) is bounded in \(L^p(X_i)\) with an operator norm \(C_{8, i} 2^{-N_i \delta_i}\), where \(\delta_i\) is the same as in (2.23). This fact was proved in [H1, p. 76]. Therefore, if we choose \(N_i \in \mathbb{N}\) such that

\[C_{8, i} 2^{-N_i \delta_i} < 1,\]

then \(T_{N_i}^{-1}\) exists and is also bounded in \(L^p(X_i)\) for \(p \in (1, \infty)\) with an operator norm

\[\sum_{j=0}^{\infty} (C_{8, i} 2^{-N_i \delta_i})^j.\]

(iii)

\[\lim_{L_i \to \infty} \left\| \sum_{|k_i| \geq L_i} D_{k_i}^{N_i} D_{k_i}(f) \right\|_{L^p(X_i)} = 0,\]

which was proved in [H1, p. 77] by a result in [DJS].

Using these facts and repeating the procedure of the proof of (2.27), we can prove (2.32) holds.

This completes the proof of Theorem 2.1.

By a procedure similar to the proof of Theorem 2.1, we can establish another continuous Calderón reproducing formulae. We leave the details to the reader.
Let $i = 1, 2$ and $\{D_{k_i}\}_{k_i \in \mathbb{Z}}$ be the same as in Theorem 2.1. Then there are families of linear operators $\{\overline{D}_{k_i}\}_{k_i \in \mathbb{Z}}$ on $X_i$ such that for all $f \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with $\beta_i, \gamma_i \in (0, \epsilon_i)$ for $i = 1, 2$,
\[
f = \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} D_{k_1} D_{k_2} \overline{D}_{k_1} \overline{D}_{k_2}(f),
\]
where the series converge in the norm of both the space $\mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2)$ with $\beta'_i \in (0, \beta_i)$ and $\gamma'_i \in (0, \gamma_i)$ for $i = 1, 2$, and $L^p(X_1 \times X_2)$ with $p \in (1, \infty)$. Moreover, $\overline{D}_{k_i}(x_i, y_i)$, the kernel of $\overline{D}_{k_i}$ for $x_i, y_i \in X_i$ and all $k_i \in \mathbb{Z}$ satisfies the conditions (1) and (3) of Definition 2.3 with $\epsilon_i$ replaced by any $\epsilon'_i \in (0, \epsilon_i)$, and
\[
\int_{X_i} \overline{D}_{k_i}(x_i, y_i) d\mu_i(y_i) = 0 = \int_{X_i} \overline{D}_{k_i}(x_i, y_i) d\mu_i(x_i),
\]
where $i = 1, 2$.

To establish the following continuous Calderón reproducing formulae in spaces of distributions, we need to use the theory of Calderón-Zygmund operators on these spaces developed in [H1]. We first recall some definitions.

Let $X$ be a space of homogeneous type as in Definition 2.1. For $\eta \in (0, \theta]$, we define $C^\eta_0(X)$ to be the set of all functions having compact support such that
\[
\|f\|_{C^\eta_0(X)} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\eta} < \infty.
\]
Endow $C^\eta_0(X)$ with the natural topology and let $(C^\eta_0(X))'$ be its dual space.

**Definition 2.5** Let $\epsilon \in (0, \theta]$ and $X$ be a space of homogeneous type as in Definition 2.1. A continuous complex-valued function $K(x, y)$ on
\[
\Omega = \{(x, y) \in X \times X : x \neq y\}
\]
is called a Calderón-Zygmund kernel of type $\epsilon$ if there exist a constant $C_{0,1} > 0$ such that
(i) $|K(x, y)| \leq C_{0,1} \rho(x, y)^{-d}$,
(ii) $|K(x, y) - K(x', y)| \leq C_{0,1} \rho(x, x')^\epsilon \rho(x, y)^{-d-\epsilon}$ for $\rho(x, x') \leq \frac{\rho(x, y)}{2A}$,
(iii) $|K(x, y) - K(x, y')| \leq C_{0,1} \rho(y, y')^\epsilon \rho(x, y)^{-d-\epsilon}$ for $\rho(y, y') \leq \frac{\rho(x, y)}{2A}$.

A continuous linear operator $T : C^\eta_0(X) \to (C^\eta_0(X))'$ for all $\eta \in (0, \theta]$ is a Calderón-Zygmund singular integral operator of type $\epsilon$ if there is a Calderón-Zygmund kernel $K(x, y)$ of the type $\epsilon$ as above such that
\[
(Tf, g) = \int_X \int_X K(x, y)f(y)g(x) \, d\mu(x) \, d\mu(y)
\]
for all $f, g \in C^\eta_0(X)$ with disjoint supports. In this case, we write $T \in CZO(\epsilon)$.
We also need the following notion of the strong weak boundedness property in [HS].

**Definition 2.6** Let $X$ be a space of homogeneous type as in Definition 2.1. A Calderón-Zygmund singular integral operator $T$ of the kernel $K$ is said to have the strong weak boundedness property, if there exist $\eta \in (0, \theta]$ and constant $C_{9,2} > 0$ such that

$$|\langle K, f \rangle| \leq C_{9,2} r^d$$

for all $r > 0$ and all continuous $f$ on $X \times X$ with $\text{supp} f \subseteq B(x_1, r) \times B(y_1, r)$, where $x_1$ and $y_1 \in X$, $\|f\|_{L^\infty(X \times X)} \leq 1$, $\|f(\cdot, y)\|_{C^\eta_0(X)} \leq r^{-\eta}$ for all $y \in X$ and $\|f(x, \cdot)\|_{C^\eta_0(X)} \leq r^{-\eta}$ for all $x \in X$. We denote this by $T \in \text{SW BP}$.

The following theorem is the variant on space of homogeneous type of Theorem 1.19 in [H1].

**Lemma 2.4** Let $\epsilon \in (0, \theta]$ and $X$ be a space of homogeneous type as in Definition 2.1. Let $T \in \text{CZO}(\epsilon)$, $T(1) = T^*(1) = 0$, and $T \in \text{SW BP}$. Furthermore, $K(x, y)$, the kernel of $T$, satisfies the following smoothness condition

$$|K(x, y) - K(x', y) - K(x, y') - K(x', y')| \leq C_{9,3} \rho(x, x')^\epsilon \rho(y, y')^\epsilon \rho(x, y)^{-d-2\epsilon}$$

for all $x, x', y, y' \in X$ such that $\rho(x, x') \leq \frac{\rho(x, y)}{4A^2}$. Then for any $x_0 \in X$, $r > 0$ and $0 < \beta, \gamma < \epsilon$, $T$ maps $G(x_0, r, \beta, \gamma)$ into itself. Moreover, if we let $\|T\| = \max\{C_{9,1}, C_{9,2}, C_{9,3}\}$, then there exists a constant $C_{9,4} > 0$ such that

$$\|Tf\|_{G(x_0, r, \beta, \gamma)} \leq C_{9,4} \|f\|_{G(x_0, r, \beta, \gamma)}.$$

We also need the following construction given by Christ in [Chr2], which provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type. A similar construction was independently given by Sawyer and Wheeden [SW].

**Lemma 2.5** Let $X$ be a space of homogeneous type as in Definition 2.1. Then there exist a collection

$$\{Q^k_\alpha \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$$

of open subsets, where $I_k$ is some index set, and constants $\delta \in (0, 1)$ and $C_{10,1}, C_{10,2} > 0$ such that

(i) $\mu(X \setminus \bigcup_\alpha Q^k_\alpha) = 0$ for each fixed $k$ and $Q^k_\alpha \cap Q^k_\beta = \emptyset$ if $\alpha \neq \beta$;

(ii) for any $\alpha, \beta, k, l$ with $l \geq k$, either $Q^l_\beta \subset Q^k_\alpha$ or $Q^l_\beta \cap Q^k_\alpha = \emptyset$;
Let all the notation be the same as in Theorem 2.1. Then for all \( k \in \mathbb{Z}_+ \),

\[
\text{diam} (Q^k) \leq C_{10,1} \delta^k;
\]

(iv) each \( Q^k \) contains some ball \( B(z^k, C_{10,2} \delta^k) \), where \( z^k \in X \).

In fact, we can think of \( Q^k \) as being a dyadic cube with diameter roughly \( \delta^k \) and
centered at \( z^k \). In what follows, we always suppose \( \delta = 1/2 \). See [HS] for how to remove
this restriction. Also, in the following, for \( k \in \mathbb{Z}_+ \) and \( \tau \in I_k \), we will denote by \( Q^k_{\tau, \nu} \),
\( \nu = 1, 2, \cdots, N(k, \tau) \), the set of all cubes \( Q^k_{\tau,j} \subset Q^k_{\tau} \), where \( j \) is a fixed large positive
integer. Denote by \( y^k_{\tau, \nu} \) a point in \( Q^k_{\tau, \nu} \). For any dyadic cube \( Q \) and any \( f \in L^1_{\text{loc}}(X) \), we set
\[
m_Q(f) = \frac{1}{\mu(Q)} \int_Q f(x) \, d\mu(x).
\]

Using Theorem 2.1, we now try to establish the following continuous Calderón re-
producing formulae in spaces of distributions.

**Theorem 2.3** Let all the notation be the same as in Theorem 2.1. Then for all \( f \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \),

\[
f = \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} D^*_{k_1} D^*_{k_2} \tilde{D}^*_{k_1} \tilde{D}^*_{k_2} (f)
\]

holds in \( \mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2) \) with \( \beta'_i \in (\beta_i, \epsilon_i) \) and \( \gamma'_i \in (\gamma_i, \epsilon_i) \) for \( i = 1, 2 \), where
\( D^*_{k_1}(x,y) = D_{k_1}(y,x) \) and \( \tilde{D}^*_{k_i}(x,y) = \tilde{D}_{k_i}(y,x) \).

**Proof.** Let \( f \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \) and \( g \in \mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2) \) with the same notation as in
the theorem. By Theorem 2.1, we have

\[
g = \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2} (g)
\]

holds in \( \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \). From this, it follows that

\[
\langle f, g \rangle = \left\langle f, \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2} (g) \right\rangle
= \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \left\langle f, \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2} (g) \right\rangle.
\]

To prove the theorem, we still need to show that

\[
\langle f, \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2} (g) \rangle = \langle D^*_{k_1} D^*_{k_2} \tilde{D}^*_{k_1} \tilde{D}^*_{k_2} (f), g \rangle.
\]

(2.34)
Let $M_1, M_2 \in \mathbb{N}$ be large enough, $B_1(x_0, M_1) = \{x_1 \in X_1 : \rho_1(x_1, x_0) < M_1\}$ and $B_2(y_0, M_2) = \{x_2 \in X_2 : \rho_2(x_2, y_0) < M_2\}$. For any fixed $k_1, k_2 \in \mathbb{Z}$, we then define

$$g_{M_1, M_2}(x_1, x_2) = \int_{B_1(x_0, M_1)} \int_{B_2(y_0, M_2)} \tilde{D}_{k_1}(x_1, y_1) \tilde{D}_{k_2}(x_2, y_2) D_{k_1}(g)(y_1, y_2) d\mu_1(y_1) d\mu_2(y_2).$$

We first claim that

$$(2.35) \quad \lim_{M_1 \to \infty, M_2 \to \infty} \left\| \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_2}(g) - g_{M_1, M_2} \right\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)} = 0.$$

To verify this, we use Lemma 2.4. To this end, we write

$$\tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_2}(g)(x_1, x_2) - g_{M_1, M_2}(x_1, x_2)$$

$$= \int_{X_1 \setminus X_2 \setminus B_2(y_0, M_2)} \tilde{D}_{k_1}(x_1, y_1) \tilde{D}_{k_2}(x_2, y_2) D_{k_1}(g)(y_1, y_2) d\mu_1(y_1) d\mu_2(y_2)$$

$$+ \int_{X_1 \setminus B_1(x_0, M_1)} \int_{B_2(y_0, M_2)} \cdots$$

$$= T_1(g)(x_1, x_2) + T_2(g)(x_1, x_2).$$

We first consider $T_1(g)(x_1, x_2)$, whose kernel is

$$K_1(x_1, x_2; z_1, z_2)$$

$$= \left\{ \int_{X_1 \setminus B_1(x_0, M_1)} \tilde{D}_{k_1}(x_1, y_1) D_{k_1}(y_1, z_1) d\mu(y_1) \right\} \times \left\{ \int_{X_2 \setminus B_2(y_0, M_2)} \tilde{D}_{k_2}(x_2, y_2) D_{k_2}(y_2, z_2) d\mu_2(y_2) \right\}$$

$$= K_{11}(x_1, z_1) K_{12}(x_2, z_2).$$

It is easy to verify that the operator $T_{11}$ with the kernel $K_{11}$ satisfies all the condition of Lemma 2.4. Thus, by Lemma 2.4, we know that there is a constant $C > 0$ independent of $M_1$ and $M_2$ such that for all $f \in \mathcal{G}(\beta_1, \gamma_1)$,

$$(2.36) \quad \|T_{11}(f)\|_{\mathcal{G}(\beta_1, \gamma_1)} \leq C \|f\|_{\mathcal{G}(\beta_1, \gamma_1)}.$$

We now verify that $K_{12}$ is a Calderón-Zygmund kernel of type $\epsilon'_2 > 0$ on $X_2$ as in Definition 2.5 with a constant $C_{0,1}$ independent of $M_2$, where $\epsilon'_2$ can be any positive
number in $(0, \epsilon_2)$. We first have

\begin{align}
(2.37) \quad |K_{12}(x_2, z_2)| &= \left| \int_{X_2 \setminus B_2(y_0, M_2)} \tilde{D}_{k_2}(x_2, y_2) D_{k_2}(y_2, z_2) \, d\mu_2(y_2) \right| \\
&\leq C_{k_2} \int_{X_2} \left| \tilde{D}_{k_2}(x_2, y_2) D_{k_2}(y_2, z_2) \right| \, d\mu_2(y_2) \\
&\leq C_{k_2} \frac{1}{(1 + \rho(x_2, z_2))^{d_2 + \epsilon'_2}} \\
&\leq C_{k_2} \frac{1}{\rho(x_2, z_2)^{d_2}},
\end{align}

where $C_{k_2}$ is independent of $M_2$.

To verify $K_{12}(x_2, z_2)$ satisfies Definition 2.5 (ii), assuming that $\rho_2(x_2, x'_2) \leq \frac{\rho_2(x_2, z_2)}{2A_2}$, we have

\begin{align}
(2.38) \quad |K_{12}(x_2, z_2) - K_{12}(x'_2, z_2)| &\leq \int_{X_2 \setminus B_2(y_0, M_2)} \left| \tilde{D}_{k_2}(x_2, y_2) - \tilde{D}_{k_2}(x'_2, y_2) \right| D_{k_2}(y_2, z_2) \, d\mu_2(y_2) \\
&\leq C_{k_2} \left[ \frac{1}{(1 + \rho_2(x_2, y_2))^{d_2 + 2\epsilon'}} \right] \frac{1}{(1 + \rho_2(y_2, z_2))^{d_2 + \epsilon'_2}} \, d\mu_2(y_2) \\
&\leq C_{k_2} \frac{1}{\rho_2(x_2, x'_2)^{\epsilon'_2}} \frac{1}{\rho_2(x_2, z_2)^{d_2 + \epsilon'_2}} \\
&\leq C_{k_2} \frac{\rho_2(x_2, z_2)^{\epsilon'_2}}{\rho_2(x_2, x'_2)^{d_2 + \epsilon'_2}},
\end{align}

where $C_{k_2}$ is independent of $M_2$.

By symmetry, similarly to the proof of (2.38), we also have that for $\rho_2(z_2, z'_2) \leq \frac{\rho_2(x_2, z_2)}{2A_2}$,

\begin{align}
(2.39) \quad |K_{12}(x_2, z_2) - K_{12}(x, z'_2)| &\leq C_{k_2} \frac{\rho_2(x, z'_2)^{\epsilon'_2}}{\rho_2(x_2, z_2)^{d_2 + \epsilon'_2}},
\end{align}

where $C_{k_2}$ is independent of $M_2$. 
We now verify that $K_{12}$ satisfies (2.33). First assuming that $\rho_2(x, x') \leq \frac{\rho_2(x, z_2)}{4A_z^2}$ and $\rho_2(z, z') \leq \frac{\rho_2(x, z_2)}{4A_z^2}$, we write

$$[K_{12}(x, z_2) - K_{12}(x', z_2)] - [K_{12}(x, z_2') - K_{12}(x', z_2')] = \int_{X_2 \setminus B_2(y_0, M_2)} \left[ \tilde{D}_{k_2}(x, y_2) - \tilde{D}_{k_2}(x', y_2) \right] \left[ D_{k_2}(y_2, z_2) - D_{k_2}(y_2, z'_2) \right] \, d\mu_2(y_2).$$

By our assumption, we now have three cases. Case 1. $\rho_2(x, x') \leq \frac{1}{2A_z^2} (1 + \rho_2(x, y_2))$ and $\rho_2(z, z') \leq \frac{1}{2A_z^2} (1 + \rho_2(y, z_2))$. In this case, we have

$$\left(2.40\right) \quad \| [K_{12}(x, z_2) - K_{12}(x', z_2)] - [K_{12}(x, z_2') - K_{12}(x', z_2')] \right| \leq C_{k_2} \int_X \frac{\rho_2(x, x')^{\epsilon_2}}{(1 + \rho_2(x, y_2))^{d_2 + 2\epsilon_2}} \frac{\rho_2(z, z')^{\epsilon_2}}{(1 + \rho_2(y, z_2))^{d_2 + 2\epsilon_2}} d\mu_2(y_2),$$

where $C_{k_2}$ is independent of $M_2$.

Case 2. $\rho_2(x, x') \leq \frac{1}{2A_z^2} (1 + \rho_2(x, y_2))$ and $\rho_2(z, z') > \frac{1}{2A_z^2} (1 + \rho_2(y, z_2))$. In this case, we in fact have $\rho_2(y, z_2) < \frac{\rho_2(x, z_2)}{2A_z^2}$, which implies that $\rho_2(x, y_2) \geq \frac{\rho_2(x, z_2)}{2A_z^2}$. The last fact and the fact that $\rho_2(z, z') > \frac{1}{2A_z^2}$ yield that

$$\left(2.41\right) \quad \| [K_{12}(x, z_2) - K_{12}(x', z_2)] - [K_{12}(x, z_2') - K_{12}(x', z_2')] \right| \leq C_{k_2} \int_X \frac{\rho_2(x, x')^{\epsilon_2}}{(1 + \rho_2(x, y_2))^{d_2 + 2\epsilon_2}} \left[ |D_{k_2}(y_2, z_2)| + |D_{k_2}(y_2, z'_2)| \right] d\mu_2(y_2),$$

where $C_{k_2}$ is independent of $M_2$.

Case 2. $\rho_2(x, x') > \frac{1}{2A_z^2} (1 + \rho_2(x, y_2))$ and $\rho_2(z, z') \leq \frac{1}{2A_z^2} (1 + \rho_2(y, z_2))$. The proof of this case is similar to the case 2 by the symmetry.

If $\frac{\rho_2(x, z_2)}{4A_z^2} < \rho_2(x, x') \leq \frac{\rho_2(x, z_2)}{3A_z^2}$ or $\frac{\rho_2(x, z_2)}{4A_z^2} < \rho_2(z, z') \leq \frac{\rho_2(x, z_2)}{3A_z^2}$, we then can deduce that $K_{12}$ satisfies (2.33) from (2.38) or (2.39), which together with (2.40) and (2.41) verifies that $K_{12}$ satisfies (2.33). We omit the details.
Finally we verify that $K_{12}$ has the strong weak boundedness property as in Definition 2.6. Let $r > 0$ and $f$ be a continuous function on $X_2 \times X_2$ with $\text{supp } f \subset B_2(x_{21}, r) \times B_2(x_{22}, r)$, where $x_{21}, x_{22} \in X_2$. \( \|f\|_{L^\infty(X_2 \times X_2)} \leq 1, \|f(\cdot, z_2)\|_{C_0^l(X_2)} \leq r^{-\eta} \) for all $z_2 \in X_2$ and $\|f(x_2, \cdot)\|_{C_0^l(X_2)} \leq r^{-\eta}$ for all $x_2 \in X_2$. From (2.37), it follows that

\[
\|\langle K_{12}, f \rangle \| \leq C_k \|f\|_{L^\infty(X_2 \times X_2)} \int_{B_2(x_{22}, r)} \left\{ \int_{X_2} \frac{1}{(1 + \rho(x_2, z_2))^{d_2 + \gamma_2}} d\mu_2(x_2) \right\} d\mu_2(z_2) \leq C_k r^{d_2},
\]

where $C_k$ is independent of $M_2$.

Let $T_{12}$ be the Calderón-Zygmund operator with the kernel $K_{12}$. It is also obvious that $T_{12}(1) = 0$. Thus, $T_{12}$ satisfies all the conditions of Lemma 2.4 with $\|T_{12}\| = C_k$ independent of $M_2$. By Lemma 2.4, we know that there is a constant $C > 0$ independent of $M_2$ such that for all $f \in G(\beta_2, \gamma_2)$,

\[
(2.42) \quad \|T_{12}(f)\|_{G(\beta_2, \gamma_2)} \leq C \|f\|_{G(\beta_2, \gamma_2)}.
\]

The estimates (2.36) and (2.42), and Lemma 2.3 tell us that $T_1$ is bounded on $G(\beta_2', \beta_2; \gamma_1', \gamma_2')$ with an operator norm independent of $M_1$ and $M_2$. Similarly, we can show that $T_2$ has the same property. Let

\[
\tilde{g}(x_1, x_2) = \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2} (g)(x_1, x_2) - g_{M_1, M_2}(x_1, x_2).
\]

Then $\tilde{g} \in G(\beta_2', \beta_2; \gamma_1', \gamma_2')$ with a norm independent of $M_1$ and $M_2$. Namely, there is a constant $C > 0$ independent of $M_1$ and $M_2$ such that

\[
(2.43) \quad |\tilde{g}(x_1, x_2) - \tilde{g}(x_1', x_2)| \leq C \left( \frac{\rho_1(x_1, x_1')}{1 + \rho_1(x_1, x_0)} \right)^{\beta_2'} \frac{1}{(1 + \rho_1(x_1, x_0))^{d_1 + \gamma_1'}} \frac{1}{(1 + \rho_2(x_2, y_0))^{d_2 + \gamma_2'}}
\]

for $\rho_1(x_1, x_1') \leq \frac{1}{2A_1} [1 + \rho_1(x_1, x_0)];$

\[
(2.44) \quad |\tilde{g}(x_1, x_2) - \tilde{g}(x_1', x_2)| \leq C \frac{1}{(1 + \rho_1(x_1, x_0))^{d_1 + \gamma_1'}} \left( \frac{\rho_2(x_2, x_2')}{1 + \rho_2(x_2, y_0)} \right)^{\beta_2'} \frac{1}{(1 + \rho_2(x_2, y_0))^{d_2 + \gamma_2'}}
\]
for \( \rho_2(x_2, x'_2) \leq \frac{1}{2A_2} [1 + \rho_2(x_2, y_0)]; \)

\[
\begin{align*}
& \quad \text{(2.45)} \quad ||\mathbf{g}(x_1, x_2) - \mathbf{g}(x'_1, x_2) - [\mathbf{g}(x_1, x'_2) - \mathbf{g}(x'_1, x'_2)] || \\
& \quad \leq C \left( \frac{\rho_1(x_1, x'_1)}{1 + \rho_1(x_1, x_0)} \right)^{\beta'_1} \left( \frac{1}{1 + \rho_1(x_1, x_0)^{d_1 + \gamma'_1}} \right) \left( \frac{\rho_2(x_2, x'_2)}{1 + \rho_2(x_2, y_0)} \right)^{\beta'_2} \left( \frac{1}{1 + \rho_2(x_2, y_0)^{d_2 + \gamma'_2}} \right)
\end{align*}
\]

for \( \rho_1(x_1, x'_1) \leq \frac{1}{2A_1} [1 + \rho_1(x_1, x_0)] \) and \( \rho_2(x_2, x'_2) \leq \frac{1}{2A_2} [1 + \rho_2(x_2, y_0)]; \)

\[
\begin{align*}
& \quad \text{(2.46)} \quad \int_{X_1} \mathbf{g}(x_1, x_2) \, d\mu_1(x_1) = 0 \\
& \quad \text{(2.47)} \quad \int_{X_2} \mathbf{g}(x_1, x_2) \, d\mu_2(x_2) = 0
\end{align*}
\]

for all \( x_2 \in X_2; \)

Moreover, we can directly compute that

\[
\begin{align*}
& \quad \text{(2.48)} \quad ||\mathbf{g}(x_1, x_2)|| \leq C_{k_1, k_2} ||\mathbf{g}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2) || \\
& \quad \times \left\{ \int_{X_1} \int_{X_2 \setminus B_2(y_0, M_2)} \int_{X_1} \int_{X_2} \frac{1}{(1 + \rho_1(x_1, y_1))^{\gamma'_1}} \frac{1}{(1 + \rho_2(x_2, y_2))^{\gamma'_2}} \right. \\
& \quad \times \frac{1}{(1 + \rho_1(y_1, z_1))^{\gamma_1}} \frac{1}{(1 + \rho_2(y_2, z_2))^{\gamma_2}} \frac{1}{(1 + \rho_1(z_1, x_0))^{\gamma'_1}} d\mu_1(z_1) \, d\mu_2(z_2) \, d\mu_1(y_1) \, d\mu_2(y_2) \\
& \quad \left. + \int_{X_1 \setminus B_1(x_0, M_1)} \int_{B_2(y_0, M_2)} \int_{X_1} \int_{X_2} \ldots \right\} \\
& \quad \leq C_{k_1, k_2} ||\mathbf{g}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2) || \\
& \quad \times \left\{ \int_{X_1} \int_{X_2 \setminus B_2(y_0, M_2)} \frac{1}{(1 + \rho_1(x_1, y_1))^{\gamma'_1}} \frac{1}{(1 + \rho_2(x_2, y_2))^{\gamma'_2}} d\mu_1(y_1) \, d\mu_2(y_2) \\
& \quad \times \frac{1}{(1 + \rho_1(y_1, x_0))^{\gamma_1}} \frac{1}{(1 + \rho_2(y_2, y_0))^{\gamma_2}} d\mu_1(y_1) \, d\mu_2(y_2) \\
& \quad + \int_{X_1 \setminus B_1(x_0, M_1)} \int_{B_2(y_0, M_2)} \ldots \right\} \\
& \quad \leq C_{k_1, k_2} \left\{ \frac{1}{M_1^{\gamma_1 - \gamma_1}} + \frac{1}{M_2^{\gamma_2 - \gamma_2}} \right\} ||\mathbf{g}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2) || \\
& \quad \times \frac{1}{1 + \rho_1(x_1, x_0)^{\gamma_1}} \frac{1}{1 + \rho_2(x_2, y_0)^{\gamma_2}}.
\end{align*}
\]
where \( C_{k_1,k_2} \) is independent of \( M_1 \) and \( M_2 \).

If \( \rho_1(x_1,x_1') \leq \frac{1}{2A_1}[1 + \rho_1(x_1,x_0)] \), from (2.48), it follows that

\[
|\bar{g}(x_1,x_2) - \bar{g}(x_1',x_2)| \leq C_{k_1,k_2} \left\{ \frac{1}{M_1^{1-\gamma_1}} + \frac{1}{M_2^{\gamma_2-\gamma_2}} \right\} \frac{\|g\|}{\|g\|_{l_2}} \left( \frac{\rho_1(x_1,x_1')}{1 + \rho_1(x_1,x_0)} \right)^{1-\alpha_1} \leq \frac{1}{1 + \rho_1(x_1,x_0)^{d_1+\gamma_2}} \frac{1}{(1 + \rho_2(x_1,x_0))^{d_2+\gamma_2}}. 
\]

Let \( \alpha_1 \in (0,1) \). The geometric means between (2.43) and (2.49) then gives that

\[
|\bar{g}(x_1,x_2) - \bar{g}(x_1',x_2)| \leq C_{k_1,k_2} \left( \frac{1}{M_1^{1-\gamma_1}} + \frac{1}{M_2^{\gamma_2-\gamma_2}} \right)^{1-\alpha_1} \frac{\|g\|}{\|g\|_{l_2}} \left( \frac{\rho_1(x_1,x_1')}{1 + \rho_1(x_1,x_0)} \right)^{1-\alpha_1} \leq \frac{1}{1 + \rho_1(x_1,x_0)^{d_1+\gamma_2}} \frac{1}{(1 + \rho_2(x_1,x_0))^{d_2+\gamma_2}}. 
\]

Let \( \alpha_2 \in (0,1) \). Similarly, from (2.48), (2.44) and the geometric means, we can deduce that if

\[
\rho_2(x_2,x_2') \leq \frac{1}{2A_2}[1 + \rho_2(x_2,y_0)],
\]

then

\[
|\bar{g}(x_1,x_2) - \bar{g}(x_1,x_2')| \leq C_{k_1,k_2} \left( \frac{1}{M_1^{1-\gamma_1}} + \frac{1}{M_2^{\gamma_2-\gamma_2}} \right)^{1-\alpha_2} \frac{\|g\|}{\|g\|_{l_2}} \left( \frac{\rho_1(x_1,x_1')}{1 + \rho_1(x_1,x_0)} \right)^{1-\alpha_2} \leq \frac{1}{1 + \rho_1(x_1,x_0)^{d_1+\gamma_2}} \frac{1}{(1 + \rho_2(x_2,y_0))^{d_2+\gamma_2}}. 
\]

Let \( \alpha_3, \alpha_4 \in (0,1) \). The estimates (2.50), (2.51) and (2.45) and the geometric means imply that if \( \rho_1(x_1,x_1') \leq \frac{1}{2A_1}[1 + \rho_1(x_1,x_0)] \) and \( \rho_2(x_2,x_2') \leq \frac{1}{2A_2}[1 + \rho_2(x_2,y_0)] \), then

\[
|\bar{g}(x_1,x_2) - \bar{g}(x_1',x_2)| - |\bar{g}(x_1,x_2') - \bar{g}(x_1',x_2')| \leq C_{k_1,k_2} \left( \frac{1}{M_1^{1-\gamma_1}} + \frac{1}{M_2^{\gamma_2-\gamma_2}} \right)^{(1-\alpha_1)(1-\alpha_2)\alpha_4 + (1-\alpha_3)(1-\alpha_4)} \frac{\|g\|}{\|g\|_{l_2}} \left( \frac{\rho_1(x_1,x_1')}{1 + \rho_1(x_1,x_0)} \right)^{(1-\alpha_1)\alpha_4 + (1-\alpha_3)\alpha_4} \left( \frac{\rho_1(x_1,x_1')}{1 + \rho_1(x_1,x_0)} \right)^{(1-\alpha_2)(1-\alpha_4)} \frac{1}{1 + \rho_1(x_1,x_0)^{d_1+\gamma_1}} \frac{1}{(1 + \rho_2(x_2,y_0))^{d_2+\gamma_2}}.
\]
From (2.48), (2.50), (2.51), (2.52), (2.46) and (2.47), it follows that $\overline{g}(x_1, x_2) \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ if we suitably chose $\alpha_i$ for $i = 1, 2, 3, 4$, and

$$\lim_{M_1, M_2 \to \infty} \|\overline{g}(\beta_1, \beta_2; \gamma_1, \gamma_2)\| = 0,$$

namely (2.35) holds, which yields that

$$(2.53) \quad \left\langle f, \check{D}_{k_1} \check{D}_{k_2} D_{k_1} D_{k_2}(g) \right\rangle = \lim_{M_1, M_2 \to \infty} (f, g_{M_1, M_2}).$$

For $J_1, J_2 \in \mathbb{N}$ and any fixed $M_1, M_2 \in \mathbb{N}$ large enough, we define

$$N_{J_1} = \left\{ i_1 \in I_{J_1} : Q_{i_1}^{J_1} \cap B_1(x_0, M_1) \neq \emptyset \right\}$$

and

$$N_{J_2} = \left\{ i_2 \in I_{J_2} : Q_{i_2}^{J_2} \cap B_2(y_0, M_2) \neq \emptyset \right\},$$

where $\{Q_{i_1}^{J_1}\}_{i_1 \in I_{J_1}}$ and $\{Q_{i_2}^{J_2}\}_{i_2 \in I_{J_2}}$ are respectively the dyadic cubes of $X_1$ and $X_2$ as in Lemma 2.5. Then the cardinal number of $N_{J_1} \sim M_1^{d_1} 2^{d_1}$ and the cardinal number of $N_{J_2} \sim M_2^{d_2} 2^{d_2}$. Write

$$g_{M_1, M_2}(x_1, x_2)$$

$$= \sum_{i_1 \in N_{J_1}} \int_{Q_{i_1}^{J_1} \cap B_1(x_0, M_1)} \int_{B_2(y_0, M_2)} \left[ \check{D}_{k_1}(x_1, y_1) - \check{D}_{k_1}(x_1, \pi_{i_1}^{J_1}) \right]$$

$$\times \check{D}_{k_2}(x_2, y_2) D_{k_1} D_{k_2}(g)(y_1, y_2) \, d\mu_1(y_1) \, d\mu_2(y_2)$$

$$+ \sum_{i_2 \in N_{J_2}} \check{D}_{k_1}(x_1, x_{Q_{i_2}^{J_2}}) \int_{B_1(x_0, M_1)} \int_{Q_{i_2}^{J_2} \cap B_2(y_0, M_2)} \left[ \check{D}_{k_2}(x_2, y_2) - \check{D}_{k_2}(x_2, \pi_{i_2}^{J_2}) \right]$$

$$\times D_{k_1} D_{k_2}(g)(y_1, y_2) \, d\mu_1(y_1) \, d\mu_2(y_2)$$

$$+ \sum_{i_1 \in N_{J_1}} \sum_{i_2 \in N_{J_2}} \check{D}_{k_1}(x_1, x_{Q_{i_1}^{J_1}}) \check{D}_{k_2}(x_2, x_{Q_{i_2}^{J_2}})$$

$$\times \int_{Q_{i_1}^{J_1} \cap B_1(x_0, M_1)} \int_{Q_{i_2}^{J_2} \cap B_2(y_0, M_2)} D_{k_1} D_{k_2}(g)(y_1, y_2) \, d\mu_1(y_1) \, d\mu_2(y_2)$$

$$= g_{M_1, M_2}^1(x_1, x_2) + g_{M_1, M_2}^2(x_1, x_2) + g_{M_1, M_2}^3(x_1, x_2),$$

where $x_{Q_{i_1}^{J_1}}$ and $x_{Q_{i_2}^{J_2}}$ are respectively any point in $Q_{i_1}^{J_1} \cap B_1(x_0, M_1)$ and $Q_{i_2}^{J_2} \cap B_2(y_0, M_2)$. Our task now is to verify that

$$(2.54) \quad \lim_{J_1, J_2 \to \infty} \|g_{M_1, M_2}^i\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)} = 0,$$
where \( i = 1, 2 \). The proof of (2.54) for \( i = 2 \) is similar to that for \( i = 1 \). We only verify (2.54) for \( i = 1 \), which can be deduced from Lemma 2.4 and Lemma 2.3 by a procedure similar to the proof of (2.35). To this end, we regard \( g_{M_1,M_2}^i(x_1,x_2) \) as an operator \( T_3 \) acts on the functions \( g \). The kernel \( K_3 \) of \( T_3 \) can be written into

\[
K_3(x_1,x_2; z_1, z_2)
= \sum_{i_1 \in N_j} \int_{Q^{j_1}_{i_1} \cap B_1(x_0, M_1)} \int_{B_2(y_0, M_2)} \left[ \tilde{D}_{k_1}(x_1, y_1) - \tilde{D}_{k_1}(x_1, x_{Q^{j_1}_{i_1}}) \right] 
\times \tilde{D}_{k_2}(x_2, y_2) D_{k_1}(y_1, z_1) D_{k_2}(y_2, z_2) \, d\mu_1(y_1) \, d\mu_2(y_2)
= \left\{ \sum_{i_1 \in N_j} \int_{Q^{j_1}_{i_1} \cap B_1(x_0, M_1)} \left[ \tilde{D}_{k_1}(x_1, y_1) - \tilde{D}_{k_1}(x_1, x_{Q^{j_1}_{i_1}}) \right] 
\times D_{k_1}(y_1, z_1) \, d\mu_1(y_1) \right\}
\times \left\{ \int_{B_2(y_0, M_2)} \tilde{D}_{k_2}(x_2, y_2) D_{k_2}(y_2, z_2) \, d\mu_2(y_2) \right\}
= K_{31}(x_1, z_1) K_{32}(x_2, z_2).
\]

Let \( T_{31} \) and \( T_{32} \) be respectively the operator corresponding to the kernel \( K_{31} \) and the kernel \( K_{32} \). Then complete similarly to the proof of (2.42), we can find a constant \( C > 0 \) independent of \( M_1, M_2, J_1 \) and \( J_2 \) such that for all \( f \in G(\beta_2, \gamma_2) \),

\[
(2.55) \quad \|T_{32}(f)\|_{G(\beta_2, \gamma_2)} \leq C \|f\|_{G(\beta_2, \gamma_2)}.
\]

We now verify the operator \( T_{31} \) satisfies all the conditions of Lemma 2.4. In what follows, let \( C_{k_1} > 0 \) be a constant independent of \( M_1, M_2, J_1 \) and \( J_2 \) and let \( \epsilon'_1 > 0 \) be any positive number in \((0, \epsilon_1)\). Noting that \( y_1, x_{Q^{j_1}_{i_1}} \in Q^{J_1}_{i_1} \) and \( J_1 \) is large enough, by Lemma 2.5, we first have

\[
(2.56) \quad |K_{31}(x_1, z_1)|
= \left| \sum_{i_1 \in N_j} \int_{Q^{j_1}_{i_1} \cap B_1(x_0, M_1)} \left[ \tilde{D}_{k_1}(x_1, y_1) - \tilde{D}_{k_1}(x_1, x_{Q^{j_1}_{i_1}}) \right] 
\times D_{k_1}(y_1, z_1) \, d\mu_1(y_1) \right|
\leq C_{k_1} \int_{B_1(x_0, M_1)} \frac{2^{-J_1 \epsilon'_1}}{(1 + \rho_1(x_1, y_1))^{d_1+2\epsilon'_1}} |D_{k_1}(y_1, z_1)| \, d\mu_1(y_1)
\]
\[ \left| K_{31}(x_1, z_1) - K_{31}(x'_1, z_1) \right| \]

\[ = \sum_{i_j \in N_{j_1}} \int_{Q_{i_1}^{j_1} \cap B_1(x_0, M_{j_1})} \left\{ \left[ \tilde{D}_{k_1}(x_1, y_1) - \tilde{D}_{k_1}(x'_1, y_1) \right] 
- \left[ \tilde{D}_{k_1} \left( x_1, x_{Q_{i_1}^{j_1}} \right) - \tilde{D}_{k_1} \left( x'_1, x_{Q_{i_1}^{j_1}} \right) \right] \right\} D_{k_1}(y_1, z_1) \, d\mu_1(y_1) \]

\[ \leq \int_{B_1(x_0, M_{j_1})} \left| \tilde{D}_{k_1}(x_1, y_1) - \tilde{D}_{k_1}(x'_1, y_1) \right| \left| D_{k_1}(y_1, z_1) \right| \, d\mu_1(y_1) \]

\[ + \sum_{i_j \in N_{j_1}} \int_{Q_{i_1}^{j_1} \cap B_1(x_0, M_{j_1})} \left| \tilde{D}_{k_1} \left( x_1, x_{Q_{i_1}^{j_1}} \right) - \tilde{D}_{k_1} \left( x'_1, x_{Q_{i_1}^{j_1}} \right) \right| \times |D_{k_1}(y_1, z_1)| \, d\mu_1(y_1) \]

\[ \leq C_{k_1} \int_{X_1} \frac{\rho_1(x_1, x'_1)^{\epsilon_1}}{(1 + \rho_1(x_1, y_1))^{d_1 + \epsilon_1}} \frac{1}{(1 + \rho_1(y_1, z_1))^{d_1 + \epsilon_1}} \, d\mu_1(y_1) \]

\[ + C_{k_1} \rho_1(x_1, x'_1)^{\epsilon_1} \int_{X_1} \left[ \frac{1}{(1 + \rho_1(x_1, y_1))^{d_1 + \epsilon_1}} + \frac{1}{(1 + \rho_1(x'_1, y_1))^{d_1 + \epsilon_1}} \right] \]
\[
\times \frac{1}{(1 + \rho_1(y_1, z_1))^{d_1 + \epsilon_1}} \, d\mu_1(y_1) \\
\leq C_{k_1} \frac{\rho_1(x_1, x_1')^{\epsilon_1'}}{(1 + \rho_1(x_1, z_1))^{d_1 + \epsilon_1'}} \\
\leq C_{k_1} \frac{\rho_1(x_1, x_1')^{\epsilon_1'}}{\rho_1(x_1, z_1)^{d_1 + \epsilon_1'}},
\]

where in third-to-last inequality, we used the following facts that for any \(y_1 \in Q_{i_1}^{J_1}\), and all \(x_1, x_1' \in X_1\),
\[
1 + \rho_1(x_1, y_1) \leq A_1 \left(1 + \rho_1(x_1, x_{Q_{i_1}^{J_1}})\right)
\]
and
\[
1 + \rho_1(x_1', y_1) \leq A_1 \left(1 + \rho_1(x_1', x_{Q_{i_1}^{J_1}})\right)
\]
by Lemma 2.5 and the large choice on \(J_1 \in \mathbb{N}\).

The estimates (2.56) and (2.57) and the geometric means then tell us that for any \(\alpha_5 \in (0, 1)\),
\[
|K_{31}(x_1, z_1) - K_{31}(x_1', z_1)| \leq C_{k_1} 2^{-J_1 \epsilon_1'(1 - \alpha_5)} \frac{\rho_1(x_1, x_1')^{\epsilon_1' \alpha_5}}{\rho_1(x_1, z_1)^{d_1 + \epsilon_1' \alpha_5}}.
\]

Assuming that \(\rho_1(z_1, z_1') \leq \rho_{1}(x_1, z_1)\), we now estimate
\[
|K_{31}(x_1, z_1) - K_{31}(x_1', z_1')| \\
= \sum_{i_1 \in \mathbb{N}_{J_1}} \int_{Q_{i_1}^{J_1} \cap B_i(x_0, M_1)} \left[\widetilde{D}_{k_1}(x_1, y_1) - \widetilde{D}_{k_1}(x_1, x_{Q_{i_1}^{J_1}})\right] \\
\times |D_{k_1}(y_1, z_1) - D_{k_1}(y_1, z_1')| \, d\mu_1(y_1)| \\
\leq C_{k_1} \int_{\rho_1(z_1, z_1') \leq \rho_1(x_1, z_1)} \frac{1}{x_1^{2d_1} (1 + \rho_1(x_1, z_1)) (1 + \rho_1(x_1, y_1))^{d_1 + 2\epsilon_1'}} \, d\mu_1(y_1) \\
\times |D_{k_1}(y_1, z_1) - D_{k_1}(y_1, z_1')| \, d\mu_1(y_1) \\
+ C_{k_1} \int_{\rho_1(z_1, z_1') > \rho_1(x_1, z_1)} \frac{1}{x_1^{2d_1} (1 + \rho_1(x_1, z_1)) (1 + \rho_1(x_1, y_1))^{d_1 + 2\epsilon_1'}} \, d\mu_1(y_1) \\
\times |D_{k_1}(y_1, z_1)| + |D_{k_1}(y_1, z_1')| \, d\mu_1(y_1) \\
\leq C_{k_1} 2^{-J_1 \epsilon_1'} \frac{\rho_1(z_1, z_1')^{\epsilon_1'}}{(1 + \rho_1(x_1, z_1))^{d_1 + \epsilon_1'}} \\
\leq C_{k_1} 2^{-J_1 \epsilon_1'} \frac{\rho_1(z_1, z_1')^{\epsilon_1'}}{\rho_1(x_1, z_1)^{d_1 + \epsilon_1'}}.
To verify that $K_{31}$ satisfies (2.33), similarly to the proof of $K_{12}$, we may assume that

$$\rho_1(x_1, x_1') \leq \frac{\rho_1(x_1, z_1)}{4A_1} \text{ and } \rho_1(z_1, z_1') \leq \frac{\rho_1(x_1, z_1)}{4A_1}.$$  

Under these assumptions, we then write

$$[[K_{31}(x_1, z_1) - K_{31}(x_1', z_1)] - [K_{31}(x_1, z_1') - K_{31}(x_1', z_1')]]$$

$$= \left| \sum_{i_1 \in N_{J_i}} \int_{Q_{i_1} \cap B_{1}(x_0, M_1)} \left\{ \left[ \tilde{D}_{k_1}(x_1, y_1) - \tilde{D}_{k_1}(x_1', y_1) \right] 
- \left[ \tilde{D}_{k_1}(x_1, x_{Q_{i_1}^\prime}) - \tilde{D}_{k_1}(x_1', x_{Q_{i_1}^\prime}) \right] \right\} \right|$$

$$\times \left| D_{k_1}(y_1, z_1) - D_{k_1}(y_1, z_1') \right| \, d\mu_1(y_1)$$

$$\leq \int_{B_{1}(x_0, M_1)} \left| \tilde{D}_{k_1}(x_1, y_1) - \tilde{D}_{k_1}(x_1', y_1) \right| \, d\mu_1(y_1)$$

$$+ \sum_{i_1 \in N_{J_i}} \int_{Q_{i_1} \cap B_{1}(x_0, M_1)} \left| \tilde{D}_{k_1}(x_1, x_{Q_{i_1}^\prime}) - \tilde{D}_{k_1}(x_1', x_{Q_{i_1}^\prime}) \right|$$

$$\times \left| D_{k_1}(y_1, z_1) - D_{k_1}(y_1, z_1') \right| \, d\mu_1(y_1)$$

$$= O_1 + O_2.$$

For $O_1$, we only have the following three cases:

(i) $\rho_1(x_1, x_1') \leq \frac{1}{2A_1}(1 + \rho_1(x_1, y_1))$ and $\rho_1(z_1, z_1') \leq \frac{1}{2A_1}(1 + \rho_1(y_1, z_1'))$;

(ii) $\rho_1(x_1, x_1') > \frac{1}{2A_1}(1 + \rho_1(x_1, y_1))$ and $\rho_1(z_1, z_1') \leq \frac{1}{2A_1}(1 + \rho_1(y_1, z_1'))$;

(iii) $\rho_1(x_1, x_1') \leq \frac{1}{2A_1}(1 + \rho_1(x_1, y_1))$ and $\rho_1(z_1, z_1') > \frac{1}{2A_1}(1 + \rho_1(y_1, z_1'))$.

For $O_2$, by (2.58), we also only have the following three cases:

(i) $\rho_1(x_1, x_1') \leq \frac{1}{2A_1}(1 + \rho_1(x_1, x_{Q_{i_1}^\prime}))$ and $\rho_1(z_1, z_1') \leq \frac{1}{2A_1}(1 + \rho_1(y_1, z_1'))$;

(ii) $\rho_1(x_1, x_1') > \frac{1}{2A_1}(1 + \rho_1(x_1, x_{Q_{i_1}^\prime}))$ and $\rho_1(z_1, z_1') \leq \frac{1}{2A_1}(1 + \rho_1(y_1, z_1'))$;

(iii) $\rho_1(x_1, x_1') \leq \frac{1}{2A_1}(1 + \rho_1(x_1, x_{Q_{i_1}^\prime}))$ and $\rho_1(z_1, z_1') > \frac{1}{2A_1}(1 + \rho_1(y_1, z_1'))$.

Then a procedure similar to that for $K_{12}$ tells us that

$$[[K_{31}(x_1, z_1) - K_{31}(x_1', z_1)] - [K_{31}(x_1, z_1') - K_{31}(x_1', z_1')]]$$

$$\leq C_{k_1} \frac{\rho_1(x_1, x_1')^{q_i} \rho_1(z_1, z_1')^{q_i}}{(1 + \rho_1(x_1, z_1))^{d_1 + 2c_1'}}$$

$$\leq C_{k_1} \frac{\rho_1(x_1, x_1')^{q_i} \rho_1(z_1, z_1')^{q_i}}{\rho_1(x_1, z_1)^{d_1 + 2c_1'}}.$$
2.6. Let $r > \sqrt{2}$ which just means (2.54) is true for $\alpha = 1$. Thus, for all $f \in G$, (2.64) tells us that

$$||K_{31}(x, z)|| \leq C_k 2^{-J_i \epsilon'_i (1-\alpha_5)} \frac{\rho_1(x, x')^{\epsilon_i 1 + \epsilon'_i}}{\rho_1(x, z)^{1 + \epsilon_i 1 + \epsilon'_i \alpha_5}}.$$  

Finally, we verify that $K_{31}$ has the strong weak boundedness property as in Definition 2.6. Let $r > 0$ and $f$ be a continuous function on $X_1 \times X_1$ with $\text{supp } f \subset B_1(x_{11}, r) \times B_1(x_{12}, r)$, where $x_{11}, x_{12} \in X_1$, $||f||_{L^\infty(X_1 \times X_1)} \leq 1$, $||f(\cdot, z_1)||_{C^0_\alpha(X_1)} \leq r^{-\eta}$ for all $z_1 \in X_1$ and $||f(x_1, \cdot)||_{C^0_\alpha(X_1)} \leq r^{-\eta}$ for all $x_1 \in X_1$. From (2.56), it follows that

$$||K_{31}, f|| \leq \int_{X_1} \int_{X_1} K_{31}(x, z_1) f(x, z_1) \, d\mu_1(x_1) \, d\mu_1(z_1) \leq C_k 2^{-J_i \epsilon'_i \alpha_5} \int_{B_1(x_{11}, r)} \int_{X_1} \frac{1}{1 + \rho(x, z_1)^{d_1 + \epsilon'_i}} \, d\mu_1(x_1) \, d\mu_1(z_1) \leq C_k 2^{-J_i \epsilon'_i \alpha_5} r^{d_1}.$$  

Obviously $T_{31}(1) = 0$, which together with the estimates (2.56), (2.60), (2.61), (2.63) and (2.64) tells us that $T_{31}$ satisfies all the conditions of Lemma 2.4 with

$$||T_{31}|| \leq C_k 2^{-J_i \epsilon'_i (1-\alpha_5)}.$$  

Thus, for all $f \in G(\beta_1, \gamma_1)$,

$$||T_{31}(f)||_{G(\beta_1, \gamma_1)} \leq C_k 2^{-J_i \epsilon'_i (1-\alpha_5)} ||f||_{G(\beta_1, \gamma_1)}.$$  

From Lemma 2.3, (2.65) and (2.55), it follows that $T_3$ is bounded on $G(\beta_1, \beta_2; \gamma_1, \gamma_2)$ and for all $g \in G(\beta_1, \beta_2; \gamma_1, \gamma_3)$,

$$||T_3(g)||_{G(\beta_1, \beta_2; \gamma_1, \gamma_3)} \leq C_k 2^{-J_i \epsilon'_i (1-\alpha_5)} ||g||_{G(\beta_1, \beta_2; \gamma_1, \gamma_3)},$$  

which just means (2.54) is true for $i = 1$, and therefore, it is also true for $i = 2$.

By (2.53), (2.54) and Lemma 2.5, we obtain

$$\langle f, \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2} (g) \rangle$$
Using the size and smooth conditions of \( \bar{D}_{k_1} \) and \( \bar{D}_{k_2} \), by the geometric means, we can verify that the functions on \( x_1 \) and \( x_2 \),

\[
\sum_{i_1 \in N_{J_1}} \sum_{i_2 \in N_{J_2}} \bar{D}_{k_1} \left( x_1, x_{Q_{i_1}^1} \right) \left[ \bar{D}_{k_2} \left( x_2, x_{Q_{i_2}^2} \right) - \bar{D}_{k_2} \left( x_2, y_2 \right) \right] \\
\times \chi_{Q_{i_1}^1 \cap B_1(x_0, M_1)} (y_1) \chi_{Q_{i_2}^2 \cap B_2(y_0, M_2)} (y_2)
\]

and

\[
\sum_{i_1 \in N_{J_1}} \sum_{i_2 \in N_{J_2}} \left[ \bar{D}_{k_1} \left( x_1, x_{Q_{i_1}^1} \right) - \bar{D}_{k_1} \left( x_1, y_1 \right) \right] \bar{D}_{k_2} \left( x_2, y_2 \right) \\
\times \chi_{Q_{i_1}^1 \cap B_1(x_0, M_1)} (y_1) \chi_{Q_{i_2}^2 \cap B_2(y_0, M_2)} (y_2)
\]

belong to the space \( G(\beta_1, \beta_2; \gamma_1, \gamma_2) \) uniformly in \( y_1 \) and \( y_2 \), and there exist some constants \( \alpha_6 > 0 \) and \( C > 0 \) independent of \( M_i \), \( J_i \) and \( y_i \) for \( i = 1, 2 \), such that

\[
\left\| \sum_{i_1 \in N_{J_1}} \sum_{i_2 \in N_{J_2}} \bar{D}_{k_1} \left( \cdot, x_{Q_{i_1}^1} \right) \left[ \bar{D}_{k_2} \left( \cdot, x_{Q_{i_2}^2} \right) - \bar{D}_{k_2} \left( \cdot, y_2 \right) \right] \\
\times \chi_{Q_{i_1}^1 \cap B_1(x_0, M_1)} (y_1) \chi_{Q_{i_2}^2 \cap B_2(y_0, M_2)} (y_2) \right\|_{G(\beta_1, \beta_2; \gamma_1, \gamma_2)} \\
\leq C 2^{-\alpha_6 J_2}
\]
Let all the notation be the same as in Theorem 2.2. Then for all \( D \) holds in \( \text{Theorem 2.4} \) producing formulae in spaces of distributions. where the last equality can be obtained by repeating the above procedure on \( C > 0 \) is independent of \( M_1, J \) and \( y_i \) for \( i = 1, 2 \). Moreover, it is easy to check that \( D_{k_1} D_{k_2}(g) \in L^1(X_1 \times X_2) \), which together with (2.67), (2.68), (2.66) and the Lebesgue dominated convergence theorem yields that

\[
\left\| \sum_{i_1 \in N_{j_1}} \sum_{i_2 \in N_{j_2}} \left[ \tilde{D}_{k_1} \left( \cdot, x_{Q_{i_1}^j} \right) - \tilde{D}_{k_1} \left( \cdot, y_1 \right) \right] \tilde{D}_{k_1} \left( \cdot, y_2 \right) \times \chi_{Q_{i_1}^j \cap B_1(x_0, M_1)}(y_1) \chi_{Q_{i_2}^j \cap B_2(y_0, M_2)}(y_2) \right\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)} \leq C^{2-\alpha_6 J_1},
\]

which imply that

\[
(2.67) \quad \left| \tilde{D}_{k_1}^* \tilde{D}_{k_2}^*(f) \left( x_{Q_{i_1}^j}, x_{Q_{j_2}^i} \right) - \tilde{D}_{k_1}^* \tilde{D}_{k_2}^*(f) \left( x_{Q_{i_1}^j}, y_2 \right) \right| \leq C^{2-\alpha_6 J_2}
\]

and

\[
(2.68) \quad \left| \tilde{D}_{k_1}^* \tilde{D}_{k_2}^*(f) \left( x_{Q_{i_1}^j}, y_2 \right) - \tilde{D}_{k_1}^* \tilde{D}_{k_2}^*(f) \left( y_1, y_2 \right) \right| \leq C^{2-\alpha_6 J_1},
\]

where \( C > 0 \) is independent of \( M_1, J \) and \( y_i \) for \( i = 1, 2 \). Moreover, it is easy to check that \( D_{k_1} D_{k_2}(g) \in L^1(X_1 \times X_2) \), which together with (2.67), (2.68), (2.66) and the Lebesgue dominated convergence theorem yields that

\[
\left\langle f, \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2} \right\rangle(g)
= \int_{X_1} \int_{X_2} \tilde{D}_{k_1}^* \tilde{D}_{k_2}^* \left( y_1, y_2 \right) D_{k_1} D_{k_2} \left( y_1, y_2 \right) d\mu_1(y_1) d\mu_2(y_2)
= \left\langle \tilde{D}_{k_1}^* D_{k_1}^* \tilde{D}_{k_2}^* D_{k_2}^* \left( f, g \right) \right\rangle,
\]

where the last equality can be obtained by repeating the above procedure on \( \tilde{D}_{k_1} \) and \( \tilde{D}_{k_2} \).

This proves (2.34) and we complete the proof of Theorem 2.3.

Similarly, from Theorem 2.2, we can deduce the following continuous Calderón reproducing formulae in spaces of distributions.

**Theorem 2.4** Let all the notation be the same as in Theorem 2.2. Then for all \( f \in \left( \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)' \),

\[
f = \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \tilde{D}_{k_1}^* \tilde{D}_{k_2}^* D_{k_1}^* D_{k_2}^*(f)
\]

holds in \( \left( \mathcal{G}(\beta_1', \beta_2'; \gamma_1', \gamma_2') \right)' \) with \( \beta_i' \in (\beta_i, \epsilon_i) \) and \( \gamma_i' \in (\gamma_i, \epsilon_i) \) for \( i = 1, 2 \), where \( D_{k_1}^*(x, y) = D_{k_1}(y, x) \) and \( \tilde{D}_{k_1}^*(x, y) = \tilde{D}_{k_1}(y, x) \).
Let $i = 1, 2$. Note that $D_{k_i}^*, \tilde{D}_{k_i}^*$ and $\overline{D}_{k_i}^*$ respectively have the same properties as $D_{k_i}, \tilde{D}_{k_i}$ and $\overline{D}_{k_i}$. From this, it is easy to see that we can re-state Theorem 2.3 and Theorem 2.4 as the following theorem, which will simplify the notation in the following applications of these formulae.

**Theorem 2.5** Let all the notation be the same as in Theorem 2.1 and Theorem 2.2. Then for all $f \in \left(\hat{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)\right)'$,

$$ f = \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \sum_{\tau \in I_{k_1}} \sum_{\nu = 1}^{N(k_1, \tau)} \mu(Q_{\tau}^{k_1, \nu}) \bar{D}_{k_1}D_{k_2}(f) = \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \sum_{\tau \in I_{k_1}} \sum_{\nu = 1}^{N(k_1, \tau)} \mu(Q_{\tau}^{k_1, \nu}) \bar{D}_{k_1}D_{k_2}(f), $$

holds in $\left(\hat{G}(\beta_1', \beta_2'; \gamma_1', \gamma_2')\right)'$ with $\beta_i' \in (\beta_i, \epsilon_i)$ and $\gamma_i' \in (\gamma_i, \epsilon_i)$ for $i = 1, 2$.

We now recall the discrete Calderón reproducing formulae on spaces of homogeneous type in [H3].

**Lemma 2.6** With all the notation as in Lemma 2.1, then for all $f \in G(\beta, \gamma)$ with $\beta, \gamma \in (0, \epsilon)$ and any $y_r^{k, \nu} \in Q_r^{k, \nu}$,

$$ (2.69) \quad f(x) = \sum_{k = -\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu = 1}^{N(k, \tau)} \mu(Q_{\tau}^{k, \nu}) \bar{D}_{k}(x, y_r^{k, \nu})D_{k}(f)(y_r^{k, \nu}) $$

$$ = \sum_{k = -\infty}^{\infty} \sum_{\tau \in I_k} \sum_{\nu = 1}^{N(k, \tau)} \mu(Q_{\tau}^{k, \nu}) D_{k}(x, y_r^{k, \nu}) \overline{D}_{k}(f)(y_r^{k, \nu}), $$

where the series converge in the norm of both the space $G(\beta', \gamma')$ with $0 < \beta' < \beta$ and $0 < \gamma' < \gamma$ and the space $L^p(X)$ with $p \in (1, \infty)$.

By an argument of duality, Han in [H3] also established the following discrete Calderón reproducing formulae on spaces of distributions, $\left(\hat{G}(\beta, \gamma)\right)'$ with $\beta, \gamma \in (0, \epsilon)$.

**Lemma 2.7** With all the notation as in Lemma 2.6, then for all $f \in \left(\hat{G}(\beta, \gamma)\right)'$ with $\beta, \gamma \in (0, \epsilon)$, (2.69) holds in $\left(\hat{G}(\beta', \gamma')\right)'$ with $\beta < \beta' < \epsilon$ and $\gamma < \gamma' < \epsilon$.

By a procedure similar to the proofs of Theorems 2.1, 2.2, 2.3 and 2.4, using Lemma 2.6 and Lemma 2.7, we can also establish the following discrete Calderón reproducing formulae on product spaces of homogeneous-type spaces. We only state the results and leave the details to the reader; see also [HY].

**Theorem 2.6** Let all the notation as in Theorems 2.1 and 2.2, and

$$ \{Q_{\tau_1}^{k_1, \nu_1} : k_1 \in Z, \tau_1 \in I_{k_1}, \nu_1 = 1, \cdots, N(k_1, \tau_1)\} $$
and \( \{Q_{k_2,\nu_2} : k_2 \in \mathbb{Z}, \tau_2 \in I_{k_2}, \nu_2 = 1, \cdots, N(k_2, \tau_2)\} \) respectively be the dyadic cubes of \( X_1 \) and \( X_2 \) defined above with \( j_1, j_2 \in \mathbb{N} \) large enough. Then for all \( f \in G(\beta_1, \beta_2; \gamma_1, \gamma_2) \) with \( \beta_i, \gamma_i \in (0, \epsilon_i) \) for \( i = 1, 2 \) and any \( y_{k_1,\nu_1} \in Q_{k_1,\nu_1}^{k_1,\nu_1} \) and \( y_{k_2,\nu_2} \in Q_{k_2,\nu_2}^{k_2,\nu_2} \),

\[
(2.70) \quad f(x_1, x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{N(k_1,\tau_1)} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{N(k_2,\tau_2)} \mu_1(Q_{k_1,\nu_1}) \mu_2(Q_{k_2,\nu_2}) \\
\times \tilde{D}_{k_1} (x_1, y_{k_1,\nu_1}) \tilde{D}_{k_2} (x_2, y_{k_2,\nu_2}) D_{k_1} D_{k_2} (f)(y_{k_1,\nu_1}, y_{k_2,\nu_2}) \\
= \sum_{k_1=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{N(k_1,\tau_1)} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{N(k_2,\tau_2)} \mu_1(Q_{k_1,\nu_1}) \mu_2(Q_{k_2,\nu_2}) \\
\times D_{k_1} (x_1, y_{k_1,\nu_1}) D_{k_2} (x_2, y_{k_2,\nu_2}) \overline{D}_{k_1} \overline{D}_{k_2} (f)(y_{k_1,\nu_1}, y_{k_2,\nu_2}),
\]

where the series converge in the norm of both the space \( G(\beta_1', \beta_2'; \gamma_1', \gamma_2') \) with \( \beta_i' \in (0, \beta_i) \) and \( \gamma_i' \in (0, \gamma_i) \) for \( i = 1, 2 \), and \( L^p(X_1 \times X_2) \) with \( p \in (1, \infty) \).

**Theorem 2.7** Let all the notation be the same as in Theorem 2.6. Then for all \( f \in \left( \hat{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)' \), (2.70) holds in \( \left( \hat{G}(\beta_1', \beta_2'; \gamma_1', \gamma_2') \right)' \) with \( \beta_i' \in (\beta_i, \epsilon_i) \) and \( \gamma_i' \in (\gamma_i, \epsilon_i) \) for \( i = 1, 2 \).
3 Littlewood-Paley theory

We first establish the Littlewood-Paley theorem on product spaces of spaces of homogeneous type. To this end, we recall the Littlewood-Paley theorem on spaces of homogeneous type in [DJS].

**Lemma 3.1** Let $X$ be a space of homogeneous type as in Definition 2.1, $\epsilon \in (0, \theta)$, $\{S_k\}_{k \in \mathbb{Z}}$ be an approximation to the identity of order $\epsilon$ as in Definition 2.3 and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$. If $1 < p < \infty$, then there is a constant $C_p > 0$ such that for all $f \in L^p(X)$,

$$C_p^{-1}\|f\|_{L^p(X)} \leq \left\{ \sum_{k=-\infty}^{\infty} |D_k(f)|^2 \right\}^{1/2} \leq C_p\|f\|_{L^p(X)}.$$

(3.1)

The Littlewood-Paley theorem on product spaces of homogeneous-type spaces can be stated as follows, whose proof can be deduced from the well-known discrete vector-valued Littlewood-Paley theorem on spaces of homogenous type, Lemma 3.1 and the Calderón reproducing formulae, Theorem 2.1; see also the proof of Theorem 2 in [FS].

**Theorem 3.1** Let $i = 1, 2, X_i$ be a space of homogeneous type as in Definition 2.1, $\epsilon_i \in (0, \theta_i)$, $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$ be an approximation to the identity of order $\epsilon_i$ on space of homogeneous type, $X_i$, and $D_{k_i} = S_{k_i} - S_{k_i-1}$ for all $k_i \in \mathbb{Z}$. If $1 < p < \infty$, then there is a constant $C_p > 0$ such that for all $f \in L^p(X_1 \times X_2)$,

$$C_p^{-1}\|f\|_{L^p(X_1 \times X_2)} \leq \|g_2(f)\|_{L^p(X_1 \times X_2)} \leq C_p\|f\|_{L^p(X_1 \times X_2)},$$

(3.2)

where $g_q(f)$ for $q \in (0, \infty)$ is called the discrete Littlewood-Paley $g$-function defined by

$$g_q(f)(x_1, x_2) = \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} |D_{k_1}D_{k_2}(f)(x_1, x_2)|^{q/2} \right\}^{1/q}$$

for $x_1 \in X_1$ and $x_2 \in X_2$.

**Proof.** We first prove the second inequality in (3.2). To do so, we use the known vector-valued Littlewood-Paley theorem on $X_2$ and Lemma 3.1 on $X_1$. Let $f \in L^p(X_1 \times X_2)$, $F(x_1, x_2) = \{F_{k_1}(x_1, x_2)\}_{k_1 \in \mathbb{Z}}$ and $F_{k_1}(x_1, x_2) = D_{k_1} [f(\cdot, x_2)](x_1)$ for $k_1 \in \mathbb{Z}$. Set

$$\|F(x_1, x_2)\|_{l^2(\mathbb{Z})} = \left\{ \sum_{k_1=-\infty}^{\infty} |F_{k_1}(x_1, x_2)|^2 \right\}^{1/2}.$$ 

For $k_2 \in \mathbb{Z}$, define $D_{k_2}F(x_1, x_2) = \{D_{k_2}F_{k_1}(x_1, \cdot)(x_2)\}_{k_1 \in \mathbb{Z}}$ and the discrete $l^2(\mathbb{Z})$-valued Littlewood-Paley $g_{2,2}$ function on $X_2$ by

$$g_{2,2}[F(x_1, \cdot)](x_2) = \left\{ \sum_{k_2=-\infty}^{\infty} \|D_{k_2}F(x_1, x_2)\|_{l^2(\mathbb{Z})}^2 \right\}^{1/2}.$$ 

(3.3)
Obviously,

\[
g_{2,2}[F(x_1,\cdot)](x_2) = g_2(f)(x_1, x_2).
\]

Taking the \(L^p(X_2)\)-norm on both sides of (3.3) and by the \(l^2(\mathbb{Z})\)-valued Littlewood-Paley theorem on \(L^p(X_2)\), we obtain

\[
\int_{X_2} |g_{2,2}[F(x_1,\cdot)](x_2)|^p \, d\mu_2(x_2) \leq C_p^p \int_{X_2} \|F(x_1, x_2)\|_{l^2(\mathbb{Z})}^p \, d\mu_2(x_2).
\]

Taking the \(L^p(X_1)\)-norm on both sides of (3.5), using (3.4) and Lemma 3.1 on \(L^p(X_1)\), and exchanging the order of the integrals on \(X_1\) and \(X_2\) yield the desired second inequality of (3.2).

We now prove the first inequality in (3.2) by using Theorem 2.1 and the second inequality of (3.2). Let \(f \in L^p(X_1 \times X_2)\) and all the notation be the same as in Theorem 2.1. By Theorem 2.1, we have

\[
f = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2}(f).
\]

Let \(1/p + 1/p' = 1\) and \(g \in L^{p'}(X_1 \times X_2)\). The Hölder inequality tells us that

\[
|\{f, g\}| = \left| \left\langle \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2}(f), g \right\rangle \right|
= \left| \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \left\langle D_{k_1} D_{k_2}(f), \tilde{D}_{k_1}^* \tilde{D}_{k_2}^*(g) \right\rangle \right|
\leq \|g_2(f)\|_{L^p(X_1 \times X_2)} \left\| \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} |\tilde{D}_{k_1}^* \tilde{D}_{k_2}^*(g)|^2 \right\}^{1/2} \right\|_{L^{p'}(X_1 \times X_2)},
\]

where \(\tilde{D}_{k_i}^*(x_i, y_i) = \tilde{D}_{k_i}(y_i, x_i)\) for \(i = 1, 2\). It is well-known that for \(k_i, k_i' \in \mathbb{Z},\)

\[
|\tilde{D}_{k_i}^* \tilde{D}_{k_i'}^*(x_i, z_i)| \leq C 2^{-|k_i - k_i'|} \frac{2^{-((k_i \wedge k_i') \wedge k_i')}}{(2^{-((k_i \wedge k_i') \wedge k_i')} + \rho_i(x_i, z_i))^{d_i + \epsilon_i}},
\]

where \(i = 1, 2, \epsilon_i \in (0, \epsilon)\) and \(a \wedge b = \min(a, b)\) for any \(a, b \in \mathbb{R};\) see \([H1]\) for a proof. Let \(M_i\) be the Hardy-Littlewood maximal function on \(X_i\). Applying (3.6) to the function \(g\) and using (3.8) and the Hölder inequality lead us that

\[
\left\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} |\tilde{D}_{k_1}^* \tilde{D}_{k_2}^*(g)|^2 \right\}^{1/2}
\]
Remark 3.1 Let $\overline{D}_{k_i}$ for $i = 1, 2$ be the same as in Theorem 3.2. In the proof of Theorem 3.1, we actually prove that the second inequality in (3.2) still holds if we replace $D_{k_1}$ there by those $\overline{D}_{k_i}$ for $i = 1, 2$; see (3.10). This is well known fact for the second inequality in (3.1); see also [DJS, H2].

Let all the notation be the same as in Theorem 3.1. We now define the Littlewood-
Paley $S$-function $S_q$ on the product space $X_1 \times X_2$ by

\[
S_q(f)(x_1, x_2) = \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \int_{\mathbb{R}} \rho_1(x_1, y_1) \leq C_{11, 1} 2^{-k_1} \int_{\mathbb{R}} \rho_2(x_2, y_2) \leq C_{11, 2} 2^{-k_2} 2^{k_1 d_1 + k_2 d_2} \times |D_{k_1} D_{k_2}(f)(y_1, y_2)|^{1/q} \ d\mu_1(y_1) \ d\mu_2(y_2) \right\}^{1/q}
\]

for $x_1 \in X_1$ and $x_2 \in X_2$.

We have the following relation theorem on the Littlewood-Paley $S$-function $S_q$ and the Littlewood-Paley $g$-function $g_q$.

**Lemma 3.2** Let $1 < p, q < \infty$. Then there exists a constant $C_{p, q} > 0$ such that for all $f \in L^p(X_1 \times X_2)$,

\[
\|S_q(f)\|_{L^p(X_1 \times X_2)} \leq C_{p, q} \|g_q(f)\|_{L^p(X_1 \times X_2)}.
\]

**Proof.** Let $f \in L^p(X_1 \times X_2)$ and all the notation be the same as in Theorem 2.1. By Theorem 2.1, we write $f$ as in (3.6). Let $\rho_1(x_1, y_1) \leq C_{11, 1} 2^{-k_1}$ and $\rho_2(x_2, y_2) \leq C_{11, 2} 2^{-k_2}$, which imply that

\[
2^{-(k_i \wedge k_i')} + \rho_i(x_i, z_i) \leq C \left( 2^{-(k_i \wedge k_i')} + \rho_i(y_i, z_i) \right)
\]

for $i = 1, 2$. From this and an estimate similar to (3.8) with $D_{k_i}^*$ there replaced by $D_{k_i}$, it follows that

\[
|D_{k_1} D_{k_2}(f)(y_1, y_2)|
\]

\[
\leq C \sum_{k_1' = -\infty}^{\infty} \sum_{k_2' = -\infty}^{\infty} 2^{-|k_1 - k_1'| \epsilon_1} 2^{-|k_2 - k_2'| \epsilon_2} \times \int_{\mathbb{R}} \int_{\mathbb{R}} (2^{-(k_1 \wedge k_1')} + \rho_1(y_1, z_1))^{d_1 + \epsilon_1'} \left( 2^{-(k_2 \wedge k_2')} + \rho_2(y_2, z_2) \right)^{d_2 + \epsilon_2'} \times |D_{k_1} D_{k_2}(f)(z_1, z_2)| \ d\mu_1(z_1) \ d\mu_2(z_2)
\]

\[
\leq C \sum_{k_1' = -\infty}^{\infty} \sum_{k_2' = -\infty}^{\infty} 2^{-|k_1 - k_1'| \epsilon_1} 2^{-|k_2 - k_2'| \epsilon_2} \times \int_{\mathbb{R}} \int_{\mathbb{R}} (2^{-(k_1 \wedge k_1')} + \rho_1(x_1, z_1))^{d_1 + \epsilon_1'} \left( 2^{-(k_2 \wedge k_2')} + \rho_2(x_2, z_2) \right)^{d_2 + \epsilon_2'} \times |D_{k_1} D_{k_2}(f)(z_1, z_2)| \ d\mu_1(z_1) \ d\mu_2(z_2)
\]
\[ \leq C \sum_{k_1' = -\infty}^{\infty} \sum_{k_2' = -\infty}^{\infty} 2^{-|k_1-k_1'| \epsilon_1} 2^{-|k_2-k_2'| \epsilon_2} M_1 M_2 [D_{k_1} D_{k_2} (f)] (x_1, x_2). \]

Using (3.12) and an iterative application of the Fefferman-Stein vector-valued inequality in \([FeS]\) on \(L^p(X_1)\) and \(L^p(X_2)\) yield that

\[ \| S_q (f) \|_{L^p(X_1 \times X_2)} \]
\[ \leq C \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \left( \sum_{k_1' = -\infty}^{\infty} \sum_{k_2' = -\infty}^{\infty} 2^{-|k_1-k_1'| \epsilon_1} 2^{-|k_2-k_2'| \epsilon_2} M_1 M_2 [D_{k_1} D_{k_2} (f)] \right)^{q} \right\}^{1/q} \| L^p(X_1 \times X_2) \]
\[ \leq C \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} (M_1 M_2 [D_{k_1} D_{k_2} (f)])^q \right\}^{1/q} \| L^p(X_1 \times X_2) \]
\[ \leq C_{p,q} \| g_q (f) \|_{L^p(X_1 \times X_2)}, \]

which completes the proof of Lemma 3.2.

**Remark 3.2** It is easy to see from the proof of Lemma 3.2 that if we replace \(D_{k_i}\) in the definition of \(S_q (f)(x_1, x_2)\), (6.32), by \( \overline{D}_{k_i} \) as in Theorem 2.2 for \(i = 1, 2\), then Lemma 3.2 still holds. This is because the contribution of \(D_{k_i}\) to the inequality in the lemma comes from an estimate similar to (3.8), which still holds if we replace \(D_{k_i}\) by \( \overline{D}_{k_i}\) for \(i = 1, 2\).

**Lemma 3.3** Let \(1 < p, q < \infty\). Then there exists a constant \(C_p > 0\) such that for all \(f \in L^p(X_1 \times X_2)\),

\[ \|f\|_{L^p(X_1 \times X_2)} \leq C_p \| S_2 (f) \|_{L^p(X_1 \times X_2)}. \]

**Proof.** Let \(f \in L^p(X_1 \times X_2)\) and all the notation be the same as in Theorem 2.1. By Theorem 2.1, we write \(f\) as in (3.6). Let \(g \in L^p(X_1 \times X_2)\). The Hölder inequality, changing the order of the integrals, Remark 3.2 and Theorem 3.1 tell us that

\[ |\langle f, g \rangle| = \left| \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \tilde{D}_{k_1} \tilde{D}_{k_2} D_{k_1} D_{k_2} (f), g \right| \]
\[ = \left| \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \left( \langle D_{k_1} D_{k_2} (f), \tilde{D}_{k_1}^* \tilde{D}_{k_2}^* (g) \rangle \right) \right| \]
\[ \leq C \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \int_{X_1 \times X_2} \left| D_{k_1} D_{k_2} (f)(y_1, y_2) \tilde{D}_{k_1}^* \tilde{D}_{k_2}^* (g)(y_1, y_2) \right| dy_1 dy_2 \]
\[ \times \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \int_{X_1} \chi_{B_1(y_1, C_{11, 2^{-k_1}})} (x_1) d\mu_1 (x_1) \right\} \]
\[ \begin{align*}
&\times \left\{ 2^{k_2d_2} \int_{X_2} x_{B_2(y_2, C_{11.2^{-k_2}})}(x_2) \, d\mu_2(x_2) \right\} \, d\mu_1(y_1) \, d\mu_2(y_2) \\
&\leq C \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \int_{X_1 \times X_2} \int_{\rho_1(x_1, y_1) \leq C_{11.2^{-k_1}}} \int_{\rho_2(x_2, y_2) \leq C_{11.2^{-k_2}}^{2k_1d_1 + k_2d_2}} \left| D_{k_1}^{*} D_{k_2}(f)(y_1, y_2) \widetilde{D}_{k_1}^{*}(g)(y_1, y_2) \right| \, d\mu_1(y_1) \, d\mu_2(y_2) \, d\mu_1(x_1) \, d\mu_2(x_2) \\
&\leq C \int_{X_1 \times X_2} S_q(f)(x_1, x_2) \\
&\times \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \int_{\rho_1(x_1, y_1) \leq C_{11.2^{-k_1}}} \int_{\rho_2(x_2, y_2) \leq C_{11.2^{-k_2}}^{2k_1d_1 + k_2d_2}} \left| \widetilde{D}_{k_1}^{*}(g)(y_1, y_2) \right|^{q'} \, d\mu_1(y_1) \, d\mu_2(y_2) \right\}^{1/q'} \, d\mu_1(x_1) \, d\mu_2(x_2) \\
&\leq C \left\| S_2(f) \right\|_{L^p(X_1 \times X_2)} \left\| g_2(g) \right\|_{L^{p'}(X_1 \times X_2)} \\
&\leq C \left\| g \right\|_{L^{p'}(X_1 \times X_2)} \left\| S_2(f) \right\|_{L^p(X_1 \times X_2)}.
\end{align*} \]

Taking the supremum on \( g \in L^p(X_1 \times X_2) \) with \( \left\| g \right\|_{L^{p'}(X_1 \times X_2)} \leq 1 \) on both sides of (3.13) yields the conclusion of the lemma, which completes the proof of Lemma 3.3.

Lemma 3.2, Lemma 3.3 and Theorem 3.1 imply the following equivalence of the Littlewood-Paley \( S \)-function and \( g \)-function in \( L^p(X_1 \times X_2) \)-norm.

**Theorem 3.2** Let all the notation be the same as in Theorem 3.1, \( g_2 \) and \( S_2 \) be defined respectively as in Theorem 3.1 and (6.32). If \( 1 < p < \infty \), then there is a constant \( C_p > 0 \) such that for all \( f \in L^p(X_1 \times X_2) \),

\[ C_p^{-1} \left\| S_2(f) \right\|_{L^p(X_1 \times X_2)} \leq \left\| g_2(f) \right\|_{L^p(X_1 \times X_2)} \leq C_p \left\| S_2(f) \right\|_{L^p(X_1 \times X_2)}. \]
4 \( H^p \) spaces

In this section, we first apply the discrete Calderón reproducing formulae, Theorem 2.7, to establish the equivalence between the Littlewood-Paley \( S \)-function and \( g \)-function in \( L^p(X_1 \times X_2) \)-norm with \( p \leq 1 \), which generalizes Theorem 3.2. Such a result for non-product spaces was already obtained in [H2] via a Plancherel-Polya inequality. We use the same ideas as in [H2] here. Thus, we first establish a product-type Plancherel-Polya inequality. To this end, we need the following lemma which can be found in [FJ, pp. 147-148] for \( \mathbb{R}^n \) and [HS, p. 93] for spaces of homogeneous type.

Lemma 4.1. Let \( X \) be a space of homogeneous type as in Definition 2.1, \( 0 < r \leq 1, k, \eta \in \mathbb{Z}_+ \) with \( \eta \leq k \) and for any dyadic cube \( Q_{\tau}^{k,\nu} \),

\[
|f_{Q_{\tau}^{k,\nu}}(x)| \leq (1 + 2^\eta \rho(x, y_{\tau}^{k,\nu}))^{-d - \gamma},
\]

where \( x \in X \), \( y_{\tau}^{k,\nu} \) is any point in \( Q_{\tau}^{k,\nu} \) and \( \gamma > d(1/r - 1) \). Then

\[
\sum_{\tau \in I_k} \sum_{\nu = 1}^{N(k,\tau)} |\lambda_{Q_{\tau}^{k,\nu}}||f_{Q_{\tau}^{k,\nu}}(x)| \leq C 2^{(k-\eta)d/r} \left[ M \left( \sum_{\tau \in I_k} \sum_{\nu = 1}^{N(k,\tau)} |\lambda_{Q_{\tau}^{k,\nu}}| r x_{Q_{\tau}^{k,\nu}}(x) \right) \right]^{1/r},
\]

where \( C \) is independent of \( x, k \) and \( \eta \), and \( M \) is the Hardy-Littlewood maximal operator on \( X \).

Theorem 4.1 Let the notation be the same as in Theorem 2.6. Moreover, let

\[
\{Q_{\tau_1}^{k_1,\nu_1'} : k_1' \in \mathbb{Z}, \tau_1' \in I_{k_1'}, \nu_1' = 1, \cdots, N(k_1', \tau_1') \}
\]

and \( \{Q_{\tau_2}^{k_2,\nu_2'} : k_2' \in \mathbb{Z}, \tau_2' \in I_{k_2'}, \nu_2' = 1, \cdots, N(k_2', \tau_2') \} \) respectively be another set of dyadic cubes of \( X_1 \) and \( X_2 \) defined above with \( j_1', j_2' \in \mathbb{N} \) large enough, let \( \{P_{k_i} \}_{k_i \in \mathbb{Z}} \) be another approximation to the identity of order \( \epsilon_i \) on homogeneous-type space \( X_i \) and \( E_{k_i} = P_{k_i} - P_{k_i-1} \) for \( k_i \in \mathbb{Z} \) and \( i = 1, 2 \). If \( \max \{ \frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2} \} < p, q \leq \infty \), then there is a constant \( C > 0 \) such that for all \( f \in \left( \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)' \) with \( \beta_i, \gamma_i \in (0, \epsilon_i) \) for \( i = 1, 2 \),

\[
(4.1) \left\| \sum_{k_1=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1 = 1}^{N(k_1,\tau_1)} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2 = 1}^{N(k_2,\tau_2)} \sup_{z_1 \in Q_{\tau_1}^{k_1,\nu_1'}, z_2 \in Q_{\tau_2}^{k_2,\nu_2'}} |D_{k_1}D_{k_2}(f)(z_1, z_2)|^q \right\|_{L^p(X_1 \times X_2)}
\]

\[
\times |x_{Q_{\tau_1}^{k_1,\nu_1'}}(\cdot)|^{1/q} \right\|_{L^q(X_1 \times X_2)} \leq C \left\| \sum_{k_1=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1 = 1}^{N(k_1,\tau_1)} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2 = 1}^{N(k_2,\tau_2)} \times \inf_{z_1 \in Q_{\tau_1}^{k_1,\nu_1'}, z_2 \in Q_{\tau_2}^{k_2,\nu_2'}} |E_{k_1}E_{k_2}(f)(z_1, z_2)|^q \right\|_{L^p(X_1 \times X_2)} \left\| x_{Q_{\tau_1}^{k_1,\nu_1'}}(\cdot) \right\|_{L^q(X_1 \times X_2)} \left\| x_{Q_{\tau_2}^{k_2,\nu_2'}}(\cdot) \right\|_{L^q(X_1 \times X_2)} \right\|_{L^p(X_1 \times X_2)}.
Proof. We first choose \( \epsilon'_i \in (0, \epsilon_i) \) for \( i = 1, 2 \) and \( r \in (0, 1] \) such that

\[
\max \left\{ \frac{d_1}{d_1 + \epsilon'_1}, \frac{d_2}{d_2 + \epsilon'_2} \right\} < r < \min(p, q).
\]

Let \( f \in \left( G(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)' \) with \( \beta_i, \gamma_i \in (0, \epsilon_i) \) for \( i = 1, 2 \). By Theorem 2.7, we have

\[
f(x_1, x_2) = \sum_{k_1' = -\infty}^{\infty} \sum_{\nu_1' = 1}^{N(k_1', \nu_1')} \sum_{k_2' = -\infty}^{\infty} \sum_{\nu_2' = 1}^{N(k_2', \nu_2')} \mu_1(Q_{\tau_1'}') \mu_2(Q_{\tau_2'}')
\]

\[
\times \bar{E}_{k_1'}(x_1, y_{\tau_1'}') \bar{E}_{k_2'}(x_2, y_{\tau_2'}') E_{k_1'} E_{k_2'} (f)(y_{\tau_1'}', y_{\tau_2'}')
\]

holds in \( \left( G(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2) \right)' \) with \( \beta'_i \in (\beta_i, \epsilon_i) \) and \( \gamma'_i \in (\gamma_i, \epsilon_i) \), where \( \bar{E}_{k_i} \) satisfies the same properties as \( \bar{D}_{k_i} \) in Theorem 2.7 and \( \nu \) and \( \Theta_j \) with \( \bar{D}_{k_i} \) and \( \bar{E}_{k_i} \) respectively replaced by \( D_{k_i} \) and \( \bar{E}_{k_i} \), it follows that for any \( k_1, k_2 \in \mathbb{Z}, \)

\[
D_{k_1} D_{k_2} (f)(z_1, z_2)
\]

\[
= \left| \sum_{k_1' = -\infty}^{\infty} \sum_{\nu_1' = 1}^{N(k_1', \nu_1')} \sum_{k_2' = -\infty}^{\infty} \sum_{\nu_2' = 1}^{N(k_2', \nu_2')} \mu_1(Q_{\tau_1'}') \mu_2(Q_{\tau_2'}')
\]

\[
\times D_{k_1} \bar{E}_{k_1'}(z_1, y_{\tau_1'}') D_{k_2} \bar{E}_{k_2'}(z_2, y_{\tau_2'}') E_{k_1'} E_{k_2'} (f)(y_{\tau_1'}', y_{\tau_2'}') \right|
\]

\[
\leq C \sum_{k_1' = -\infty}^{\infty} \sum_{\nu_1' = 1}^{N(k_1', \nu_1')} \sum_{k_2' = -\infty}^{\infty} \sum_{\nu_2' = 1}^{N(k_2', \nu_2')} 2^{-k_1' d_1 - k_2' d_2} 2^{-|k_1' - k_2'|} 2^{-|k_2|}
\]

\[
\times \left| \left( 2^{-(k_1' \wedge k_1')} + \rho_1 (z_1, y_{\tau_1'}') \right) + \left( 2^{-(k_2' \wedge k_2')} + \rho_2 (z_2, y_{\tau_2'}') \right) \right| d_1 + \epsilon_1 \left( 2^{-(k_1' \wedge k_1')} + \rho_1 (z_1, y_{\tau_1'}') \right) d_2 + \epsilon_2
\]

\[
\times \left| E_{k_1'} E_{k_2'} (f)(y_{\tau_1'}', y_{\tau_2'}') \right|.
\]

Lemma 4.1 with (4.2) and the estimate (4.4) tell us that

\[
\left\{ \sum_{k_1' = -\infty}^{\infty} \sum_{\nu_1' = 1}^{N(k_1', \nu_1')} \sum_{k_2' = -\infty}^{\infty} \sum_{\nu_2' = 1}^{N(k_2', \nu_2')} \sup_{z_1 \in Q_1, z_2 \in Q_2} |D_{k_1} D_{k_2} (f)(z_1, z_2)|^q \right\}
\]

\[
\leq C \left\{ \sum_{k_1' = -\infty}^{\infty} \sum_{\nu_1' = 1}^{N(k_1', \nu_1')} \sum_{k_2' = -\infty}^{\infty} \sum_{\nu_2' = 1}^{N(k_2', \nu_2')} \mu_1(Q_{\tau_1'}') \mu_2(Q_{\tau_2'}')
\]

\[
\times \chi_{Q_{\tau_1'}'}(x_1) \chi_{Q_{\tau_2'}'}(x_2) \right\}
\]

\[
\leq C \left\{ \sum_{k_1' = -\infty}^{\infty} \sum_{\nu_1' = 1}^{N(k_1', \nu_1')} \sum_{k_2' = -\infty}^{\infty} \sum_{\nu_2' = 1}^{N(k_2', \nu_2')} \mu_1(Q_{\tau_1'}') \mu_2(Q_{\tau_2'}')
\]

\[
\times \chi_{Q_{\tau_1'}'}(x_1) \chi_{Q_{\tau_2'}'}(x_2) \right\}.
Product $H^p$ Theory on homogeneous spaces

(4.6) \[
\left( \sum_i |a_i| \right)^q \leq \sum_i |a_i|^q,
\]
and for $i = 1, 2$,

\[
\sum_{k_i = -\infty}^{\infty} 2^{-k_i'd_1 - |k_i - k_i'|} \leq C;
\]

if $q > 1$, we used the Hölder inequality and the facts that for $i = 1, 2$,

\[
\left( \sum_{k_i = -\infty}^{\infty} + \sum_{k_i' = -\infty}^{\infty} \right) 2^{-k_i'd_1 - |k_i - k_i'|} \leq C.
\]

Taking the $L^p(X_1 \times X_2)$-norm on both sides of (4.5) and an iterative application of the Fefferman-Stein vector-valued inequality in [FeS] on $L^{p/r}(X_1)$ and $L^{p/r}(X_2)$ together with the arbitrariness of $y_{r_1}^{k_i',\nu_i'}$ and $y_{r_2}^{k_i'',\nu_i''}$ give us the desired (4.1), which completes the proof of Theorem 4.1.
Remark 4.1 If we replace $D_{k_i}$ in (4.1) by $\overline{D}_{k_i}$ for $i = 1, 2$ as in Theorem 2.2, then (4.1) still holds. The reason for this is that the contribution to (4.1) of $D_{k_i}$ is given by an estimate similar to (3.8), which still holds if we replace $D_{k_i}$ by $\overline{D}_{k_i}$ for $i = 1, 2$.

We now can generalize Theorem 3.2 to the case $p, q \leq 1$.

Theorem 4.2 Let all the notation be the same as in Theorem 3.2. If

$$\max \left\{ \frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2} \right\} < p, q \leq \infty,$$

then there is a constant $C_{p,q} > 0$ such that for all $f \in \left( \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)'$ with $\beta_i, \gamma_i \in (0, \epsilon_i)$ for $i = 1, 2$,

$$C_{p,q}^{-1} \|S_q(f)\|_{L^p(X_1 \times X_2)} \leq \|g_q(f)\|_{L^p(X_1 \times X_2)} \leq C_{p,q} \|S_q(f)\|_{L^p(X_1 \times X_2)}. \tag{4.7}$$

Proof. We begin with proving the first inequality in (4.7). Let $f \in \left( \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)'$ with $\beta_i, \gamma_i \in (0, \epsilon_i)$ for $i = 1, 2$. By Theorem 2.7, we have

$$f(x_1, x_2) = \sum_{k_1' = -\infty}^{\infty} \sum_{\nu_1' = 1}^{\infty} \sum_{k_2' = -\infty}^{\infty} \sum_{\nu_2' = 1}^{\infty} \mu_1(Q_{k_1', \nu_1'}^1) \mu_2(Q_{k_2', \nu_2'}^2)$$

$$\times \overline{D}_{k_1'}(x_1, y_{k_1', \nu_1'}) \overline{D}_{k_2'}(x_2, y_{k_2', \nu_2'}) D_{k_1} D_{k_2}(f)(y_{k_1', \nu_1'}, y_{k_2', \nu_2'}) \tag{4.8}$$

holds in $\left( \mathcal{G}(\beta_1', \beta_2'; \gamma_1', \gamma_2') \right)'$ with $\beta_i' \in (\beta_i, \epsilon_i)$ and $\gamma_i' \in (\gamma_i, \epsilon_i)$, where $\overline{D}_{k_i'}$ satisfies the same properties as $\overline{D}_{k_i}$ in Theorem 2.7 and $i = 1, 2$.

In what follows, if $Q$ is a dyadic cube and $C > 0$ is a constant, let $CQ$ be the dyadic cube with the same center as $Q$ and diameter $C \text{diam}(Q)$.

From (4.8) and an estimate similar to (3.8) with $D_{k_i}$ and $\overline{D}_{k_i'}$ respectively replaced by $D_{k_i}$ and $\overline{D}_{k_i'}$, it follows that for some given constant $C_{12} > 0$ and any $k_1, k_2 \in \mathbb{Z},$

$$\sup_{z_1 \in C_{12}Q_{k_1', \nu_1'}, z_2 \in C_{12}Q_{k_2', \nu_2'}} |D_{k_1} D_{k_2}(f)(z_1, z_2)| \chi_{D_{k_1', \nu_1'}}(x_1) \chi_{Q_{k_2', \nu_2'}}(x_2) \tag{4.9}$$

$$\leq C \sum_{k_1' = -\infty}^{\infty} \sum_{\nu_1' = 1}^{\infty} \sum_{k_2' = -\infty}^{\infty} \sum_{\nu_2' = 1}^{\infty} \mu_1(Q_{k_1', \nu_1'}^1) \mu_2(Q_{k_2', \nu_2'}^2)$$

$$\times \overline{D}_{k_1'}(z_1, y_{k_1', \nu_1'}) \overline{D}_{k_2'}(z_2, y_{k_2', \nu_2'}) D_{k_1} D_{k_2}(f)(y_{k_1', \nu_1'}, y_{k_2', \nu_2'}) \times \chi_{Q_{k_1', \nu_1'}}(x_1) \chi_{Q_{k_2', \nu_2'}}(x_2)$$

$$\leq C \sum_{k_1' = -\infty}^{\infty} \sum_{\nu_1' = 1}^{\infty} \sum_{k_2' = -\infty}^{\infty} \sum_{\nu_2' = 1}^{\infty} 2^{-k_1' d_1 - k_2' d_2 - |k_1' - k_1| - |k_2' - k_2|}$$
\[ \times D_{k_1'} D_{k_2'}(f)(y_{\tau_1'}, y_{\tau_2'}) ; \]

see also the proof of (4.4).

Instead of (4.4) by (4.9) and repeating the proof of (4.1) yield a variant of (4.1), namely, there is a constant \( C > 0 \) such that for all \( f \in \left( \mathcal{C}(\beta_1, \beta_2 ; \gamma_1, \gamma_2) \right)' \) with \( \beta_i, \gamma_i \in (0, \epsilon_i) \) for \( i = 1, 2, \)

\[
(4.10) \quad \left\| \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1 = 1}^{\infty} \sum_{k_2 = -\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2 = 1}^{\infty} N(k_1, \tau_1) \sum_{k_2, \tau_2} N(k_2, \tau_2) \times \sup_{z_1 \in C_{12} k_1^{1/2} \|1, z_2 \in C_{12} k_2^{1/2} \|2} |D_{k_1} D_{k_2}(f)(z_1, z_2)|^q \right\|_{LP(X_1 \times X_2)} \right\|_{LP(X_1 \times X_2)} \leq C \left\| \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1 = 1}^{\infty} \sum_{k_2 = -\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2 = 1}^{\infty} N(k_1, \tau_1) \sum_{k_2, \tau_2} N(k_2, \tau_2) \times \inf_{z_1 \in Q_{\tau_1}^{k_1, \nu_1}, z_2 \in Q_{\tau_2}^{k_2, \nu_2}} |D_{k_1} D_{k_2}(f)(z_1, z_2)|^q \chi_{Q_{\tau_1}^{k_1, \nu_1}} \chi_{Q_{\tau_2}^{k_2, \nu_2}} \right\} \right\|_{LP(X_1 \times X_2)},
\]

where \( p, q \) are the same as in the theorem.

From (4.10) with suitably chosen \( C_{12} \) and the definition of the Littlewood-Paley \( S \)-function, (3.13), it follows that

\[
\| S_q(f) \|_{LP(X_1 \times X_2)} = \left\| \sum_{k_1 = -\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1 = 1}^{\infty} \sum_{k_2 = -\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2 = 1}^{\infty} N(k_1, \tau_1) \sum_{k_2, \tau_2} N(k_2, \tau_2) \times \int_{\rho_2(-\tau_2, \nu_2) \leq C_{12} \|2^{k_2} d_1 + d_2} |D_{k_1} D_{k_2}(f)(y_1, y_2)|^q d\mu_1(y_1) d\mu_2(y_2) \right\|_{LP(X_1 \times X_2)} \leq C \left\| \sum_{k_1 = -\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1 = 1}^{\infty} \sum_{k_2 = -\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2 = 1}^{\infty} N(k_1, \tau_1) \sum_{k_2, \tau_2} N(k_2, \tau_2) \times \sup_{z_1 \in C_{12} k_1^{1/2} \|1, z_2 \in C_{12} k_2^{1/2} \|2} |D_{k_1} D_{k_2}(f)(z_1, z_2)|^q \chi_{Q_{\tau_1}^{k_1, \nu_1}} \chi_{Q_{\tau_2}^{k_2, \nu_2}} \right\|_{LP(X_1 \times X_2)} \]
where in the second inequality, we used the following fact that if $x$, $y$ \in $Q_{k_1}^{x_1,y_1} \cap Q_{k_2}^{x_2,y_2}$, then

$$
\leq C \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{\infty} \frac{N(k_1, \tau_1)}{N(k_2, \tau_2)} \right\}^{1/q} \left\| \inf_{z_1 \in Q_{k_1}^{x_1,y_1}, z_2 \in Q_{k_2}^{x_2,y_2}} |D_{k_1} D_{k_2} (f) (z_1, z_2)|^q \chi_{Q_{k_1}^{x_1,y_1}} \chi_{Q_{k_2}^{x_2,y_2}} \right\|_{L^p(X_1 \times X_2)}
$$

which proves the first inequality in (4.7).

We now turn to the proof of the second inequality in (4.7). By Theorem 4.1, we have

$$
\|S_q(f)\|_{L^p(X_1 \times X_2)}
$$

$$
= \left\| \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{\infty} \frac{N(k_1, \tau_1)}{N(k_2, \tau_2)} \right\}^{1/q} \left\| \inf_{z_1 \in Q_{k_1}^{x_1,y_1}, z_2 \in Q_{k_2}^{x_2,y_2}} |D_{k_1} D_{k_2} (f) (y_1, y_2)|^q \chi_{Q_{k_1}^{x_1,y_1}} \chi_{Q_{k_2}^{x_2,y_2}} \right\|_{L^p(X_1 \times X_2)}
$$

$$
\geq C \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{\infty} \frac{N(k_1, \tau_1)}{N(k_2, \tau_2)} \right\}^{1/q} \left\| \inf_{z_1 \in Q_{k_1}^{x_1,y_1}, z_2 \in Q_{k_2}^{x_2,y_2}} |D_{k_1} D_{k_2} (f) (z_1, z_2)|^q \chi_{Q_{k_1}^{x_1,y_1}} \chi_{Q_{k_2}^{x_2,y_2}} \right\|_{L^p(X_1 \times X_2)}
$$

$$
\geq C \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{\infty} \frac{N(k_1, \tau_1)}{N(k_2, \tau_2)} \right\}^{1/q} \left\| \sup_{z_1 \in Q_{k_1}^{x_1,y_1}, z_2 \in Q_{k_2}^{x_2,y_2}} |D_{k_1} D_{k_2} (f) (z_1, z_2)|^q \chi_{Q_{k_1}^{x_1,y_1}} \chi_{Q_{k_2}^{x_2,y_2}} \right\|_{L^p(X_1 \times X_2)}
$$

$$
\geq C \|g_q(f)\|_{L^p(X_1 \times X_2)},
$$

where in the second inequality, we used the following fact that if $x_1, y_1 \in Q_{k_1}^{x_1,y_1}$, then
Remark 4.1 By Remark 4.1, it is easy to see that the first inequality in (4.7) still holds if we replace $D_{k_i}$ in the definition of $S_q(f)$ by $\overline{D}_{k_i}$ for $i = 1, 2$ as in Theorem 2.2, which is useful in applications.

We can now introduce the Hardy spaces $H^p(X_1 \times X_2)$ for some $p \leq 1$ and establish their atomic decomposition characterization.

Definition 4.1 Let $X_i$ be a homogeneous-type space as in Definition 2.1, $\epsilon_i \in (0, \theta_i]$ and $\{D_{k_i}\}_{k_i \in \mathbb{Z}}$ be the same as in Theorem 3.1 for $i = 1, 2$. Let

$$\max \left\{ \frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2} \right\} < p < \infty$$

and for $i = 1, 2$,

$$(4.11) \quad d_i(1/p - 1) + \beta_i, \gamma_i < \epsilon_i.$$  

The Hardy space $H^p(X_1 \times X_2)$ is defined to be the set of all $f \in \left( \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)'$ such that $\|g_2(f)\|_{L^p(X_1 \times X_2)} < \infty$, and we define

$$\|f\|_{H^p(X_1 \times X_2)} = \|g_2(f)\|_{L^p(X_1 \times X_2)},$$

where $g_2(f)$ is defined as in Theorem 3.1.

We first consider the reasonability of the definition of the Hardy space $H^p(X_1 \times X_2)$.

Proposition 4.1 Let all the notation be the same as in Definition 4.1. Then the definition of the Hardy space $H^p(X_1 \times X_2)$ is independent of the choice of approximations to the identity and the spaces of distributions with $\beta_i$ and $\gamma_i$ satisfying (4.11), where $i = 1, 2$.

Proof. We first verify that the definition of the Hardy space $H^p(X_1 \times X_2)$ is independent of the choice of approximations to the identity, which is a corollary of Theorem 2.1. In fact, let all the notation be the same as in Theorem 4.1. Then, Theorem 4.1 tells us that

$$\left\| \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} |D_{k_1}D_{k_2}(f)|^2 \right\}^{1/2} \right\|_{L^p(X_1 \times X_2)} \leq \left\| \sum_{k_1 = -\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{n_1 = 1}^{N(k_1, \tau_1)} \sum_{k_2 = -\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{n_2 = 1}^{N(k_2, \tau_2)} \sup_{z_1 \in Q^{k_1,n_1}_{\tau_1}, z_2 \in Q^{k_2,n_2}_{\tau_2}} |D_{k_1}D_{k_2}(f)(z_1, z_2)|^q \times \chi_{Q^{k_1,n_1}_{\tau_1}}(\cdot) \chi_{Q^{k_2,n_2}_{\tau_2}}(\cdot) \right\|_{L^p(X_1 \times X_2)}^{1/q},$$

where $\chi_{Q^{k,n}_{\tau}}$ denotes the characteristic function of the cube $Q^{k,n}_{\tau}$. The second inequality in (4.7) and completes the proof of Theorem 4.2.

Remark 4.2 By Remark 4.1, it is easy to see that the first inequality in (4.7) still holds if we replace $D_{k_i}$ in the definition of $S_q(f)$ by $\overline{D}_{k_i}$ for $i = 1, 2$ as in Theorem 2.2, which is useful in applications. 

for $j_i \in \mathbb{N}$ is large enough, then $\rho_i(x_i, y_i) \leq C_{10,1} 2^{-k_i - j_i}$, and therefore, if $j_i \in \mathbb{N}$ is large enough, then $\rho_i(x_i, y_i) \leq C_{11,1} 2^{-k_i}$ for $i = 1, 2$. This verifies the second inequality in (4.7) and completes the proof of Theorem 4.2.

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\[ \leq C \left\| \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \left\| D_{k_1} D_{k_2}(f) \right\|^2 \right\|^{1/2}_{L^p(X_1 \times X_2)} \]

By the symmetry, we further obtain

\[ \sim \left\| \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \left\| E_{k_1} E_{k_2}(f) \right\|^2 \right\|^{1/2}_{L^p(X_1 \times X_2)}. \]

Thus, the definition of the Hardy space \( H^p(X_1 \times X_2) \) is independent of the choice of approximations to the identity.

Let \( \beta_i, \gamma_i \) and \( \beta'_i, \gamma'_i \) for \( i = 1, 2 \) both satisfy (4.11) and \( f \in \left( G(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)' \) with \( \| g_2(f) \|_{L^p(X_1 \times X_2)} < \infty \). We now verify that \( f \in \left( G(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)' \). To this end, let \( \psi \in G(\epsilon_1, \epsilon_2) \) and the notation be the same as in Theorem 2.7. Let \( \gamma'_1 \in (0, \gamma_1), \gamma'_2 \in (0, \gamma_2) \) and \( y_1 \in X_1, y_2 \in X_2 \). We claim that for any \( k_1, k_2 \in \mathbb{Z}_+ \),

\[ (4.12) \quad \left\| \left( \tilde{D}_{k_1}(\cdot, y_1) \tilde{D}_{k_2}(\cdot, y_2), \psi \right) \right\| \leq C 2^{-k_1 \beta_1 - k_2 \beta_2} \left\| \psi \right\|_{G(\beta_1, \beta_2; \gamma_1, \gamma_2)} \frac{1}{(1 + \rho_1(y_1, x_0))^{d_1 + \gamma_1}} \frac{1}{(1 + \rho_2(y_2, y_0))^{d_2 + \gamma_2}}; \]

for any \( k_1 \in \mathbb{Z}_+ \) and any \( k_2 \in \mathbb{Z} \setminus \mathbb{Z}_+ \),

\[ (4.13) \quad \left\| \left( \tilde{D}_{k_1}(\cdot, y_1) \tilde{D}_{k_2}(\cdot, y_2), \psi \right) \right\| \leq C 2^{k_2 \gamma_2 - k_2 z_1} \left\| \psi \right\|_{G(\beta_1, \beta_2; \gamma_1, \gamma_2)} \frac{1}{(1 + \rho_1(y_1, x_0))^{d_1 + \gamma_1}} \frac{2^{-k_2 \gamma_2}}{(2-k_2 + \rho_2(y_2, y_0))^{d_2 + \gamma_2}}; \]

for any \( k_1 \in \mathbb{Z} \setminus \mathbb{Z}_+ \) and any \( k_2 \in \mathbb{Z}_+ \),

\[ (4.14) \quad \left\| \left( \tilde{D}_{k_1}(\cdot, y_1) \tilde{D}_{k_2}(\cdot, y_2), \psi \right) \right\| \leq C 2^{k_1 \gamma_1 - k_2 \beta_2} \left\| \psi \right\|_{G(\beta_1, \beta_2; \gamma_1, \gamma_2)} \frac{2^{-k_1 \gamma_1}}{(2-k_1 + \rho_1(y_1, x_0))^{d_1 + \gamma_1}} \frac{1}{(1 + \rho_2(y_2, y_0))^{d_2 + \gamma_2}}; \]
and for any $k_1, k_2 \in \mathbb{Z} \setminus \mathbb{Z}_+$, 

\begin{equation}
\left| \left\langle \bar{D}_{k_1}(\cdot, y_1) \bar{D}_{k_2}(\cdot, y_2), \psi \right\rangle \right| 
\leq C 2^{k_1 \gamma_1 + k_2 \gamma_2} \| \psi \| \| g(\beta_1, \beta_2; \gamma_1, \gamma_2) \| \frac{2^{-k_1 \gamma_1}}{(2-k_1 + \rho_1(y_1, x_0))^{d_1 + \gamma_1}} \times \frac{2^{-k_2 \gamma_2}}{(2-k_2 + \rho_2(y_2, y_0))^{d_2 + \gamma_2}}.
\end{equation}

We first verify (4.12). In this case, we have 

\begin{align*}
\left| \left\langle \bar{D}_{k_1}(\cdot, y_1) \bar{D}_{k_2}(\cdot, y_2), \psi \right\rangle \right|
&= \int_{X_1 \times X_2} \bar{D}_{k_1}(z_1, y_1) \bar{D}_{k_2}(z_2, y_2) \psi(z_1, z_2) \, d\mu_1(z_1) \, d\mu_2(z_2) \\
&= \int_{X_1 \times X_2} \bar{D}_{k_1}(z_1, y_1) \bar{D}_{k_2}(z_2, y_2) \\
&\quad \times \{ \left[ \psi(z_1, z_2) - \psi(y_1, z_2) \right] - \left[ \psi(z_1, y_2) - \psi(y_1, y_2) \right] \} \, d\mu_1(z_1) \, d\mu_2(z_2) \\
&\leq \int_{\rho_1(z_1, y_1) \leq \frac{1}{\pi_1} \left( 1 + \rho_1(y_1, x_0) \right) \rho_2(z_2, y_2) \leq \frac{1}{\pi_2} \left( 1 + \rho_2(y_2, y_0) \right)} \bar{D}_{k_1}(z_1, y_1) \bar{D}_{k_2}(z_2, y_2) \\
&\quad \times \left\{ \left| \psi(z_1, z_2) - \psi(y_1, z_2) \right| - \left| \psi(z_1, y_2) - \psi(y_1, y_2) \right| \right\} \, d\mu_1(z_1) \, d\mu_2(z_2) \\
&\quad + \int_{\rho_1(z_1, y_1) > \frac{1}{\pi_1} \left( 1 + \rho_1(y_1, x_0) \right) \rho_2(z_2, y_2) \leq \frac{1}{\pi_2} \left( 1 + \rho_2(y_2, y_0) \right)} \cdots + \int_{\rho_1(z_1, y_1) > \frac{1}{\pi_1} \left( 1 + \rho_1(y_1, x_0) \right) \rho_2(z_2, y_2) > \frac{1}{\pi_2} \left( 1 + \rho_2(y_2, y_0) \right)} \cdots \\
&\quad + \int_{\rho_1(z_1, y_1) > \frac{1}{\pi_1} \left( 1 + \rho_1(y_1, x_0) \right) \rho_2(z_2, y_2) > \frac{1}{\pi_2} \left( 1 + \rho_2(y_2, y_0) \right)} \cdots \\
&= Q_{11} + Q_{12} + Q_{13} + Q_{14}.
\end{align*}

For $Q_{11}$, by the second difference condition (iv) satisfied by $\psi$ as in Definition 2.4, we have 

\begin{align*}
Q_{11} &\leq C \| \psi \| \| g(\beta_1, \beta_2; \gamma_1, \gamma_2) \| \frac{1}{\rho_1(z_1, y_1)} \frac{1}{\rho_2(z_2, y_2)} \left( 1 + \rho_1(y_1, x_0) \right)^{d_1 + \gamma_1} \\
&\quad \times \left\{ \beta_1 \frac{1}{\left( 1 + \rho_1(y_1, x_0) \right)^{d_1 + \gamma_1}} \right\} \\
&\quad \times \left\{ \beta_2 \frac{1}{\left( 1 + \rho_2(y_2, y_0) \right)^{d_2 + \gamma_2}} \right\} \, d\mu_1(z_1) \, d\mu_2(z_2) \\
&\leq C 2^{-k_1 \gamma_1 - k_2 \gamma_2} \| \psi \| \| g(\beta_1, \beta_2; \gamma_1, \gamma_2) \| \left( 1 + \rho_1(y_1, x_0) \right)^{d_1 + \gamma_1} \left( 1 + \rho_2(y_2, y_0) \right)^{d_2 + \gamma_2},
\end{align*}
which is a desired estimate.

Definition 2.4 (ii) tells us that

\[
Q_{12} \leq C \|\psi\| g(\beta_1, \beta_2; \gamma_1, \gamma_2) \int_{\rho_1(z_1, y_1) \leq \frac{1}{2^{12}} (1 + \rho_1(y_1, x_0))} |\tilde{D}_{k_1}(z_1, y_1)\tilde{D}_{k_2}(z_2, y_2)|
\]

\[
\times \left( \frac{\rho_1(z_1, y_1)}{1 + \rho_1(y_1, x_0)} \right)^{\beta_1} \frac{1}{(1 + \rho_1(y_1, x_0))^{d_1 + \gamma_1}}
\]

\[
\times \left\{ \frac{1}{(1 + \rho_2(z_2, y_0))^{d_2 + \gamma_2} + (1 + \rho_2(y_2, y_0))^{d_2 + \gamma_2}} \right\} d\mu_1(z_1) d\mu_2(z_2)
\]

\[
\leq C 2^{-k_1 \beta_1 - k_2 \beta_2} \|\psi\| g(\beta_1, \beta_2; \gamma_1, \gamma_2) \frac{1}{(1 + \rho_1(y_1, x_0))^{d_1 + \gamma_1}} \frac{1}{(1 + \rho_2(y_2, y_0))^{d_2 + \gamma_2}},
\]

which is also a desired estimate.

The estimate for \(Q_{13}\) is similar to that for \(Q_{12}\). We omit the details.

For \(Q_{14}\), the size condition satisfied by \(\psi\) tells us that

\[
Q_{14} \leq C \|\psi\| g(\beta_1, \beta_2; \gamma_1, \gamma_2) \int_{\rho_1(z_1, y_1) \leq \frac{1}{2^{12}} (1 + \rho_1(y_1, x_0))} |\tilde{D}_{k_1}(z_1, y_1)\tilde{D}_{k_2}(z_2, y_2)|
\]

\[
\times \left\{ \frac{1}{(1 + \rho_2(z_2, y_0))^{d_2 + \gamma_2}} \left[ \frac{1}{(1 + \rho_1(z_1, x_0))^{d_1 + \gamma_1}} + \frac{1}{(1 + \rho_1(y_1, x_0))^{d_1 + \gamma_1}} \right] \right\}
\]

\[
\times \frac{1}{(1 + \rho_1(z_1, x_0))^{d_1 + \gamma_1}} \frac{1}{(1 + \rho_2(y_2, y_0))^{d_2 + \gamma_2}}
\]

\[
\leq C 2^{-k_1 \beta_1 - k_2 \beta_2} \|\psi\| g(\beta_1, \beta_2; \gamma_1, \gamma_2) \frac{1}{(1 + \rho_1(y_1, x_0))^{d_1 + \gamma_1}} \frac{1}{(1 + \rho_2(y_2, y_0))^{d_2 + \gamma_2}},
\]

which completes the proof of (4.12).

We now prove (4.13). In this case, we write

\[
\left| \langle \tilde{D}_{k_1} (\cdot, y_1) \tilde{D}_{k_2} (\cdot, y_2), \psi \rangle \right|
\]

\[
= \left| \int_{X_1 \times X_2} \tilde{D}_{k_1}(z_1, y_1)\tilde{D}_{k_2}(z_2, y_2) \psi(z_1, z_2) d\mu_1(z_1) d\mu_2(z_2) \right|
\]

\[
= \left| \int_{X_1 \times X_2} \tilde{D}_{k_1}(z_1, y_1) \left[ \tilde{D}_{k_2}(z_2, y_2) - \tilde{D}_{k_2}(y_0, y_2) \right]
\]

\[
\times \left[ \psi(z_1, z_2) - \psi(y_1, z_2) \right] d\mu_1(z_1) d\mu_2(z_2) \right|
\]

\[
\leq \int_{\rho_1(z_1, y_1) \leq \frac{1}{2^{12}} (1 + \rho_1(y_1, x_0))} \left| \tilde{D}_{k_1}(z_1, y_1) \right| \left| \tilde{D}_{k_2}(z_2, y_2) - \tilde{D}_{k_2}(y_0, y_2) \right|
\]

\[
\times \left[ \psi(z_1, z_2) - \psi(y_1, z_2) \right] d\mu_1(z_1) d\mu_2(z_2) \right|
\]
\[ \times |\psi(z_1, z_2) - \psi(y_1, z_2)| \, d\mu_1(z_1) \, d\mu_2(z_2) \]

\[ + \int_{\rho_1(z_1, y_1) \leq \frac{1}{\pi \epsilon} (1 + \rho_1(y_1, x_0))} \cdots + \int_{\rho_2(z_2, y_0) \leq \frac{1}{\pi \epsilon} (2^{-k_2} + \rho_2(y_2, y_0))} \cdots \]

\[ + \int_{\rho_2(z_2, y_0) > \frac{1}{\pi \epsilon} (1 + \rho_1(y_1, x_0))} \cdots \]

\[ = Q_{21} + Q_{22} + Q_{23} + Q_{24}. \]

The regularity of \( \tilde{D}_{k_2} \) and \( \psi \) and the size condition of \( \tilde{D}_{k_1} \) yield that

\[ Q_{21} \leq C \| \psi \|_G(\beta_1, \beta_2, \gamma_1, \gamma_2) \int_{\rho_1(z_1, y_1) \leq \frac{1}{\pi \epsilon} (1 + \rho_1(y_1, x_0))} \cdots \int_{\rho_2(z_2, y_0) \leq \frac{1}{\pi \epsilon} (2^{-k_2} + \rho_2(y_2, y_0))} \cdots \int_{\rho_2(z_2, y_0) > \frac{1}{\pi \epsilon} (1 + \rho_1(y_1, x_0))} \cdots \]

\[ \times \left( \frac{\rho_2(z_2, y_0)}{1 + \rho_1(y_1, x_0)} \right)^\beta_1 \left( 1 + \rho_1(y_1, x_0) \right)^{d_1 + \gamma_1} \]

\[ \leq C 2^{k_2 \gamma_2 - k_1 \beta_1} \| \psi \|_G(\beta_1, \beta_2; \gamma_1, \gamma_2) \left( 1 + \rho_1(y_1, x_0) \right)^{d_1 + \gamma_1} \left( 2^{-k_2} + \rho_2(y_2, y_0) \right)^{d_2 + \gamma_2}, \]

which is a desired estimate.

Similarly we have

\[ Q_{22} \leq C \| \psi \|_G(\beta_1, \beta_2; \gamma_1, \gamma_2) \int_{\rho_1(z_1, y_1) \leq \frac{1}{\pi \epsilon} (1 + \rho_1(y_1, x_0))} \cdots \int_{\rho_2(z_2, y_0) > \frac{1}{\pi \epsilon} (2^{-k_2} + \rho_2(y_2, y_0))} \cdots \int_{\rho_2(z_2, y_0) > \frac{1}{\pi \epsilon} (1 + \rho_1(y_1, x_0))} \cdots \]

\[ \times \left\{ \frac{2^{-k_2 \epsilon_2}}{(2^{-k_2} + \rho_2(y_2, z_2))^{d_2 + \epsilon_2}} + \frac{2^{-k_2 \epsilon_2}}{(2^{-k_2} + \rho_2(y_2, y_2))^{d_2 + \epsilon_2}} \right\} \]

\[ \times \left( \frac{\rho_1(z_1, y_1)}{1 + \rho_1(y_1, x_0)} \right)^\beta_1 \left( 1 + \rho_1(y_1, x_0) \right)^{d_1 + \gamma_1} \]

\[ \leq C 2^{k_2 \gamma_2 - k_1 \beta_1} \| \psi \|_G(\beta_1, \beta_2; \gamma_1, \gamma_2) \left( 1 + \rho_1(y_1, x_0) \right)^{d_1 + \gamma_1} \left( 2^{-k_2} + \rho_2(y_2, y_0) \right)^{d_2 + \gamma_2}, \]

which is also a desired estimate.

The estimate for \( Q_{23} \) is similar to that for \( Q_{22} \) by symmetry.
Finally the size conditions of $\tilde{D}_{k_1}$, $\tilde{D}_{k_2}$ and $\psi$ imply that

$$Q_{24} \leq C \|\psi\| |\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)\| \int_{\rho_1(z_1,y_1) > \frac{1}{K_1}(1 + \rho_1(y_1,x_0))} \frac{2^{-k_2e'_2}}{(2^{-k_2} + \rho_2(y_2,z_2))d_2 + e'_2} \frac{2^{-k_2e'_2}}{(2^{-k_2} + \rho_2(y_0,y_2))d_2 + e'_2}$$

$$\times \left\{ \frac{2^{-k_2e'_2}}{(2^{-k_2} + \rho_2(y_2,z_2))d_2 + e'_2} \frac{2^{-k_2e'_2}}{(2^{-k_2} + \rho_2(y_0,y_2))d_2 + e'_2} \right\} \left\{ \frac{1}{(1 + \rho_1(z_1,x_0))^{d_1 + \gamma_1}} + \frac{1}{(1 + \rho_1(y_1,x_0))^{d_1 + \gamma_1}} \right\} \frac{1}{(1 + \rho_2(z_2,y_0))^{d_2 + \gamma_2}} d\mu_1(z_1) d\mu_2(z_2)$$

$$\leq C 2^{k_2\gamma_2 - k_1 \beta_1} \|\psi\| |\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)\| \left( \frac{1}{(1 + \rho_1(y_1,x_0))^{d_1 + \gamma_1}} (2^{-k_2} + \rho_2(y_0,y_2))^{d_2 + \gamma_2} \right)$$

which completes the proof of (4.13).

The verification of (4.14) is similar to that for (4.13) by symmetry.

We now prove (4.15). Write

$$\left| \left\langle \tilde{D}_{k_1}(\cdot, y_1) \tilde{D}_{k_2}(\cdot, y_2), \psi \right\rangle \right|$$

$$= \int_{X_1 \times X_2} \tilde{D}_{k_1}(z_1, y_1) \tilde{D}_{k_2}(z_2, y_2) \psi(z_1, z_2) d\mu_1(z_1) d\mu_2(z_2)$$

$$= \int_{X_1 \times X_2} \left[ \tilde{D}_{k_1}(z_1, y_1) - \tilde{D}_{k_1}(x_0, y_1) \right] \left[ \tilde{D}_{k_2}(z_2, y_2) - \tilde{D}_{k_2}(y_0, y_2) \right]$$

$$\times \psi(z_1, z_2) d\mu_1(z_1) d\mu_2(z_2)$$

$$\leq \int_{\rho_1(z_1,x_0) > \frac{1}{K_1}(2^{-k_1} + \rho_1(y_1,x_0))} \frac{2^{-k_2e'_2}}{(2^{-k_2} + \rho_2(y_2,z_2))d_2 + e'_2} \frac{2^{-k_2e'_2}}{(2^{-k_2} + \rho_2(y_0,y_2))d_2 + e'_2}$$

$$\times \left\{ \frac{2^{-k_2e'_2}}{(2^{-k_2} + \rho_2(y_2,z_2))d_2 + e'_2} \frac{2^{-k_2e'_2}}{(2^{-k_2} + \rho_2(y_0,y_2))d_2 + e'_2} \right\} \left\{ \frac{1}{(1 + \rho_1(z_1,x_0))^{d_1 + \gamma_1}} + \frac{1}{(1 + \rho_1(y_1,x_0))^{d_1 + \gamma_1}} \right\} \frac{1}{(1 + \rho_2(z_2,y_0))^{d_2 + \gamma_2}} d\mu_1(z_1) d\mu_2(z_2)$$

$$\times \psi(z_1, z_2) d\mu_1(z_1) d\mu_2(z_2) + \int_{\rho_1(z_1,x_0) > \frac{1}{K_1}(2^{-k_1} + \rho_1(y_1,x_0))} \frac{2^{-k_2e'_2}}{(2^{-k_2} + \rho_2(y_2,z_2))d_2 + e'_2} \frac{2^{-k_2e'_2}}{(2^{-k_2} + \rho_2(y_0,y_2))d_2 + e'_2}$$

$$\times \left\{ \frac{2^{-k_2e'_2}}{(2^{-k_2} + \rho_2(y_2,z_2))d_2 + e'_2} \frac{2^{-k_2e'_2}}{(2^{-k_2} + \rho_2(y_0,y_2))d_2 + e'_2} \right\} \left\{ \frac{1}{(1 + \rho_1(z_1,x_0))^{d_1 + \gamma_1}} + \frac{1}{(1 + \rho_1(y_1,x_0))^{d_1 + \gamma_1}} \right\} \frac{1}{(1 + \rho_2(z_2,y_0))^{d_2 + \gamma_2}} d\mu_1(z_1) d\mu_2(z_2)$$

$$= Q_{31} + Q_{32} + Q_{33} + Q_{34}.$$
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\[ \times \frac{2^{-k_1\epsilon_1^2}}{(2^{-k_1} + \rho_1(x_1, y_1))^{d_1 + \gamma_1}} \left( \frac{\rho_2(z_2, y_0)}{2^{-k_2} + \rho_2(y_0, y_2)} \right) \gamma_2 \quad \frac{2^{-k_2\epsilon_2^2}}{(2^{-k_2} + \rho_2(y_0, y_2))^{d_2 + \gamma_2}} \]

which verified (4.15).

\[ \leq C 2^{k_1\gamma_1^2 + k_2\gamma_2^2} \| \psi \| g(\beta_1, \beta_2; \gamma_1, \gamma_2) \left( \frac{2^{-k_1} + \rho_1(y_1, x_0)}{2^{-k_1} + \rho_1(y_1, x_0)} \right)^{d_1 + \gamma_1} \]

which is a desired estimate.

Similarly, for $Q_{32}$, we have

\[ Q_{32} \leq C \| \psi \| g(\beta_1, \beta_2; \gamma_1, \gamma_2) \int_{\rho_1(z_1, x_0)}^{\frac{1}{2^{-k_1} + \rho_1(y_1, x_0)}} \left( \frac{\rho_1(z_1, x_0)}{2^{-k_1} + \rho_1(y_1, x_0)} \right)^{\gamma_1} \]

\[ \times \frac{2^{-k_1\epsilon_1^2}}{(2^{-k_1} + \rho_1(x_0, y_1))^{d_1 + \gamma_1}} \left( \frac{2^{-k_2\epsilon_2^2}}{(2^{-k_2} + \rho_2(y_0, y_2))^{d_2 + \gamma_2}} + \frac{2^{-k_2\epsilon_2^2}}{(2^{-k_2} + \rho_2(y_0, y_2))^{d_2 + \gamma_2}} \right) \]

\[ \times \frac{1}{(1 + \rho_1(z_1, x_0))^{d_1 + \gamma_1}} \frac{1}{(1 + \rho_2(z_2, y_0))^{d_2 + \gamma_2}} \]

\[ \leq C 2^{k_1\gamma_1^2 + k_2\gamma_2^2} \| \psi \| g(\beta_1, \beta_2; \gamma_1, \gamma_2) \left( \frac{2^{-k_1} + \rho_1(y_1, x_0)}{2^{-k_1} + \rho_1(y_1, x_0)} \right)^{d_1 + \gamma_1} \]

which is also a desired estimate.

The symmetry of $Q_{33}$ with $Q_{32}$ implies a desired estimate for $Q_{33}$.

We now estimate $Q_{34}$ by

\[ Q_{34} \leq C \| \psi \| g(\beta_1, \beta_2; \gamma_1, \gamma_2) \int_{\rho_1(z_1, x_0)}^{\frac{1}{2^{-k_1} + \rho_1(y_1, x_0)}} \left( \frac{\rho_1(z_1, x_0)}{2^{-k_1} + \rho_1(y_1, x_0)} \right)^{\gamma_1} \]

\[ \times \left\{ \frac{2^{-k_1\epsilon_1^2}}{(2^{-k_1} + \rho_1(x_0, y_1))^{d_1 + \gamma_1}} + \frac{2^{-k_2\epsilon_2^2}}{(2^{-k_2} + \rho_2(y_0, y_2))^{d_2 + \gamma_2}} \right\} \]

\[ \times \frac{1}{(1 + \rho_1(z_1, x_0))^{d_1 + \gamma_1}} \frac{1}{(1 + \rho_2(z_2, y_0))^{d_2 + \gamma_2}} \]

\[ \leq C 2^{k_1\gamma_1^2 + k_2\gamma_2^2} \| \psi \| g(\beta_1, \beta_2; \gamma_1, \gamma_2) \left( \frac{2^{-k_1} + \rho_1(y_1, x_0)}{2^{-k_1} + \rho_1(y_1, x_0)} \right)^{d_1 + \gamma_1} \]

which verified (4.15).
Theorem 2.7 now tells us that
\[
|\langle f, \psi \rangle| = \left| \sum_{k_1=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{\infty} \mu_1(Q_{\tau_1}^{k_1,\nu_1}) \mu_2(Q_{\tau_2}^{k_2,\nu_2})
\times D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2}) \langle \tilde{D}_{k_1}(\cdot, y_{\tau_1}^{k_1,\nu_1}) \tilde{D}_{k_2}(\cdot, y_{\tau_2}^{k_2,\nu_2}), \psi \rangle \right|
\leq \sum_{k_1=0}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{\infty} \sum_{k_2=0}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{\infty} \mu_1(Q_{\tau_1}^{k_1,\nu_1}) \mu_2(Q_{\tau_2}^{k_2,\nu_2})
\times |D_{k_1}D_{k_2}(f)(y_{\tau_1}^{k_1,\nu_1}, y_{\tau_2}^{k_2,\nu_2})| \left| \langle \tilde{D}_{k_1}(\cdot, y_{\tau_1}^{k_1,\nu_1}) \tilde{D}_{k_2}(\cdot, y_{\tau_2}^{k_2,\nu_2}), \psi \rangle \right|
\leq Q_{41} + Q_{42} + Q_{43} + Q_{44}.
\]

If \( p \leq 1 \), the estimate (4.12), the inequality (4.6), the arbitrariness of \( y_{\tau_1}^{k_1,\nu_1} \) and \( y_{\tau_2}^{k_2,\nu_2} \), the assumption on \( \beta_1 \) and \( \beta_2 \) and the Hölder inequality imply that
\[
Q_{41} \leq C\|\psi\|_2 \sum_{k_1=0}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{\infty} \sum_{k_2=0}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{\infty} 2^{-k_1\beta_1-k_2\beta_2}
\times \mu_1(Q_{\tau_1}^{k_1,\nu_1}) \mu_2(Q_{\tau_2}^{k_2,\nu_2})
\times \frac{1}{(1 + \rho_1(y_{\tau_1}^{k_1,\nu_1}, x_0))^{d_1+\gamma_1}} \frac{1}{(1 + \rho_2(y_{\tau_2}^{k_2,\nu_2}, y_0))^{d_2+\gamma_2}}
\leq C\|\psi\|_2 \sum_{k_1=0}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{\infty} \sum_{k_2=0}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{\infty} \mu_1(Q_{\tau_1}^{k_1,\nu_1}) \mu_2(Q_{\tau_2}^{k_2,\nu_2})
\times \left( \sum_{\tau_1 \in I_{k_1}} \sum_{\nu_1=1}^{\infty} \sum_{\tau_2 \in I_{k_2}} \sum_{\nu_2=1}^{\infty} \mu_1(Q_{\tau_1}^{k_1,\nu_1}) \mu_2(Q_{\tau_2}^{k_2,\nu_2}) \right)^{1/p}
\leq C\|\psi\|_2 \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \|D_{k_1}D_{k_2}(f)\|_{L^p(X_1 \times X_2)}^2 \right)^{1/2};
while if $p > 1$, similarly we have

$$Q_{41} \leq C\|\psi\|\|g(\beta_1, \beta_2; \gamma_1, \gamma_2)\| \left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} 2^{-k_1 \beta_1 - k_2 \beta_2} \right. $$

$$\times \mu_1(Q^{k_1, \nu_1}) \mu_2(Q^{k_2, \nu_2}) \left| D_{k_1} D_{k_2}(f)(y_{k_1, \nu_1}, y_{k_2, \nu_2}) \right|^p \times \frac{1}{(1 + \rho_1(y_{k_1, \nu_1}, x_0))^{d_1 + \gamma_1}} \left. \frac{1}{(1 + \rho_2(y_{k_2, \nu_2}, y_0))^{d_2 + \gamma_2}} \right\}^{1/p} \times \left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} 2^{-k_1 \beta_1 - k_2 \beta_2} \int_{X_1 \times X_2} \frac{1}{(1 + \rho_1(y_1, x_0))^{d_1 + \gamma_1}} \right. $$

$$\times \frac{1}{(1 + \rho_2(y_2, y_0))^{d_2 + \gamma_2}} d\mu_1(y_1) d\mu_2(y_2) \right\}^{1/p'} \leq C\|\psi\|\|g(\beta_1, \beta_2; \gamma_1, \gamma_2)\| \left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} 2^{-k_1 \beta_1 - k_2 \beta_2} \right. $$

$$\left. \int_{X_1 \times X_2} D_{k_1} D_{k_2}(f) \right\|^p_{L^p(X_1 \times X_2)} \left. \right\}^{1/p} \leq C\|\psi\|\|g(\beta_1, \beta_2; \gamma_1, \gamma_2)\| \left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \left\| D_{k_1} D_{k_2}(f) \right\|_{L^p(X_1 \times X_2)}^{\max(p,2)} \right\}^{1/\max(p,2)}. $$

Thus, we always have

(4.16) \hspace{1cm} Q_{41} \leq C\|\psi\|\|g(\beta_1, \beta_2; \gamma_1, \gamma_2)\| \left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \left\| D_{k_1} D_{k_2}(f) \right\|_{L^p(X_1 \times X_2)}^{\max(p,2)} \right\}^{1/\max(p,2)}.

If $p \leq 1$, instead of (4.12) by (4.13), similarly to the estimate for $Q_{41}$, we have

$$Q_{42} \leq C\|\psi\|\|g(\beta_1, \beta_2; \gamma_1, \gamma_2)\| \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} 2^{k_2 \gamma_2 - k_1 \beta_1} \right. $$

$$\times \mu_1(Q^{k_1, \nu_1}) \mu_2(Q^{k_2, \nu_2}) \left| D_{k_1} D_{k_2}(f)(y_{k_1, \nu_1}, y_{k_2, \nu_2}) \right|^p \times \frac{1}{(1 + \rho_1(y_{k_1, \nu_1}, x_0))^{d_1 + \gamma_1}} \left. \frac{1}{(1 + \rho_2(y_{k_2, \nu_2}, y_0))^{d_2 + \gamma_2}} \right\}^{1/p} \times \left[ \sum_{\gamma_1 \in K_{\gamma_1}} \sum_{\gamma_2 \in K_{\gamma_2}} \int_{X_1 \times X_2} D_{k_1} D_{k_2}(f)(y_{k_1, \nu_1}, y_{k_2, \nu_2}) \right]^{1/p} \leq C\|\psi\|\|g(\beta_1, \beta_2; \gamma_1, \gamma_2)\| \left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=-\infty}^{\infty} \left\| D_{k_1} D_{k_2}(f) \right\|_{L^p(X_1 \times X_2)}^2 \right\}^{1/2}.
while if $p > 1$, similarly we have

\[
Q_{42} \leq C \|\psi\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)} \left\{ \sum_{k_1 = 0}^{\infty} \sum_{k_2 = -\infty}^{\infty} \sum_{\nu_1 = 1}^{N(k_1, \tau_1)} \sum_{\nu_2 = 1}^{N(k_2, \tau_2)} 2^{k_2 \gamma_2 - k_1 \beta_1} \times \mu_1(Q_{T_1}^{k_1, \nu_1}) \mu_2(Q_{T_2}^{k_2, \nu_2}) \left| D_{k_1} D_{k_2}(f)(y_{T_1}^{k_1, \nu_1}, y_{T_2}^{k_2, \nu_2}) \right|^p \right\}^{1/p} \\
\times \frac{1}{(1 + \rho_1(y_{T_1}^{k_1, \nu_1}, x_0))^{d_1 + \gamma_1}} \left\{ \int_{X_1 \times X_2} \frac{1}{(1 + \rho_1(y_1, x_0))^{d_1 + \gamma_1}} \times \frac{1}{(2^{-k_2} + \rho_2(y_{T_2}^{k_2, \nu_2}, y_0))^{d_2 + \gamma_2}} \times (2^{-k_2} + \rho_2(y_{T_2}^{k_2, \nu_2}, y_0)) d \mu_1(y_1) d \mu_2(y_2) \right\}^{1/p'} \\
\leq C \|\psi\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)} \left\{ \sum_{k_1 = 0}^{\infty} \sum_{k_2 = -\infty}^{\infty} \sum_{\nu_1 = 1}^{N(k_1, \tau_1)} \sum_{\nu_2 = 1}^{N(k_2, \tau_2)} 2^{k_2 \gamma_2 - k_1 \beta_1} \left\| D_{k_1} D_{k_2}(f) \right\|_{L^p(X_1 \times X_2)}^{\max(p, 2)} \right\}^{1/p} \\
\leq C \|\psi\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)} \left\{ \sum_{k_1 = 0}^{\infty} \sum_{k_2 = -\infty}^{\infty} \left\| D_{k_1} D_{k_2}(f) \right\|_{L^p(X_1 \times X_2)}^{\max(p, 2)} \right\}^{1/\max(p, 2)}.
\]

Thus, we always have

\[
(4.17) \quad Q_{42} \leq C \|\psi\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)} \left\{ \sum_{k_1 = 0}^{\infty} \sum_{k_2 = -\infty}^{\infty} \left\| D_{k_1} D_{k_2}(f) \right\|_{L^p(X_1 \times X_2)}^{\max(p, 2)} \right\}^{1/\max(p, 2)}.
\]

By instead of (4.13) by (4.14) and the symmetry with $Q_{42}$, we can verify that

\[
(4.18) \quad Q_{43} \leq C \|\psi\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)} \left\{ \sum_{k_1 = 0}^{\infty} \sum_{k_2 = 0}^{\infty} \left\| D_{k_1} D_{k_2}(f) \right\|_{L^p(X_1 \times X_2)}^{\max(p, 2)} \right\}^{1/\max(p, 2)}.
\]

If $p \leq 1$, the estimate (4.15) and some similar computation to the above yield that

\[
Q_{44} \leq C \|\psi\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)} \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}}^{N(k_1, \tau_1)} \sum_{\nu_1 = 1}^{N(k_2, \tau_2)} \sum_{\nu_2 = 1}^{N(k_2, \tau_2)} 2^{k_1 \gamma_1 + k_2 \gamma_2} \times \mu_1(Q_{T_1}^{k_1, \nu_1}) \mu_2(Q_{T_2}^{k_2, \nu_2}) \left| D_{k_1} D_{k_2}(f)(y_{T_1}^{k_1, \nu_1}, y_{T_2}^{k_2, \nu_2}) \right| \right\}^{1/\max(p, 2)}
\]

\[
\times \frac{2^{-k_1 \gamma_1}}{(2^{-k_1} + \rho_1(y_{T_1}^{k_1, \nu_1}, x_0))^{d_1 + \gamma_1}} \times \frac{2^{-k_2 \gamma_2}}{(2^{-k_2} + \rho_2(y_{T_2}^{k_2, \nu_2}, y_0))^{d_2 + \gamma_2}}.
\]
while if $p > 1$, similarly we have

$$Q_{44} \leq C \|\psi\|g(\beta_1, \beta_2; \gamma_1, \gamma_2) \left\{ \sum_{k_1 = -\infty}^{-1} \sum_{k_2 = -\infty}^{-1} \frac{d \mu_1(y_1)}{|2^{-k_1 \gamma_1} + 2^{-k_2 \gamma_2}|} \sum_{k_1 = -\infty}^{-1} \sum_{k_2 = -\infty}^{-1} \frac{d \mu_2(y_2)}{|2^{-k_1 \gamma_1} + 2^{-k_2 \gamma_2}|} \right\}^{1/p}$$

Thus, we always have

$$Q_{44} \leq C \|\psi\|g(\beta_1, \beta_2; \gamma_1, \gamma_2) \left\{ \sum_{k_1 = -\infty}^{-1} \sum_{k_2 = -\infty}^{-1} \frac{d \mu_1(y_1)}{|2^{-k_1 \gamma_1} + 2^{-k_2 \gamma_2}|} \sum_{k_1 = -\infty}^{-1} \sum_{k_2 = -\infty}^{-1} \frac{d \mu_2(y_2)}{|2^{-k_1 \gamma_1} + 2^{-k_2 \gamma_2}|} \right\}^{1/p}$$

Combining (4.16), (4.17), (4.18) and (4.19) tells us that

$$\langle f, \psi \rangle \leq C \|\psi\|g(\beta_1, \beta_2; \gamma_1, \gamma_2) \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \frac{d \mu_1(y_1)}{|2^{-k_1 \gamma_1} + 2^{-k_2 \gamma_2}|} \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \frac{d \mu_2(y_2)}{|2^{-k_1 \gamma_1} + 2^{-k_2 \gamma_2}|} \right\}^{1/2}$$

Thus, we always have

$$Q_{44} \leq C \|\psi\|g(\beta_1, \beta_2; \gamma_1, \gamma_2) \left\{ \sum_{k_1 = -\infty}^{-1} \sum_{k_2 = -\infty}^{-1} \frac{d \mu_1(y_1)}{|2^{-k_1 \gamma_1} + 2^{-k_2 \gamma_2}|} \sum_{k_1 = -\infty}^{-1} \sum_{k_2 = -\infty}^{-1} \frac{d \mu_2(y_2)}{|2^{-k_1 \gamma_1} + 2^{-k_2 \gamma_2}|} \right\}^{1/2}$$
where in the second equality we used the Minkowski inequality on the series and integral if $p \leq 2$ and we used (4.6) if $p \geq 2$.

Suppose now $\psi \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$. Then there is a sequence of test functions, \( \{\psi_n\}_{n \in \mathbb{N}}, h_n \in \mathcal{G}(\epsilon_1, \epsilon_2) \) such that

\[
\|\psi_n - \psi\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)} \to 0
\]
as $n \to \infty$. The estimate (4.20) then implies that for any $n, m \in \mathbb{N}$,

\[
|\langle f, \psi_n - \psi_m \rangle| \leq C \|g_2(f)\|_{L^p(X_1 \times X_2)} \|\psi_n - \psi_m\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)},
\]
which shows $\lim_{n \to \infty} \langle f, \psi_n \rangle$ exists and the limit is independent of the choice of $\{\psi_n\}_{n \in \mathbb{N}}$. Therefore, we define

\[
\langle f, \psi \rangle = \lim_{n \to \infty} \langle f, \psi_n \rangle.
\]
Then, the estimate (4.20) again tells us that for all $\psi \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$,

\[
|\langle f, \psi \rangle| \leq C \|g_2(f)\|_{L^p(X_1 \times X_2)} \|\psi\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)},
\]
which indicates that $f \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)'$ and we complete the proof of Proposition 4.1.

Thus, Definition 4.1 is reasonable by Proposition 4.1. We remark that in the proof of Proposition 4.1, we actually only require that $0 < \gamma_i < \epsilon_i$ for $i = 1, 2$. However, if $\gamma_i$ and $\beta_i$ for $i = 1, 2$ are as in (4.11), we then can verify that the space of test functions, $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$, is contained in the Hardy space $H^p(X_1 \times X_2)$. To be precise, we have the following proposition.

**Proposition 4.2** Let $p$ and the space $H^p(X_1 \times X_2)$ be the same as in Definition 4.1. If $0 < \beta_i < \epsilon_i$ and $d_i(1/p - 1)_+ < \gamma_i < \epsilon_i$ for $i = 1, 2$, then

\[
\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \subset H^p(X_1 \times X_2).
\]

**Proof.** Let $\psi \in \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ and $\gamma'_i \in (0, \gamma_i)$ such that $d_i(1/p - 1)_+ < \gamma'_i < \epsilon_i$ for $i = 1, 2$. It is easy to see that the estimates (4.12), (4.13), (4.14) and (4.15) with $\tilde{D}_{k_i}$ replaced by $D_{k_i}$ for $i = 1, 2$ still holds. By these estimates and a proof similar to that for (4.20), we obtain

\[
\|\psi\|_{H^p(X_1 \times X_2)} = \left\| \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} |D_{k_1} D_{k_2}(f)|^2 \right\}^{1/2} \right\|_{L^p(X_1 \times X_2)}^{1/\min(p,2)} \leq C \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \|D_{k_1} D_{k_2}(f)\|_{L^p(X_1 \times X_2)}^{\min(p,2)} \right\}^{1/\min(p,2)}.
\]
Let \( \hat{\Omega} \) be the “longest” \( \hat{S}_2(f) \) in \( L^p(X_1 \times X_2) \).

We now use Proposition 4.4 to obtain the atomic decomposition of the Hardy space \( H^p(X_1 \times X_2) \). First, we need to establish Journé’s covering lemma in the setting of homogeneous-type spaces.

We recall some notation. Let \( \{ Q_{\alpha_i}^{k_i} \subset X_i : k_i \in \mathbb{Z}, \alpha_i \in I_{k_i} \} \) for \( i = 1, 2 \) be the same as in Lemma 2.5. Then the open set \( Q_{\alpha_i}^{k_i} \times Q_{\alpha_2}^{k_2} \) for \( k_1, k_2 \in \mathbb{Z}, \alpha_1 \in I_{k_1} \) and \( \alpha_2 \in I_{k_2} \) is called a dyadic rectangle of \( X_1 \times X_2 \). Let \( \Omega \subset X_1 \times X_2 \) be an open set of finite measure and \( \mathcal{M}_1(\Omega) \) denote the family of dyadic rectangles \( R \subset \Omega \) which are maximal in the \( x_i \) “direction”, where \( i = 1, 2 \). In what follows, we denote by \( R = B_1 \times B_2 \) any dyadic rectangle of \( X_1 \times X_2 \). Given \( R = B_1 \times B_2 \in \mathcal{M}_1(\Omega) \), let \( \hat{B}_2 = \hat{B}_2(B_1) \) be the “longest” dyadic cube containing \( B_2 \) such that

\[
(4.21) \quad (\mu_1 \times \mu_2)(B_1 \times \hat{B}_2 \cap \Omega) > \frac{1}{2} (\mu_1 \times \mu_2)(B_1 \times \hat{B}_2);
\]
and given $R = B_1 \times B_2 \in \mathcal{M}_2(\Omega)$, let $\widehat{B}_1 = \widehat{B}_1(B_2)$ be the “longest” dyadic cube containing $B_1$ such that

\[(4.22) \quad (\mu_1 \times \mu_2)(\widehat{B}_1 \times B_2 \cap \Omega) > \frac{1}{2} (\mu_1 \times \mu_2)(\widehat{B}_1 \times B_2).\]

If $B_i = Q^k_{\alpha_i} \subset X_i$ for some $k_i \in \mathbb{Z}$ and some $\alpha_i \in I_{k_i}$, $(B_i)_k$ for $k \in \mathbb{N}$ is used to denote any dyadic cube $Q_{\alpha_i}^{k_i-k}$ containing $Q_{\alpha_i}^{k_i}$ and $(B_i)_0 = B_i$, where $i = 1, 2$. Also, let $w(x)$ be any increasing function such that $\sum_{j=0}^{\infty} jw(C_{13}2^{-j}) < \infty$, where $C_{13} > 0$ is any given constant. In particular, we may take $w(x) = x^{\delta}$ for any $\delta > 0$.

The main idea of the following variant of Journé’s covering lemma in the setting of homogeneous type comes from Pipher [P].

**Lemma 4.2** Assume that $\Omega \subset X_1 \times X_2$ is an open set with finite measure. Let all the notation be the same as above and $\mu = \mu_1 \times \mu_2$. Then

\[(4.23) \quad \sum_{R = B_1 \times B_2 \in \mathcal{M}_1(\Omega)} \mu(R)w\left(\frac{\mu_2(B_2)}{\mu_2(\widehat{B}_2)}\right) \leq C\mu(\Omega)\]

and

\[(4.24) \quad \sum_{R = B_1 \times B_2 \in \mathcal{M}_2(\Omega)} \mu(R)w\left(\frac{\mu_1(B_1)}{\mu_1(\widehat{B}_1)}\right) \leq C\mu(\Omega).\]

**Proof.** We only verify (4.24) and the proof of (4.23) is similar. Let $R = B_1 \times B_2 \in \mathcal{M}_2(\Omega)$ and for $k \in \mathbb{N}$, let

\[(4.25) \quad A_{B_1,k} = \bigcup \left\{ B_2 : B_1 \times B_2 \in \mathcal{M}_2(\Omega) \text{ and } \widehat{B}_1 = (B_1)_{k-1} \right\}.

Then

\[(4.26) \quad \sum_{R = B_1 \times B_2 \in \mathcal{M}_2(\Omega)} \mu(R)w\left(\frac{\mu_1(B_1)}{\mu_1(\widehat{B}_1)}\right)
= \sum_{\{B_1 : B_1 \times B_2 \in \mathcal{M}_2(\Omega)\}} \mu_1(B_1)\mu_2(B_2)w\left(\frac{\mu_1(B_1)}{\mu_1(\widehat{B}_1)}\right)
= \sum_{\{B_1 : B_1 \times B_2 \in \mathcal{M}_2(\Omega)\}} \mu_1(B_1)\sum_{k=1}^{\infty} \sum_{\{B_2 : B_2 \in A_{B_1,k}\}} \mu_2(B_2)w\left(\frac{\mu_1(B_1)}{\mu_1(\widehat{B}_1)}\right)
\leq \sum_{\{B_1 : B_1 \times B_2 \in \mathcal{M}_2(\Omega)\}} \mu_1(B_1)\sum_{k=1}^{\infty} \sum_{\{B_2 : B_2 \in A_{B_1,k}\}} \mu_2(B_2)
\leq \sum_{\{B_1 : B_1 \times B_2 \in \mathcal{M}_2(\Omega)\}} \mu_1(B_1)\sum_{k=1}^{\infty} w\left(C_{13}2^{-k}\right) \sum_{\{B_2 : B_2 \in A_{B_1,k}\}} \mu_2(B_2)\]

\[= \sum_{\{B_1 : B_1 \times B_2 \in \mathcal{M}_2(\Omega)\}} \mu_1(B_1)\sum_{k=1}^{\infty} w\left(C_{13}2^{-k}\right) \mu_2(A_{B_1,k}),\]
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since $\{B_2 : B_2 \in A_{B_1,k}\}$ are disjoint by their “maximality”, where $C_{13} > 0$ depends only on the constants of $\mu_1$ appearing in (2.2) and the constants $C_{10,1}$ and $C_{10,2}$ in Lemma 2.5 for $X_1$.

Set

$$E_{B_1}(\Omega) = \bigcup \{B_2 : B_1 \times B_2 \subset \Omega\}.$$  

If $x_2 \in A_{B_1,k}$, then there is some dyadic cube $B_1 \times B_2 \in \mathcal{M}_2(\Omega)$ and some $k \in \mathbb{N}$ such that $x_2 \in B_2$ and $\hat{B}_1 = (B_1)_{k-1}$ by (4.25). By (4.22) and the maximality of $\hat{B}_1$, we have

$$\mu \left( (B_1)_{k-1} \times B_2 \cap \Omega \right) > \frac{1}{2} \mu \left( (B_1)_{k-1} \times B_2 \right)$$

and

$$\mu \left( (B_1)_{k} \times B_2 \cap \Omega \right) \leq \frac{1}{2} \mu \left( (B_1)_{k} \times B_2 \right),$$

which implies that

$$\mu \left( (B_1)_{k} \times B_2 \cap \left( (B_1)_{k} \times E_{(B_1)_{k}} \right) \right) \leq \frac{1}{2} \mu \left( (B_1)_{k} \times B_2 \right)$$

and further

$$\mu \left( (B_1)_{k} \times (B_2 \cap E_{(B_1)_{k}}) \right) \leq \frac{1}{2} \mu \left( (B_1)_{k} \times B_2 \right).$$

Therefore,

$$\mu_2 \left( B_2 \cap E_{(B_1)_{k}} \right) \leq \frac{1}{2} \mu_2 (B_2),$$

which in turn tells us that

(4.27) \hspace{1cm} $$\mu_2 \left( B_2 \cap \left( E_{(B_1)_{k}} \right)^c \right) > \frac{1}{2} \mu_2 (B_2),$$

where $\left( E_{(B_1)_{k}} \right)^c = X_2 \setminus E_{(B_1)_{k}}$. From (4.27), it follows that

$$M_2 \left( \chi_{E_{(B_1)_{k}}} \right)(x_2) > \frac{1}{2}$$

and therefore

$$A_{B_1,k} \subset \left\{ x_2 \in X_2 : M_2 \left( \chi_{E_{(B_1)_{k}}} \right)(x_2) > \frac{1}{2} \right\},$$

which implies that

(4.28) \hspace{1cm} $$\mu_2 (A_{B_1,k}) \leq \mu_2 \left( \left\{ x_2 \in X_2 : M_2 \left( \chi_{E_{(B_1)_{k}}} \right)(x_2) > \frac{1}{2} \right\} \right)$$

$$\leq C \mu_2 \left( E_{B_1} \setminus E_{(B_1)_{k}} \right).$$
Combining (4.26) with (4.28) yields that
\[
\sum_{R = B_1 \times B_2 \in \mathcal{M}_2(\Omega)} \mu(R) w \left( \frac{\mu_1(B_1)}{\mu_1(B)} \right) \\
\leq C \sum_{\{B_1 : B_1 \times B_2 \in \mathcal{M}_2(\Omega)\}} \mu_1(B_1) \sum_{k=1}^{\infty} w \left( C_{13} 2^{-k} \right) \mu_2 \left( E_{B_1} \setminus E_{(B_1)_k} \right) \\
\leq C \sum_{\{B_1 : B_1 \times B_2 \in \mathcal{M}_2(\Omega)\}} \mu_1(B_1) \sum_{k=1}^{\infty} w \left( C_{13} 2^{-k} \right) \\
\times \left\{ \mu_2 \left( E_{B_1} \setminus E_{(B_1)_1} \right) + \cdots + \mu_2 \left( E_{(B_1)_{k-1}} \setminus E_{(B_1)_k} \right) \right\} \\
\leq C \sum_{\{B_1 : B_1 \times B_2 \in \mathcal{M}_2(\Omega)\}} \mu_1(B_1) \sum_{k=1}^{\infty} w \left( C_{13} 2^{-k} \right) \\
\times \sum_{\{B_0 \text{ dyadic cube}: B_1 \subset B_0 \subset (B_1)_k \} \Omega} \mu_2 \left( E_{B_0} \setminus E_{(B_0)_1} \right) \\
\leq C \sum_{k=1}^{\infty} w \left( C_{13} 2^{-k} \right) \sum_{\{B_0 \text{ dyadic cube} : B_0 \times (E_{B_0} \setminus E_{(B_0)_1}) \subset \Omega} \mu_1(B_0) \mu_2 \left( E_{B_0} \setminus E_{(B_0)_1} \right) \\
\times \sum_{\{B_1 \text{ dyadic cube} : B_1 \subset B_0 \subset (B_1)_k \} \Omega} \mu_1(B_1) \mu_1(B_0)} \\
\leq C \frac{\sum_{k=1}^{\infty} k w \left( C_{13} 2^{-k} \right) \sum_{\{B_0 \text{ dyadic cube} : B_0 \times (E_{B_0} \setminus E_{(B_0)_1}) \subset \Omega} \mu_1(B_0) \mu_2 \left( E_{B_0} \setminus E_{(B_0)_1} \right) \\
\times \sum_{\{B_1 \text{ dyadic cube} : B_1 \subset B_0 \subset (B_1)_k \} \Omega} \mu_1(B_1) \mu_1(B_0)}{\sum_{\{B_0 \text{ dyadic cube} : B_0 \times (E_{B_0} \setminus E_{(B_0)_1}) \subset \Omega} \mu_1(B_0) \mu_2 \left( E_{B_0} \setminus E_{(B_0)_1} \right)} \\
\leq C \frac{\sum_{k=1}^{\infty} k w \left( C_{13} 2^{-k} \right) \sum_{\{B_0 \text{ dyadic cube} : B_0 \times (E_{B_0} \setminus E_{(B_0)_1}) \subset \Omega} \mu_1(B_0) \mu_2 \left( E_{B_0} \setminus E_{(B_0)_1} \right) \mu(\Omega),}{\sum_{\{B_0 \text{ dyadic cube} : B_0 \times (E_{B_0} \setminus E_{(B_0)_1}) \subset \Omega} \mu_1(B_0) \mu_2 \left( E_{B_0} \setminus E_{(B_0)_1} \right)} \\
\leq C \mu(\Omega),
\]

since
\[
\sum_{\{B_0 \text{ dyadic cube} : B_0 \times (E_{B_0} \setminus E_{(B_0)_1}) \subset \Omega} \mu_1(B_0) \mu_2 \left( E_{B_0} \setminus E_{(B_0)_1} \right) \leq C \mu(\Omega)
\]

by noting that the sets \{B_0 \times (E_{B_0} \setminus E_{(B_0)_1}) \subset \Omega : B_0 \text{ is any dyadic cube}\} are disjoint, which finishes the proof of Lemma 4.2.

We now introduce the $H^p(X_1 \times X_2)$-atom. In what follows, for any open set $\Omega$, we denote by $\mathcal{M}(\Omega)$ the set of all maximal dyadic rectangles contained in $\Omega$. 
Definition 4.2 Let all the notation be the same as in Definition 4.1 and \( \mu = \mu_1 \times \mu_2 \). A function \( a(x_1, x_2) \) on \( X_1 \times X_2 \) is called a \((p, 2)\)-atom of \( H^p(X_1 \times X_2) \), if it satisfies

1. \( \text{supp } a \subset \Omega \), where \( \Omega \) is an open set of \( X_1 \times X_2 \) with finite measure;
2. \( a \) can be further decomposed into

\[
a = \sum_{R \in \mathcal{M}(\Omega)} a_R,
\]

where

(i) supposing \( R = Q_1 \times Q_2 \) with \( \text{diam } Q_1 \sim 2^{-k_1} \) and \( \text{diam } Q_2 \sim 2^{-k_2} \), then 

\[
\text{supp } a_R \subset B_1(z_1, A_1(C_{2,1} + C_{10,1}^1)2^{-k_1}) \times B_2(z_2, A_2(C_{2,2} + C_{10,2}^2)2^{-k_2}),
\]

where \( z_i \) is the center of \( Q_i \) for \( i = 1, 2 \), \( C_{10,1}^1 \) and \( C_{10,1}^2 \) mean the constant \( C_{10,1} \) in Lemma 2.5, respectively, for \( X_1 \) and \( X_2 \), and \( C_{2,1} \) and \( C_{2,2} \) means the constant \( C_2 \) in Definition 2.3, respectively, for \( X_1 \) and \( X_2 \).

(ii) for all \( x_1 \in X_1 \),

\[
\int_{X_2} a_R(x_1, x_2) \, d\mu_2(x_2) = 0
\]

and for all \( x_2 \in X_2 \),

\[
\int_{X_1} a_R(x_1, x_2) \, d\mu_1(x_1) = 0;
\]

(iii) \( \|a\|_{L^2(X_1 \times X_2)} \leq \mu(\Omega)^{1/2-1/p} \) and

\[
\left\{ \sum_{R \in \mathcal{M}(\Omega)} \|a_R\|_{L^2(X_1 \times X_2)}^2 \right\}^{1/2} \leq \mu(\Omega)^{1/2-1/p}.
\]

Moreover, \( a_R \) is called an \( H^p(X_1 \times X_2) \) \((p, 2)\)-rectangle atom, if \( a_R \) satisfies (i), (ii) and

(iv) \( \|a_R\|_{L^2(X_1 \times X_2)} \leq \mu(R)^{1/2-1/p} \).

The atomic decomposition of the Hardy space \( H^p(X_1 \times X_2) \) is stated in the following theorem.

Theorem 4.3 Let \( i = 1, 2, X_i \) be a homogeneous-type space as in Definition 2.1, \( \epsilon_i \in (0, \theta_i] \) and

\[
\max \left\{ \frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2} \right\} < p \leq 1.
\]
Then \( f \in H^p(X_1 \times X_2) \) if and only if \( f \in \left( \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)' \) for some \( \beta_i, \gamma_i \) satisfying (4.11), where \( i = 1, 2 \), and there is a sequence of numbers, \( \{\lambda_k\}_{k \in \mathbb{Z}} \), and a sequence of \((p,2)\)-atoms of \( H^p(X_1 \times X_2) \), \( \{a_k\}_{k \in \mathbb{Z}} \), such that \( \sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty \) and

\[
f = \sum_{k=-\infty}^{\infty} \lambda_k a_k
\]
in \( \left( \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)' \). Moreover, in this case,

\[
\|f\|_{H^p(X_1 \times X_2)} \sim \inf \left\{ \left[ \sum_{k=-\infty}^{\infty} |\lambda_k|^p \right]^{1/p} \right\},
\]

where the infimum is taken over all the decompositions as above.

**Proof.** Let \( f \in H^p(X_1 \times X_2) \). By Definition 4.1, \( f \in \left( \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \right)' \) for some \( \beta_i, \gamma_i \) satisfying (4.11), where \( i = 1, 2 \). We use Theorem 2.5 and Proposition 4.4 to get the atomic decomposition of \( f \). To this end, for \( i = 1, 2 \), let \( \{S_{k_i}\}_{k_i \in \mathbb{Z}} \) be an approximation to the identity of order \( \epsilon_i \) having compact support as in Definition 2.3 on space of homogeneous type, \( X_i \), and \( D_{k_i} = S_{k_i} - S_{k_i-1} \) for all \( k_i \in \mathbb{Z} \). Then, by Theorem 2.5, there exist two families of linear operators \( \{D_{k_i}\}_{k_i \in \mathbb{Z}} \) on \( X_i \) as in Theorem 2.2 such that

\[
f = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} D_{k_1} D_{k_2} \mathcal{D}_{k_1} \mathcal{D}_{k_2} (f)
\]
in \( \left( \mathcal{G}(\beta'_1, \beta'_2; \gamma'_1, \gamma'_2) \right)' \) with \( \beta'_i \in (\beta_i, \epsilon_i) \) and \( \gamma'_i \in (\gamma_i, \epsilon_i) \) for \( i = 1, 2 \). By Definition 4.1, Theorem 4.2 and Remark 4.1, we have

\[
\|S_2(f)\|_{L^p(X_1 \times X_2)} \leq C \|f\|_{H^p(X_1 \times X_2)},
\]

where \( S_2(f) \) is defined by (6.32) with \( D_{k_i} \) replaced by \( \mathcal{D}_{k_i} \) for \( i = 1, 2 \) as in Theorem 2.2, \( q = 2 \), \( C_{1,1} \geq 1 \) and \( C_{1,2} \geq 2 \). For any \( k \in \mathbb{Z} \), let

\[
\Omega_k = \left\{ (x_1, x_2) \in X_1 \times X_2 : S_2(f)(x_1, x_2) > 2^k \right\}.
\]

Let \( \mu = \mu_1 \times \mu_2 \),

\[
\mathcal{R} = \{ R = Q_1 \times Q_2 : Q_1 \text{ and } Q_2 \text{ are dyadic cubes, respectively, of } X_1 \text{ and } X_2 \},
\]

and for \( k \in \mathbb{Z} \),

\[
\mathcal{R}_k = \left\{ R \in \mathcal{R} : \mu(R \cap \Omega_k) > \frac{1}{2} \mu(R) \text{ and } \mu(R \cap \Omega_{k+1}) \leq \frac{1}{2} \mu(R) \right\}.
\]
Obviously, for any \( R \in \mathcal{R} \), there is a unique \( k \in \mathbb{Z} \) such that \( R \in k \). Thus, we can reclassify the set of all dyadic cubes in \( X_1 \times X_2 \) by

\[
\bigcup_{R \in \mathcal{R}} R = \bigcup_{k \in \mathbb{Z}} \bigcup_{R \in k} R.
\]

In what follows, for \( i = 1, 2 \), if \( Q_{k_i} \) is a dyadic cube and \( \text{diam} \ Q_{k_i} \sim 2^{-k_i} \), we rewrite \( D_{k_i} \) and \( \overline{D}_{k_i} \), respectively, by \( D_{Q_{k_i}} \) and \( \overline{D}_{Q_{k_i}} \). Then, from (4.29) and (4.30), it follows that

\[
f = \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} D_{k_1} D_{k_2} \overline{D}_{k_1} \overline{D}_{k_2} (f)
\]

\[
= \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \sum_{\text{diam} \ Q_{k_1} \sim 2^{-k_1}} \sum_{\text{diam} \ Q_{k_2} \sim 2^{-k_2}} \int_{Q_{k_1}} \int_{Q_{k_2}} D_{Q_{k_1}}(x_1, y_1) D_{Q_{k_2}}(x_2, y_2)
\]

\[
\times \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_1, y_2) \, d\mu_1(y_1) \, d\mu_2(y_2)
\]

\[
= \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}} \int_{R} D_{Q_{k_1}}(x_1, y_1) D_{Q_{k_2}}(x_2, y_2)
\]

\[
\times \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_1, y_2) \, d\mu_1(y_1) \, d\mu_2(y_2)
\]

\[
= \sum_{k = -\infty}^{\infty} \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_{R} D_{Q_{k_1}}(x_1, y_1) D_{Q_{k_2}}(x_2, y_2)
\]

\[
\times \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_1, y_2) \, d\mu_1(y_1) \, d\mu_2(y_2)
\]

\[
= \sum_{k = -\infty}^{\infty} \lambda_k a_k(x_1, x_2),
\]

where

\[
\lambda_k = \frac{2^k \mu(\Omega_k)^{1/p}}{C_{14,1}}
\]

and

\[
a_k(x_1, x_2) = \frac{C_{14,1}}{2^k \mu(\Omega_k)^{1/p}} \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_{R} D_{Q_{k_1}}(x_1, y_1) D_{Q_{k_2}}(x_2, y_2)
\]

\[
\times \overline{D}_{Q_{k_1}} \overline{D}_{Q_{k_2}}(f)(y_1, y_2) \, d\mu_1(y_1) \, d\mu_2(y_2),
\]

where \( C_{14,1} > 0 \) is a constant which will be determined later.
We now verify that \( \{\lambda_k\}_{k \in \mathbb{Z}} \) and \( \{a_k\}_{k \in \mathbb{Z}} \) satisfy the requirement of the theorem. First, some trivial computation tells us that

\[
\sum_{k=-\infty}^{\infty} |\lambda_k|^p = \frac{1}{C_{14,1}^{p}} \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\Omega_k) \\
\leq C \sum_{k=-\infty}^{\infty} 2^{kp} \mu(\Omega_k \setminus \Omega_{k+1}) \\
\leq C \|S_2(f)\|_{L^p(X_1 \times X_2)}^p,
\]

which is a desired estimate.

Let \( C_{14,2} \in (0, 1/2) \) be a small constant which will be determined later and for \( k \in \mathbb{Z} \),

\[
\tilde{\Omega}_k = \{(x_1, x_2) \in X_1 \times X_2 : M_s \chi_{\Omega_k}(x_1, x_2) > C_{14,2}\}.
\]

Then, the \( L^q(X_1 \times X_2) \)-boundedness of \( M_s \) with \( q \in (1, \infty) \) implies that

\[
\mu(\tilde{\Omega}_k) \leq C \mu(\Omega_k).
\]

Moreover, if \( C_{14,2} \) is chosen to be small enough which depends on \( A_1, A_2, C_2 \) in Definition 2.3, the constants concealed in (2.2), \( C_{10,1}^1 \) and \( C_{10,1}^2 \), then it is easy to check that

\[
\text{supp} a_k \subset \tilde{\Omega}_k.
\]

Let now \( h \in L^2(X_1 \times X_2) \) with \( \|h\|_{L^2(X_1 \times X_2)} \leq 1 \). The Hölder inequality, (4.30) and Theorem 3.1 tell us that

\[
|\langle a_k, h \rangle| = \left| \frac{C_{14,1}}{2^{kp} \mu(\Omega_k)^{1/p}} \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_{X_1 \times X_2} \int_R D_{Q_{k_1}} f(x_1, y_1) D_{Q_{k_2}} h(x_2, y_2) \\
\times \mathcal{D}_{Q_{k_1}} D_{Q_{k_2}} f(y_1, y_2) d\mu_1(y_1) d\mu_2(y_2) d\mu_1(x_1) d\mu_2(x_2) \right| \\
= \frac{C_{14,1}}{2^{kp} \mu(\Omega_k)^{1/p}} \left| \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_R \mathcal{D}_{Q_{k_1}} D_{Q_{k_2}} f(y_1, y_2) \\
\times \mathcal{D}_{Q_{k_1}}^* D_{Q_{k_2}}^* h(y_1, y_2) d\mu_1(y_1) d\mu_2(y_2) \right|
\]
On the other hand, we have

\[ \sum_{\Omega} \int_{Q} \left| \mathcal{D}_{Q_{k_1}} \mathcal{D}_{Q_{k_2}} (f) (y_1, y_2) \right|^2 d\mu_1(y_1) d\mu_2(y_2) \]

To see this, we note that the estimate (4.35) indicates that

\[ \mathcal{D}_{Q_{k_1}} \mathcal{D}_{Q_{k_2}} (h) (y_1, y_2), \]

We now claim that

\[ \leq \frac{C_{14.1}}{2^k \mu(\Omega_k)^{1/p}} \]

\[ \times \left\{ \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_{R} \left| \mathcal{D}_{Q_{k_1}} \mathcal{D}_{Q_{k_2}} (f) (y_1, y_2) \right|^2 d\mu_1(y_1) d\mu_2(y_2) \right\}^{1/2} \]

\[ \times \left\{ \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_{R} \left| \mathcal{D}_{Q_{k_1}} \mathcal{D}_{Q_{k_2}} (h) (y_1, y_2) \right|^2 d\mu_1(y_1) d\mu_2(y_2) \right\}^{1/2} \]

\[ \leq \frac{C_{14.1}}{2^k \mu(\Omega_k)^{1/p}} \]

\[ \times \left\{ \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_{R} \left| \mathcal{D}_{Q_{k_1}} \mathcal{D}_{Q_{k_2}} (f) (y_1, y_2) \right|^2 d\mu_1(y_1) d\mu_2(y_2) \right\}^{1/2} \]

\[ \times \left\{ \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_{R} \left| \mathcal{D}_{Q_{k_1}} \mathcal{D}_{Q_{k_2}} (h) (y_1, y_2) \right|^2 d\mu_1(y_1) d\mu_2(y_2) \right\}^{1/2} \].

We now claim that

\[ (4.38) \left\{ \sum_{R=Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_{R} \left| \mathcal{D}_{Q_{k_1}} \mathcal{D}_{Q_{k_2}} (f) (y_1, y_2) \right|^2 d\mu_1(y_1) d\mu_2(y_2) \right\}^{1/2} \leq C 2^k \mu(\Omega_k)^{1/2}. \]

To see this, we note that the estimate (4.35) indicates that

\[ (4.39) \int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} [S_2 (f) (x_1, x_2)]^2 d\mu_1(x_1) d\mu_2(x_2) \leq 2^{2(k+1)} \mu(\tilde{\Omega}_k \setminus \Omega_{k+1}) \leq C 2^k \mu(\Omega_k). \]

On the other hand, we have

\[ (4.40) \int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} [S_2 (f) (x_1, x_2)]^2 d\mu_1(x_1) d\mu_2(x_2) \]

\[ = \int_{\tilde{\Omega}_k \setminus \Omega_{k+1}} \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \int_{\rho_1 (x_1, y_1) \leq C_{11.1} 2^{-k_1}} \right\} \]
\[ \times \int_{\rho_2(x_2,y_2) \leq C_{11,2}^{-2k_2}} 2^{k_1d_1+k_2d_2} |\overline{D}_{k_1} \overline{D}_{k_2}(f)(y_1, y_2)|^2 \, d\mu_1(y_1) \, d\mu_2(y_2) \]
\[ \times d\mu_1(x_1) \, d\mu_2(x_2) \]
\[ = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \int_{X_1 \times X_2} 2^{k_1d_1+k_2d_2} \mu \left( \left\{ (x_1, x_2) \in \tilde{\Omega}_k \setminus \Omega_{k+1} : \right. \right.
\[ \times \rho_1(x_1, y_1) \leq C_{11,1} 2^{-k_1}, \rho_2(x_2, y_2) \leq C_{11,2} 2^{-k_2} \right\} \bigg) \bigg( \left| \overline{D}_{k_1} \overline{D}_{k_2}(f)(y_1, y_2) \right|^2 \, d\mu_1(y_1) \, d\mu_2(y_2) \bigg) \bigg) \bigg( \left( \overline{D}_{k_1} \overline{D}_{k_2}(f)(y_1, y_2) \right|^2 \, d\mu_1(y_1) \, d\mu_2(y_2) \bigg). \]

If \((y_1, y_2), (x_1, x_2) \in R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k\), by Lemma 2.5 and our choice that \(C_{11,1} \geq C_{10,1}^1\) and \(C_{11,2} \geq C_{10,1}^2\), we have \(\rho_1(x_1, y_1) \leq C_{11,1} 2^{-k_1}\) and \(\rho_2(x_2, y_2) \leq C_{11,2} 2^{-k_2}\), and moreover, \(R \in \mathcal{R}_k\) implies that \(R \subset \tilde{\Omega}_k\), since \(C_{14,2} < 1/2\). These facts lead us that

\[ (4.41) \quad \mu \left( \left\{ (x_1, x_2) \in \tilde{\Omega}_k \setminus \Omega_{k+1} : \right. \right.
\[ \times \rho_1(x_1, y_1) \leq C_{11,1} 2^{-k_1}, \rho_2(x_2, y_2) \leq C_{11,2} 2^{-k_2} \right\} \bigg) \bigg( \left| \overline{D}_{k_1} \overline{D}_{k_2}(f)(y_1, y_2) \right|^2 \, d\mu_1(y_1) \, d\mu_2(y_2) \bigg) \bigg( \left( \overline{D}_{k_1} \overline{D}_{k_2}(f)(y_1, y_2) \right|^2 \, d\mu_1(y_1) \, d\mu_2(y_2) \bigg). \]

Combining (4.40), (4.41) with (4.39) yield our claim (4.38). Using (4.38) and (4.35) and taking the supremum in both sides of (4.37) on \(h \in L^2(X_1 \times X_2)\) with \(\|h\|_{L^2(X_1 \times X_2)} \leq 1\) tells us that

\[ (4.42) \quad \|a_k\|_{L^2(X_1 \times X_2)} \leq CC_{14,1} \mu(\Omega_k)^{1/2-1/p} \leq \mu \left( \tilde{\Omega}_k \right)^{1/2-1/p}, \]

if we choose \(C_{14,1} > 0\) such that \(CC_{14,1} < 1\), which is a desired estimate.
Obviously if \( R \in \mathcal{R}_k \), then \( R \subset \mathcal{O}_k \). From this, it is easy to see that we can further decompose \( a_k(x_1, x_2) \) into

\[
(4.43) \quad a_k(x_1, x_2) = \frac{C_{14,1}}{2^k \mu(\Omega_k)^{1/p}} \sum_{R=Q_k \times Q'_{k_2} \in \mathcal{R}_k} \int_R D_{Q_k_1}(x_1, y_1) D_{Q_k_2}(x_2, y_2)
\]

\[
\times \overline{D}_{Q_k_1} \overline{D}_{Q_k_2}(f)(y_1, y_2) \ d\mu_1(y_1) \ d\mu_2(y_2)
\]

\[
= \frac{C_{14,1}}{2^k \mu(\Omega_k)^{1/p}} \sum_{\tilde{R} \in \mathcal{M}(\mathcal{O}_k)} \sum_{R \subseteq \tilde{R}} \sum_{R=Q_k \times Q'_{k_2} \in \mathcal{R}_k} \int_R D_{Q_k_1}(x_1, y_1) D_{Q_k_2}(x_2, y_2)
\]

\[
\times \overline{D}_{Q_k_1} \overline{D}_{Q_k_2}(f)(y_1, y_2) \ d\mu_1(y_1) \ d\mu_2(y_2)
\]

\[
= \sum_{\tilde{R} \in \mathcal{M}(\mathcal{O}_k)} \alpha_{\tilde{R}}(x_1, x_2).
\]

Let \( \tilde{R} = Q_1 \times Q_2 \) with \( \text{diam} \ Q_1 \sim 2^{-k_1} \) and \( \text{diam} \ Q_2 \sim 2^{-k_2} \) and \( z_i \) be the center of \( Q_i \) with \( i = 1, 2 \). Then \( k_i' \leq k_i \) for \( i = 1, 2 \). From this, it is easy to verify that

\[
(4.44) \quad \text{supp} \ a_{\tilde{R}} \subset B_1(z_1, A_1(C_{2,1} + C_{10,1}) 2^{-k_1}) \times B_2(z_2, A_2(C_{2,2} + C_{10,2}) 2^{-k_2}).
\]

Obviously, we have that for all \( x_2 \in X_2 \),

\[
(4.45) \quad \int_{X_1} \alpha_{\tilde{R}}(x_1, x_2) \ d\mu_1(x_1) = 0,
\]

and for all \( x_1 \in X_1 \),

\[
(4.46) \quad \int_{X_2} \alpha_{\tilde{R}}(x_1, x_2) \ d\mu_2(x_2) = 0.
\]

Let \( h \) be the same as in (4.37). Similarly to the estimate for (4.37), we have

\[
\left| \langle \alpha_{\tilde{R}}, h \rangle \right|
\]

\[
= \frac{C_{14,1}}{2^k \mu(\Omega_k)^{1/p}} \left| \int_{X_1 \times X_2} \sum_{R=Q_k \times Q'_{k_2} \in \mathcal{R}_k} \int_R D_{Q_k_1}(x_1, y_1) D_{Q_k_2}(x_2, y_2)
\]

\[
\times \overline{D}_{Q_k_1} \overline{D}_{Q_k_2}(f)(y_1, y_2) h(x_1, x_2) \ d\mu_1(y_1) \ d\mu_2(y_2) \ d\mu_1(x_1) \ d\mu_2(x_2) \right|
\]
From this, it follows that

\[
\leq \frac{C_{14,1}}{2^k \mu(\Omega_k)^{1/p}} \left\{ \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_R |D_{Q_{k_1}} D_{Q_{k_2}}^* (f)(y_1, y_2)|^2 \, d\mu_1(y_1) \, d\mu_2(y_2) \right\}^{1/2}
\times \left\{ \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_R |D_{Q_{k_1}}^* D_{Q_{k_2}} (h)(y_1, y_2)|^2 \, d\mu_1(y_1) \, d\mu_2(y_2) \right\}^{1/2}
\leq \frac{CC_{14,1} \|h\|_{L^2(X_1 \times X_2)}}{2^k \mu(\Omega_k)^{1/p}}
\times \left\{ \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_R |D_{Q_{k_1}} D_{Q_{k_2}} (f)(y_1, y_2)|^2 \, d\mu_1(y_1) \, d\mu_2(y_2) \right\}^{1/2}
\leq \frac{CC_{14,1}}{2^k \mu(\Omega_k)^{1/p}} \left\{ \sum_{R = Q_{k_1} \times Q_{k_2} \in \mathcal{R}_k} \int_R |D_{Q_{k_1}} D_{Q_{k_2}} (f)(y_1, y_2)|^2 \, d\mu_1(y_1) \, d\mu_2(y_2) \right\}^{1/2}.
\]
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\[ \leq \mu \left( \Omega_k \right)^{1/2 - 1/p}, \]

if we chose $C_{14,1}$ small enough such that $CC_{14,1} < 1$. This, together with (4.34), (4.36), (4.42), (4.44), (4.45) and (4.46), tells us that we have obtained a desired atomic decomposition for $f$.

We now consider the converse. To this end, by Definition 4.1 and (4.6), we easily see that it suffices to verify that there is a constant $C > 0$ such that for any $(p, 2)$-atom of $H^p(X_1 \times X_2)$, $a$,

\[ (4.47) \quad \|g_2(a)\|_{L^p(X_1 \times X_2)} \leq C, \]

where

\[ g_2(a)(x_1, x_2) = \left\{ \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} |D_{k_1} D_{k_2}(a)(x_1, x_2)|^2 \right\}^{1/2} \]

and we choose \( \{D_{k_i}\}_{k_i \in \mathbb{Z}} \) as in (4.29).

We suppose $\text{supp} \ a \subset \Omega$ and define

\[ \tilde{\Omega} = \{(x_1, x_2) \in X_1 \times X_2 : M_s \chi_\Omega(x_1, x_2) > 1/2\} \]

and

\[ \bar{\Omega} = \{(x_1, x_2) \in X_1 \times X_2 : M_s \chi_{\tilde{\Omega}}(x_1, x_2) > 1/2\}. \]

Moreover, suppose $a = \sum_{R \in \mathcal{M}(\Omega)} a_R$. For any $R = Q_1 \times Q_2 \in \mathcal{M}(\Omega)$, we define $\tilde{R} = \tilde{Q}_1 \times \tilde{Q}_2 \in \mathcal{M}_1(\tilde{\Omega})$ such that

\[ \mu(\tilde{R} \cap \Omega) > \frac{1}{2} \mu(\tilde{R}) \]

and $\bar{R} = \bar{Q}_1 \times \bar{Q}_2 \in \mathcal{M}_2(\bar{\Omega})$ such that

\[ \mu(\bar{R} \cap \tilde{\Omega}) > \frac{1}{2} \mu(\bar{R}) \]

Let $C_{15,1} \geq 1$ and $C_{15,2} \geq 1$ be two constants which will be determined later and we set

\[ 100C_R = 100C_{15,1} \tilde{Q}_1 \times 100C_{15,2} \tilde{Q}_2, \]

where $100C_{15,i} \tilde{Q}_i$ means the “cube” with the same center as $\tilde{Q}_i$ but with diameter $100C_{15,i}$ times the diameter of $\tilde{Q}_i$. We also denote by $\tilde{z}_i$ the center of $\tilde{Q}_i$ for $i = 1, 2$.

We now control $\|g_2(a)\|_{L^p(X_1 \times X_2)}$ by

\[ \|g_2(a)\|_{L^p(X_1 \times X_2)}^p = \int_{\bigcup_{R \in \mathcal{M}(\Omega)} 100C_R \pi} g_2(a)(x_1, x_2)^p \, d\mu_1(x_1) \, d\mu_2(x_2) + \int_{\bigcup_{R \in \mathcal{M}(\Omega)} 100C_R \pi} \cdots \]

\[ = U_{11} + U_{12}, \]
where \( \left( \bigcup_{R' \in M(\Omega)} 100C^2R \right)^c = (X_1 \times X_2) \setminus \left( \bigcup_{R' \in M(\Omega)} 100C^2R \right) \). The Hölder inequality, Lemma 2.5 and Theorem 3.1 imply that

\[
(4.48) \quad U_{11} \leq \mu \left( \bigcup_{R' \in M(\Omega)} 100C^2R \right)^{1-p/2} \left\{ \int_{X_1 \times X_2} g_2(a)(x_1, x_2)^2 \, d\mu_1(x_1) \, d\mu_2(x_2) \right\}^{p/2}
\]

\[
\leq C \mu(\Omega)^{1-p/2} \|a\|_{L^2(X_1 \times X_2)}^p
\]

\[
\leq C,
\]

which is a desired estimate.

We further control \( U_{12} \) by

\[
(4.49) \quad U_{12} = \int_{\left( \bigcup_{R' \in M(\Omega)} 100C^2R \right)^c} g_2(a)(x_1, x_2)^p \, d\mu_1(x_1) \, d\mu_2(x_2)
\]

\[
\leq \sum_{R \in M(\Omega)} \int_{\left( \bigcup_{R' \in M(\Omega)} 100C^2R \right)^c} g_2(a_R)(x_1, x_2)^p \, d\mu_1(x_1) \, d\mu_2(x_2)
\]

\[
\leq \sum_{R \in M(\Omega)} \left\{ \int_{x_1 \notin 100C_{15,1}Q_1} \int_{X_2} g_2(a_R)(x_1, x_2)^p \, d\mu_1(x_1) \, d\mu_2(x_2) \right. 
\]

\[
+ \int_{X_1} \int_{x_2 \notin 100C_{15,2}Q_2} \cdots \right\}
\]

\[
= \sum_{R \in M(\Omega)} (U_{1R} + U_{2R}).
\]

The estimate for \( U_{2R} \) is similar to the estimate for \( U_{1R} \) by symmetry. Thus, we only estimate \( U_{1R} \) and leave the details for the estimate of \( U_{2R} \) to the reader. To estimate \( U_{1R} \), we further decompose it into

\[
U_{1R} = \int_{x_1 \notin 100C_{15,1}Q_1} \int_{X_2} g_2(a_R)(x_1, x_2)^p \, d\mu_1(x_1) \, d\mu_2(x_2)
\]

\[
= \int_{x_1 \notin 100C_{15,1}Q_1} \int_{x_2 \in 100C_{15,2}Q_2} g_2(a_R)(x_1, x_2)^p \, d\mu_1(x_1) \, d\mu_2(x_2)
\]

\[
+ \int_{x_1 \notin 100C_{15,1}Q_1} \int_{x_2 \notin 100C_{15,2}Q_2} \cdots
\]

\[
= U_{1R1} + U_{1R2}.
\]

The Hölder inequality implies that

\[
(4.50) \quad U_{1R1} \leq \mu_2 \left( 100C_{15,2}Q_2 \right)^{1-p/2}
\]

\[
\times \int_{x_1 \notin 100C_{15,1}Q_1} \left[ \int_{x_2 \in 100C_{15,2}Q_2} g_2(a_R)(x_1, x_2)^2 \, d\mu_2(x_2) \right]^{p/2} \, d\mu_1(x_1)
\]
\[ \leq C \mu_2(Q_2)^{1-p/2} \int_{x_1 \notin 100C_{15,1} \tilde{Q}_1} \left[ \int_{X_2} g_2(a_R)(x_1, x_2)^2 \, d\mu_2(x_2) \right]^{p/2} \, d\mu_1(x_1). \]

By Lemma 3.1 for \( X_2 \), we obtain

\[
(4.51) \quad \int_{X_2} g_2(a_R)(x_1, x_2)^2 \, d\mu_2(x_2)
\]
\[ = \int_{X_2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} |D_{k_1} D_{k_2}(a_R)(x_1, x_2)|^2 \, d\mu_2(x_2) \]
\[ \leq C \sum_{k_1=-\infty}^{\infty} \int_{X_2} |D_{k_1} [a_R(\cdot, x_2)](x_1)|^2 \, d\mu_2(x_2) \]
\[ = C \sum_{k_1=-\infty}^{\infty} \int_{X_2} \int_{X_1} |D_{k_1}(x_1, y_1) a_R(y_1, x_2) \, d\mu_1(y_1)|^2 \, d\mu_2(x_2). \]

Suppose \( \text{diam } Q_i \sim 2^{-k_{i,0}} \) and \( \text{diam } \tilde{Q}_i \sim 2^{-\tilde{k}_{i,0}} \) for some \( k_{i,0}, \tilde{k}_{i,0} \in \mathbb{Z} \) and \( i = 1, 2 \). Then \( \tilde{k}_{i,0} \leq k_{i,0} \). From

\[ \text{supp } a_R \subset B_1(z_1, A_1(C_{2,1} + C_{10,1}^1)2^{-k_{1,0}}) \times B_2(z_2, A_2(C_{2,2} + C_{10,2}^2)2^{-k_{2,0}}), \]

where \( z_i \) is the center of \( Q_i \), it follows that

\[ \rho_1(y_1, z_1) \leq A_1(C_{2,1} + C_{10,1}^1)2^{-k_{1,0}}, \]

which combines the fact that \( \rho_1(x_1, y_1) \leq C_{2,1}2^{-k_1} \) tells us that

\[
(4.52) \quad \rho_1(x_1, z_1) \leq A_1 C_{2,1}2^{-k_1} + A_1(C_{2,1} + C_{10,1}^1)2^{-k_{1,0}} \]
\[ \leq A_1 C_{2,1}2^{-k_1} + A_1(C_{2,1} + C_{10,1}^1)2^{-\tilde{k}_{1,0}}. \]

On another hand, since \( z_1 \in Q_1 \subset \tilde{Q}_1 \) and \( x_1 \notin 100C_{15,1} \tilde{Q}_1 \), we then have

\[
(4.53) \quad \rho_1(x_1, z_1) \geq \left( 1 - \frac{C_{10,1}^1}{100C_{10,2}^1C_{15,1}} \right) 100C_{10,2}^1C_{15,1}2^{-\tilde{k}_{1,0}}. \]

If we choose \( C_{15,1} \) large enough, then (4.52) and (4.53) tell us that \( k_1 \leq \tilde{k}_{1,0} \) in (4.51). Thus, by the Hölder inequality, we further have

\[
(4.54) \quad \int_{X_2} g_2(a_R)(x_1, x_2)^2 \, d\mu_2(x_2) \]
\[
\leq C \sum_{k_1 = -\infty}^{\tilde{k}_{1,0}} \int_{X_1} \int_{X_2} D_{k_1}(x_1, y_1) a_R(y_1, x_2) \, d\mu_1(y_1) \int_{X_2} \, d\mu_2(x_2)
\]

\[
= C \sum_{k_1 = -\infty}^{\tilde{k}_{1,0}} \int_{X_1} \int_{X_1} |D_{k_1}(x_1, y_1) - D_{k_1}(x_1, z_1)| a_R(y_1, x_2) \, d\mu_1(y_1) \int_{X_2} \, d\mu_2(x_2)
\]

\[
\leq C \sum_{k_1 = -\infty}^{\tilde{k}_{1,0}} \frac{2^{-2k_1,0\epsilon_1} 2^{2k_1\epsilon_1'}}{\rho_1(x_1, z_1)^{2d_1 + 2(\epsilon_1 - \epsilon_1')}} \int_{X_2} \left[ \int_{X_1} |a_R(y_1, x_2)| \, d\mu_1(y_1) \right] \int_{X_2} \, d\mu_2(x_2)
\]

\[
\leq C \frac{2^{-2k_1,0\epsilon_1} 2^{2\tilde{k}_{1,0}\epsilon'}}{\rho_1(x_1, z_1)^{2d_1 + 2(\epsilon_1 - \epsilon_1')}} \mu_1(Q_1) \mu_2(Q_1),
\]

where we chose \( \epsilon'_1 \in (0, \epsilon_1) \) such that \( d_1 p + (\epsilon_1 - \epsilon'_1)p > d_1 \). Noting that if \( C_{15,1} \) is large enough, then \( x_1 \notin 100C_{15,1}Q_1 \) implies that \( \rho_1(x_1, z_1) \geq \rho_1(x_1, \bar{z}_1) \), which, together (4.54) with (4.50) indicates that

\[
U_{1R1} \leq C \mu_2(Q_2)^{1-p/2} \mu_1(Q_1)^{p/2} \|a_R\|^p_{L^2(X_1 \times X_2)}
\]

\[
\times \int_{x_1 \notin 100C_{15,1}Q_1} \frac{2^{-p\tilde{k}_{1,0}\epsilon_1} 2^{p\tilde{k}_{1,0}\epsilon'_1}}{\rho_1(x_1, z_1)^{pd_1 + p(\epsilon_1 - \epsilon'_1)}} \, d\mu_1(x_1)
\]

\[
\leq C \mu_2(Q_2)^{1-p/2} \mu_1(Q_1)^{p/2} \|a_R\|^p_{L^2(X_1 \times X_2)}
\]

\[
\times \int_{x_1 \notin 100C_{15,1}Q_1} \frac{2^{-p\tilde{k}_{1,0}\epsilon_1} 2^{p\tilde{k}_{1,0}\epsilon'_1}}{\rho_1(x_1, \bar{z}_1)^{pd_1 + p(\epsilon_1 - \epsilon'_1)}} \, d\mu_1(x_1)
\]

\[
\leq C \mu(R)^{1-p/2} \|a_R\|^p_{L^2(X_1 \times X_2)} \left( \frac{\mu_1(Q_1)}{\mu_1(Q_1)} \right)^{p(d_1 + \epsilon_1)/d_1 - 1}.
\]

From this, the Hölder inequality and Lemma 4.2, it follows that

\[
(4.55) \quad \sum_{R \in \mathcal{M}(\Omega)} U_{1R1} \leq C \left\{ \sum_{R \in \mathcal{M}(\Omega)} \|a_R\|_{L^2(X_1 \times X_2)}^2 \right\}^{p/2}
\]

\[
\times \left\{ \sum_{R \in \mathcal{M}(\Omega)} \mu(R) \left( \frac{\mu_1(Q_1)}{\mu_1(Q_1)} \right)^{p(d_1 + \epsilon_1)/d_1 - 1/2 - p} \right\}^{1-p/2}
\]

\[
\leq C \mu(\Omega)^{p/2 - 1} \mu(\Omega)^{1-p/2}
\]

\[
\leq C,
\]

which is a desired estimate.

We now estimate \( U_{1R2} \). For \( x_1 \notin 100C_{15,1}\bar{Q}_1 \) and \( x_2 \notin 100C_{15,2}Q_2 \), similarly to the
estimate for $U_{1R1}$, if we choose $C_{15,1}$ and $C_{15,2}$ large enough, we then have

$$g_2(a_R)(x_1, x_2)$$

$$= \left\{ \sum_{k_1=\infty}^\infty \sum_{k_2=\infty}^\infty \left| \int_{X_1 \times X_2} D_{k_1}(x_1, y_1) D_{k_2}(x_2, y_2) a_R(y_1, y_2) \, d\mu_1(y_1) \, d\mu_2(y_2) \right|^2 \right\}^{1/2}$$

$$\leq \left\{ \sum_{\tilde{k}_1,0} \sum_{k_2,0} \left| \int_{X_1 \times X_2} D_{k_1}(x_1, y_1) D_{k_2}(x_2, y_2) a_R(y_1, y_2) \, d\mu_1(y_1) \, d\mu_2(y_2) \right|^2 \right\}^{1/2}$$

$$= \left\{ \sum_{\tilde{k}_1,0} \sum_{k_2,0} \left| \int_{X_1 \times X_2} [D_{k_1}(x_1, y_1) - D_{k_1}(x_1, z_1)] \right| \, d\mu_1(y_1) \, d\mu_2(y_2) \right\}^{1/2}$$

$$\times \left[ \int_{X_1 \times X_2} |a_R(y_1, y_2)| \, d\mu_1(y_1) \, d\mu_2(y_2) \right]^2$$

$$\leq C \left\{ \sum_{\tilde{k}_1,0} \sum_{k_2,0} \left| \int_{X_1 \times X_2} [D_{k_1}(x_1, y_1) - D_{k_1}(x_1, z_1)] \right| \, d\mu_1(y_1) \, d\mu_2(y_2) \right\}^{1/2}$$

$$\times \left[ \int_{X_1 \times X_2} |a_R(y_1, y_2)| \, d\mu_1(y_1) \, d\mu_2(y_2) \right]^2$$

$$\leq C \|a_R\|_{L^2(X_1 \times X_2)} \|\mu(R)\|^{1/2} \frac{2\tilde{k}_{1,0}^{-\epsilon_1}}{\rho_1(x_1, z_1)^{d_1+\epsilon_1-\epsilon_1'}} \frac{2\tilde{k_{2,0}}^{-\epsilon_2}}{\rho_2(x_2, z_2)^{d_2+\epsilon_2-\epsilon_2'}}$$

where we choose $\epsilon_1' \in (0, \epsilon_1)$ and $\epsilon_2' \in (0, \epsilon_2)$ such that $p(d_1+\epsilon_1-\epsilon_1') > d_1$ and $p(d_2+\epsilon_2-\epsilon_2') > d_2$. From this and the fact $\rho_1(x_1, z_1) \geq C \rho_1(x_1, \tilde{z}_1)$, it follows that

$$U_{1R2} \leq C \|a_R\|_{L^2(X_1 \times X_2)} \|\mu(R)\|^{p/2} \int_{x \in 100C_{15,1}Q_1} \frac{2\tilde{k}_{1,0}^{-\epsilon_1}}{\rho_1(x_1, z_1)^{d_1+\epsilon_1-\epsilon_1'}} \, d\mu_1(x_1)$$

$$\times \int_{x_2 \notin 100C_{15,2}Q_2} \frac{2\tilde{k_{2,0}}^{-\epsilon_2'}}{\rho_2(x_2, z_2)^{d_2+\epsilon_2-\epsilon_2'}} \, d\mu_2(x_2)$$

$$\leq C \|a_R\|_{L^2(X_1 \times X_2)} \|\mu(R)\|^{p/2} \left( \frac{\mu_1(Q_1)}{\mu_1(Q_1)} \right)^{p/(d_1+\epsilon_1)/d_1-1}$$

Thus, similarly to the estimate for (4.55), the estimate (4.56), the Hölder inequality, and Lemma 4.2 tell us that

$$U_{1R2} \leq C.$$
5 Singular integrals

We first recall some notation. Let $\Omega$ be an open set in $X_1 \times X_2$. As in the proof of Theorem 4.3, we define
\[
\overline{\Omega} = \{(x_1, x_2) \in X_1 \times X_2 : M_s \chi_{\Omega}(x_1, x_2) > 1/2\}
\]
and
\[
\overline{\Omega} = \{(x_1, x_2) \in X_1 \times X_2 : M_s \chi_{\overline{\Omega}}(x_1, x_2) > 1/2\}.
\]
For any $R = Q_1 \times Q_2 \in \mathcal{M}(\Omega)$, we define $\tilde{R} = \tilde{Q}_1 \times \tilde{Q}_2 \in \mathcal{M}_1(\overline{\Omega})$ such that
\[
\mu(\tilde{R} \cap \Omega) > \frac{1}{2} \mu(\tilde{R})
\]
and $\overline{R} = \tilde{Q}_1 \times \tilde{Q}_2 \in \mathcal{M}_2(\overline{\Omega})$ such that
\[
\mu(\overline{R} \cap \Omega) > \frac{1}{2} \mu(\overline{R}).
\]
Let $C_{16,1} \geq 1$ and $C_{16,2} \geq 1$ be two constants which are large enough and we set
\[
\overline{C}R = C_{16,1} \tilde{Q}_1 \times C_{16,2} \tilde{Q}_2,
\]
where $C_{16,i} \tilde{Q}_i$ means the “cube” with the same center as $\tilde{Q}_i$ but with diameter $C_{16,i}$ times the diameter of $\tilde{Q}_i$. We also denote by $\tilde{z}_i$ the center of $\tilde{Q}_i$ for $i = 1$, $2$.

We first have the following general theorem on the boundedness of linear operators from $H^p(X_1 \times X_2)$ to $L^p(X_1 \times X_2)$ with $p \in (p_0, 1]$, when the linear operators are assumed to be bounded on $L^2(X_1 \times X_2)$. This is a generalization of R. Fefferman’s theorem in pure product setting in Euclidean spaces, see Theorem 1 in [F4]. Here $p_0$ is some positive number less than 1.

**Theorem 5.1** Suppose that $T$ is a bounded linear operator on $L^2(X_1 \times X_2)$. Let $\epsilon_i \in (0, \theta_i]$ and
\[
\max \left\{ \frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2} \right\} < p \leq 1.
\]
Suppose further that if $a_R$ is an $H^p(X_1 \times X_2)$ $(p, 2)$-rectangle atom as in Definition 4.2 and $R = Q_1 \times Q_2$. Let $\tilde{Q}_1$ and $\tilde{Q}_2$ be the same as in (5.1) and (5.2). If there exist fixed constant $\delta > 0$ and some fixed large enough constants $C_{16,1} \geq 1$ and $C_{16,2} \geq 1$ such that for all $R = Q_1 \times Q_2$,
\[
\int_{X_2} \int_{(C_{16,1} \tilde{Q}_1)^c} |T(a_R)(x_1, x_2)|^p \, d\mu_1(x_1) \, d\mu_2(x_2) \leq C \left( \frac{\mu_1(Q_1)}{\mu_1(\tilde{Q}_1)} \right)^\delta
\]
and
\[
\int_{(C_{16,2} \tilde{Q}_2)^c} \int_{X_1} |T(a_R)(x_1, x_2)|^p \, d\mu_1(x_1) \, d\mu_2(x_2) \leq C \left( \frac{\mu_2(Q_2)}{\mu_2(\tilde{Q}_2)} \right)^\delta,
\]
then $T$ is a bounded operator from $H^p(X_1 \times X_2)$ to $L^p(X_1 \times X_2)$, where

$$\left( C_{16,i} \tilde{Q}_i \right)^c = X_i \setminus C_{16,i} \tilde{Q}_i, \ i = 1, 2. $$

**Proof.** It suffices to prove that there is a constant $C > 0$ such that for all $(p, 2)$-atoms of $H^p(X_1 \times X_2)$,

$$\|T(a)\|_{L^p(X_1 \times X_2)} \leq C. $$

(5.6)

Use all the notation the same as in Definition 4.2, in particular, suppose $\text{supp } a \subset \Omega$ and $\ a = \sum_{R \in \mathcal{M}(\Omega)} a_R.$

For $R = Q_1 \times Q_2$, let $\tilde{Q}_1, \tilde{Q}_2$ and $\mathcal{C} R$ be the same as in (5.1), (5.2) and (5.3). By the Hölder inequality and $L^2(X_1 \times X_2)$-boundedness of $T$, we can estimate

(5.7) \[ \int_{\bigcup_{R' \in \mathcal{M}(\Omega)}} |T(a)(x_1, x_2)|^p \, d\mu_1(x_1) \, d\mu_2(x_2) \]

\[ \leq \mu \left( \bigcup_{R' \in \mathcal{M}(\Omega)} \mathcal{C} R' \right)^{1-p/2} \left\{ \int_{X_1 \times X_2} |T(a)(x_1, x_2)|^p \, d\mu_1(x_1) \, d\mu_2(x_2) \right\}^{p/2} \]

\[ \leq C \mu(\Omega)^{1-p/2} \|a\|_{L^2(X_1 \times X_2)}^p \]

\[ \leq C \mu(\Omega)^{1-p/2} \mu(\Omega)^{(1/2-1/p)p} \]

\[ \leq C, \]

which is a desired estimate.

We now write

\[ \int_{\bigcup_{R' \in \mathcal{M}(\Omega)}} |T(a)(x_1, x_2)|^p \, d\mu_1(x_1) \, d\mu_2(x_2) \]

\[ \leq \sum_{R \in \mathcal{M}(\Omega)} \int_{\bigcup_{R' \in \mathcal{M}(\Omega)}} \mathcal{C} R' \] $\mathcal{C} R' \]$^p \%

\[ \leq \sum_{R \in \mathcal{M}(\Omega)} \int_{C_{16,2} \tilde{Q}_2} \int_{X_1} |T(a_R)(x_1, x_2)|^p \, d\mu_1(x_1) \, d\mu_2(x_2) \]

\[ + \sum_{R \in \mathcal{M}(\Omega)} \int_{C_{16,1} \tilde{Q}_1} \int_{C_{16,2} \tilde{Q}_2} |T(a_R)(x_1, x_2)|^p \, d\mu_1(x_1) \, d\mu_2(x_2) \]

\[ = J_1 + J_2. \]
Note that \(a_R \mu(R)^{1/2 - 1/p} \|a_R\|_{L^2(X_1 \times X_2)}^{-1}\) is an \(H^p(X_1 \times X_2)\) \((p, 2)\)-rectangle atom. The assumption (5.1), the Hölder inequality and Lemma 4.2 tell us that

\[
(5.8) \quad J_1 \leq C \sum_{R \in \mathcal{M}(\Omega)} \|a_R\|_{L^2(X_1 \times X_2)}^p \mu(R)^{(1/p - 1/2)p} \left( \frac{\mu_1(Q_1)}{\mu_1(Q_1)} \right) \delta
\]

\[
\leq C \sum_{R \in \mathcal{M}(\Omega)} \|a_R\|_{L^2(X_1 \times X_2)}^2 \left[ \sum_{R \in \mathcal{M}(\Omega)} \mu(R) \left( \frac{\mu_1(Q_1)}{\mu_1(Q_1)} \right)^{2\delta/(2-p)} \right]^{1-p/2}
\]

\[
\leq C \mu(\Omega)^{(1/2 - 1/p)p} \left[ \sum_{R \in \mathcal{M}(\Omega)} \mu(R) \left( \frac{\mu_1(Q_1)}{\mu_1(Q_1)} \right)^{2\delta/(2-p)} \right]^{1-p/2}
\]

\[
\leq C \mu(\Omega)^{p/2 - 1} \mu(\Omega)^{1-p/2}
\]

which is a desired estimate.

Finally, we note that if \(R' = Q_1' \times Q_2' \in \mathcal{M}(\Omega)\) and \(R = Q_1 \times Q_2 \in \mathcal{M}(\Omega)\) such that \(\widetilde{R}' = \widetilde{R} \in \mathcal{M}_1(\tilde{\Omega})\), then \(R' = R\) or \(R' \cap R = \emptyset\). From this fact, the assumption (5.2), the Hölder inequality and Lemma 4.2, it follows that

\[
(5.9) \quad J_2 \leq C \sum_{R \in \mathcal{M}(\Omega)} \|a_R\|_{L^2(X_1 \times X_2)}^p \mu(R)^{(1/p - 1/2)p} \left( \frac{\mu_2(Q_2)}{\mu_2(Q_2)} \right) \delta
\]

\[
\leq C \sum_{R \in \mathcal{M}(\Omega)} \|a_R\|_{L^2(X_1 \times X_2)}^2 \left[ \sum_{R \in \mathcal{M}(\Omega)} \mu(R) \left( \frac{\mu_2(Q_2)}{\mu_2(Q_2)} \right)^{2\delta/(2-p)} \right]^{1-p/2}
\]

\[
\leq C \mu(\Omega)^{p/2 - 1} \left\{ \sum_{S \in \mathcal{M}_1(\tilde{\Omega})} \left[ \sum_{R \in \mathcal{M}(\Omega)} \mu(R) \left( \frac{\mu_2(Q_2)}{\mu_2(Q_2)} \right)^{2\delta/(2-p)} \right]^{1-p/2} \right\}
\]

\[
\leq C \mu(\Omega)^{p/2 - 1} \mu(\tilde{\Omega})^{1-p/2}
\]

\[
\leq C \mu(\Omega)^{p/2 - 1} \mu(\Omega)^{1-p/2}
\]

\[
\leq C,
\]

which is a desired estimate.

Combining (5.7), (5.8) and (5.9) gives us (5.6) which completes the proof of Theorem 5.1.
Remark 5.1 We mention that the examples where Theorem 5.1 applies, if $X_1$ and $X_2$ are Euclidean spaces, are the double Hilbert transform, product versions of commutators as in [F6], and the class introduced by Fefferman and Stein in [FS]; see also [F4].

We now consider the boundedness on $H^p$ space for a certain range of $p \in (p_0, 1]$ for a class of singular integrals similar to [NS3].

Let $\eta_i \in (0, \theta_i], \ i = 1, 2$. We define $C_0^{\eta_1, \eta_2}(X_1 \times X_2) = C_0^{\eta_1}(X_1) \otimes C_0^{\eta_2}(X_2)$. Also, for $i = 1, 2$, we say $\varphi$ is a bump function on $X_i$ associated to a ball $B(x_i, \delta_i)$, if it is supported in that ball, and satisfies $\|\varphi\|_{L^\infty(X_i)} \leq 1$ and $\|\varphi\|_{C_0^{\eta_1}(X_i)} \leq C \delta_i^\eta$ for all $\eta \in (0, \theta_i]$, where $C \geq 0$ is independent of $\delta_i$ and $x_i$. In what follows, for its convenience, if $f \in L^\infty(X_i)$, we write $f \in C_0(X_i)$ and define

$$\|f\|_{C_0(X_i)} = \|f\|_{L^\infty(X_i)},$$

and for $\eta_i \in (0, \theta_i]$,

$$\|f\|_{C_0^{\eta_i}(X_i)} = \sup_{x_i, y_i \in X_i} \frac{|f(x_i) - f(y_i)|}{\rho_i(x_i, y_i)^{\eta_i}}, \ i = 1, 2.$$

Definition 5.1 Let $\eta_i \in (0, \theta_i], \ i = 1, 2$. A linear operator $T$ initially defined from $C_0^{\eta_1, \eta_2}(X_1 \times X_2) = C_0^{\eta_1}(X_1) \otimes C_0^{\eta_2}(X_2)$ to its dual is called a singular integral if $T$ has an associated distribution kernel $K(x_1, x_2; y_1, y_2)$ which is locally integrable away from the "cross"

$$\{(x_1, x_2; y_1, y_2) : x_1 = y_1, \ or \ x_2 = y_2\}$$

satisfying the following additional properties

(i) 

$$\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2 \rangle$$

$$= \int_{X_1 \times X_2 \times X_1 \times X_2} K(x_1, x_2; y_1, y_2) \varphi_1(y_1) \varphi_2(y_2)$$

$$\times \psi_1(x_1) \psi_2(x_2) \ d\mu_1(y_1) \ d\mu_2(y_2) \ d\mu_1(x_1) \ d\mu_2(x_2)$$

whenever $\varphi_1, \ \psi_1 \in C_0^{\eta_1}(X_1)$ and have disjoint supports, and $\varphi_2, \ \psi_2 \in C_0^{\eta_2}(X_2)$ and have disjoint supports;

(ii) For each bump function $\varphi_2$ on $X_2$ and each $x_2 \in X_2$, there exists a singular integral $T^{\varphi_2; x_2}$ (of the one-factor type) on $X_1$, so that $x_2 \rightarrow T^{\varphi_2; x_2}$ is smooth in the sense make precise below, and so that

$$\langle T(\varphi_1 \otimes \varphi_2), \psi_1 \otimes \psi_2 \rangle = \int_{X_1} \langle T^{\varphi_2, x_2} \varphi_1, \psi_1 \rangle \psi_2(x_2) \ d\mu_2(x_2).$$

Moreover, we require that $T^{\varphi_2, x_2}$ uniformly satisfies the following conditions that $T^{\varphi_2, x_2}$ has a distribution kernel $K^{\varphi_2, x_2}(x_1, y_1)$ having the following properties:
(ii) If $\varphi_1, \psi_1 \in C_0^\infty(X_1)$ have disjoint supports, then
\[
\langle T^{\varphi_2 \times x_2} \varphi_1, \psi_1 \rangle = \int_{X_1 \times X_1} K^{\varphi_2 \times x_2}(x_1, y_1) \varphi_1(x_1) \psi_1(y_1) \, d\mu_1(x_1) \, d\mu_1(y_1);
\]

(ii) If $\varphi_1$ is a bump function associated to the ball $B(\bar{x}_1, r_1)$, then
\[
\| T^{\varphi_2 \times x_2} \varphi_1 \|_{C^\alpha(X_1)} \leq C r_1^{-\alpha_1}
\]
for all $a_1 \in (0, \theta_1]$, where $C \geq 0$ is independent of $\varphi_2$, $x_2$, and $r_1$. Precisely, this means that for each $a_1 \geq 0$, there is a $b_1 \geq 0$ and a constant $C_{a_1, b_1}$, independent of $\varphi_2$, $x_2$ and $r_1$, so that whenever $\varphi \in C_0^\infty(X_1)$ supported in a ball $B(\bar{x}_1, r_1)$, then
\[
r_1^{a_1} \| T^{\varphi_2 \times x_2} \varphi_1 \|_{C^\alpha(X_1)} \leq C_{a_1, b_1} \sup_{a_1 \leq b_1} r_1^{a_1} \| T^{\varphi_2 \times x_2} \varphi_1 \|_{C^\alpha(X_1)};
\]

(ii) There is a constant $C > 0$ independent of $\varphi_2$, $x_2$, and $r_1$ such that

(ii) If $\varphi_2$ is a bump function associated to $B(\bar{x}_2, r_2)$, then for $a_2 \in (0, \theta_2)$,
\[
r_2^{a_2} \rho_2(x_2, u_2)^{-a_2} \| T^{\varphi_2 \times x_2} - T^{\varphi_2 \times u_2} \|
\]
also uniformly satisfies properties (ii) through (ii)3;

(ii) Properties (ii) through (ii)4 also hold with $x_1$ and $y_1$ interchanged. That is, there properties also hold for the adjoint operator $(T^{\varphi_2 \times x_2})^t$ defined by
\[
\langle (T^{\varphi_2 \times x_2})^t \varphi, \psi \rangle = \langle T\psi, \varphi \rangle;
\]

(iii) The property (ii) hold when the index 1 and 2 are interchanged, namely, if the roles of $X_1$ and $X_2$ are interchanged;

(iv) There is a constant $C > 0$ such that for all bump functions $\varphi_1$ and $\varphi_2$, respectively, associated to $B(\bar{x}_1, r_1)$ and $B(\bar{x}_2, r_2)$,
\[
\| [T(\varphi_1 \otimes \varphi_2)(x_1, x_2) - T(\varphi_1 \otimes \varphi_2)(u_1, x_2)]
- [T(\varphi_1 \otimes \varphi_2)(x_1, u_2) - T(\varphi_1 \otimes \varphi_2)(u_1, u_2)]
\| \leq C r_1^{-a_1} r_2^{-a_2} \rho_1(x_1, u_1)^{a_1} \rho_2(x_2, u_2)^{a_2}
\]
for all $a_1 \in (0, \theta_1]$ and all $a_2 \in (0, \theta_2)$;
(v) The kernel $K(x_1, x_2; y_1, y_2)$ satisfies the following conditions:

\[(v)_1 \quad |K(x_1, x_2; y_1, y_2)| \leq C \rho_1(x_1, y_1)^{-d_1} \rho_2(x_2, y_2)^{-d_2},
\]

\[(v)_2 \quad |K(x_1, x_2; y_1, y_2) - K(x_1', x_2'; y_1, y_2)| \leq C \frac{1}{\rho_1(x_1, y_1)^{d_1}} \frac{\rho_2(x_2, x_2')^{\eta_2}}{\rho_2(x_2, y_2)^{d_2 + \eta_2}} \quad \text{for}
\]

\[\rho_2(x_2, x_2') \leq \frac{\rho_2(x_2, y_2)}{2A_2},
\]

\[(v)_3 \quad |K(x_1, x_2; y_1, y_2) - K(x_1, x_2; y_1', y_2')| \leq C \frac{1}{\rho_1(x_1, y_1)^{d_1}} \frac{\rho_2(y_2, y_2')^{\eta_2}}{\rho_2(x_2, y_2)^{d_2 + \eta_2}} \quad \text{for}
\]

\[\rho_2(y_2, y_2') \leq \frac{\rho_2(x_2, y_2)}{2A_2},
\]

\[(v)_4 \quad \left| [K(x_1, x_2; y_1, y_2) - K(x_1', x_2; y_1, y_2)] - [K(x_1, x_2; y_1, y_2') - K(x_1', x_2; y_1, y_2')] \right|
\]

\[\leq C \frac{\rho_1(x_1, x_1')^{\eta_1}}{\rho_1(x_1, y_1)^{d_1 + \eta_1}} \frac{\rho_2(x_2, x_2')^{\eta_2}}{\rho_2(x_2, y_2)^{d_2 + \eta_2}} \quad \text{for} \quad \rho_1(x_1, x_1') \leq \frac{\rho_1(x_1, y_1)}{2A_1} \quad \text{and} \quad \rho_2(x_2, x_2') \leq \frac{\rho_2(x_2, y_2)}{2A_2},
\]

\[(v)_5 \quad \left| [K(x_1, x_2; y_1, y_2) - K(x_1', x_2; y_1, y_2)] - [K(x_1, x_2; y_1, y_2') - K(x_1', x_2; y_1, y_2')] \right|
\]

\[\leq C \frac{\rho_1(x_1, y_1')^{\eta_1}}{\rho_1(x_1, y_1)^{d_1 + \eta_1}} \frac{\rho_2(y_2, y_2')^{\eta_2}}{\rho_2(x_2, y_2)^{d_2 + \eta_2}} \quad \text{for} \quad \rho_1(x_1, y_1') \leq \frac{\rho_1(x_1, y_1)}{2A_1} \quad \text{and} \quad \rho_2(y_2, y_2') \leq \frac{\rho_2(x_2, y_2)}{2A_2},
\]

\[(v)_6 \quad \left| [K(x_1, x_2; y_1, y_2) - K(x_1, x_2'; y_1, y_2')] - [K(x_1, x_2; y_1, y_2') - K(x_1, x_2'; y_1, y_2')] \right|
\]

\[\leq C \frac{\rho_1(y_1, y_1')^{\eta_1}}{\rho_1(x_1, y_1)^{d_1 + \eta_1}} \frac{\rho_2(y_2, y_2')^{\eta_2}}{\rho_2(x_2, y_2)^{d_2 + \eta_2}} \quad \text{for} \quad \rho_1(y_1, y_1') \leq \frac{\rho_1(x_1, y_1)}{2A_1} \quad \text{and} \quad \rho_2(y_2, y_2') \leq \frac{\rho_2(x_2, y_2)}{2A_2},
\]
(v) The properties (iii) to (iii) hold when the index 1 and 2 are interchanged, that is, if the roles of $X_1$ and $X_2$ are interchanged.

(vi) The same properties are assumed to hold for the 3 “transposes” of $T$, i.e. those operators which arise by interchanging $x_1$ and $y_1$, or interchanging $x_2$ and $y_2$, or doing both interchanges.

We can now establish the $H^p$-boundedness of these singular operators as defined in Definition 5.1 as follows.

**Theorem 5.2** Let $0 < \epsilon_i, \eta_i \leq \theta_i, i = 1, 2$, and

$$\max \left\{ \frac{d_1}{d_1 + \epsilon_1}, \frac{d_2}{d_2 + \epsilon_2}, \frac{d_1}{d_1 + \eta_1}, \frac{d_2}{d_2 + \eta_2} \right\} < p < \infty.$$ 

Each product singular integral as in Definition 5.1 extends to a bounded operator on $H^p(X_1 \times X_2)$ to itself.

**Proof.** Let all the notation be the same as in Theorem 3.1 and Theorem 2.2. For $f \in H^p(X_1 \times X_2)$, by Theorem 2.7, for $k_1, k_2 \in \mathbb{Z}$, we have

$$D_{k_1}D_{k_2}Tf = \sum_{k_1' \in I_{k_1}} \sum_{k_2' \in I_{k_2}} \mu_1(Q_{k_1'}(Q_{k_2'})) \mu_2(Q_{k_1'}(Q_{k_2'}))$$

$$\times D_{k_1}D_{k_2}T_D_{k_1'}D_{k_2'}(x_1, y_1, y_2).$$

We now prove that there is constants $C > 0$, $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $k_1, k_2, k_1', k_2' \in \mathbb{Z}$ and all $x_i, y_i \in X_i, i = 1, 2$,

$$\left| D_{k_1}D_{k_2}T_D_{k_1'}D_{k_2'}(x_1, x_2; y_1, y_2) \right| \leq C 2^{-|k_1-k_1'|\delta_1} 2^{-|k_2-k_2'|\delta_2} \frac{2^{-(k_1 \wedge k_1')} \phi_1(x_1, y_1)}{(2^{-(k_1 \wedge k_1')} + \phi_1(x_1, y_1))^{d_1 + \eta_1}} \times \frac{2^{-(k_2 \wedge k_2')} \phi_2(x_2, y_2)}{(2^{-(k_2 \wedge k_2')} + \phi_2(x_2, y_2))^{d_2 + \eta_2}}.$$ 

To verify (5.11), we need to consider several cases. We only prove the case that $k_1 \geq k_1'$ and $k_2 \geq k_2'$ and leave the other cases to the reader. Under this assumption, we again need to consider several cases. Let $l_1, l_2 \in \mathbb{N}$ be large enough which will be decided later.

**Case 1.** $\rho_1(x_1, y_1) \geq 2^{l_1 - k_1'}$ and $\rho_2(x_2, y_2) \geq 2^{l_2 - k_2'}$. In this case, by

$$\int_{X_i} D_{k_i}(x_i, u_i) d\mu_i(u_i) = 0, i = 1, 2,$$
we can write

\[
D_{k_1} D_{k_2} T D_{k'_1} D_{k'_2}(x_1, x_2; y_1, y_2)
\]

\[
= \int_{X_1 \times X_2} \int_{X_1 \times X_2} D_{k_1}(x_1, u_1) D_{k_2}(x_2, u_2) K(u_1, u_2; z_1, z_2)
\]

\[
\times D_{k'_1}(z_1, y_1) D_{k'_2}(z_2, y_2) \, d\mu_1(u_1) \, d\mu_2(u_2) \, d\mu_1(z_1) \, d\mu_2(z_2)
\]

\[
= \int_{X_1 \times X_2} \int_{X_1 \times X_2} D_{k_1}(x_1, u_1) D_{k_2}(x_2, u_2) \{[K(u_1, u_2; z_1, z_2) - K(x_1, u_2; z_1, z_2)]
\]

\[
- [K(u_1, x_2; z_1, z_2) - K(x_1, x_2; z_1, z_2)]\}
\]

\[
\times D_{k'_1}(z_1, y_1) D_{k'_2}(z_2, y_2) \, d\mu_1(u_1) \, d\mu_2(u_2) \, d\mu_1(z_1) \, d\mu_2(z_2).
\]

We choose \(l_1, l_2 \in \mathbb{N}\) large enough, depending on \(A_1\) and \(A_2\), such that in this case, we have

\[
(5.13) \quad \rho_i(u_i, z_i) \geq C \rho_i(x_i, y_i),
\]

where \(i = 1, 2\). A property similar to \((v)_6\) in Definition 5.1 tells us that

\[
\left| D_{k_1} D_{k_2} T D_{k'_1} D_{k'_2}(x_1, x_2; y_1, y_2) \right|
\]

\[
\leq C \rho_1(x_1, y_1)^{d_1 + \eta_1} \rho_2(x_2, y_2)^{d_2 + \eta_2}
\]

\[
\times \int_{X_1 \times X_2} \int_{X_1 \times X_2} \left| D_{k_1}(x_1, u_1) D_{k_2}(x_2, u_2) D_{k'_1}(z_1, y_1) D_{k'_2}(z_2, y_2) \right|
\]

\[
\times \rho_1(u_1, x_1)^{\eta_1} \rho_2(u_2, x_2)^{\eta_2} \, d\mu_1(u_1) \, d\mu_2(u_2) \, d\mu_1(z_1) \, d\mu_2(z_2)
\]

\[
\leq C \rho_1(x_1, y_1)^{d_1 + \eta_1} \rho_2(x_2, y_2)^{d_2 + \eta_2}
\]

\[
\leq C 2^{-(k_1 - k'_1) \eta_1} 2^{-(k_2 - k'_2) \eta_2} \rho_1(x_1, y_1)^{d_1 + \eta_1} \rho_2(x_2, y_2)^{d_2 + \eta_2},
\]

which is what expect to derive.

**Case 2.** \(\rho_1(x_1, y_1) < 2^{l_1 - k'_1}\) and \(\rho_2(x_2, y_2) < 2^{l_2 - k'_2}\). In this case, by (5.12), we can write

\[
D_{k_1} D_{k_2} T D_{k'_1} D_{k'_2}(x_1, x_2; y_1, y_2)
\]

\[
= \int_{X_1 \times X_2} \int_{X_1 \times X_2} D_{k_1}(x_1, u_1) D_{k_2}(x_2, u_2)
\]

\[
\times \left\{ \left[T D_{k'_1} D_{k'_2}(u_1, u_2; y_1, y_2) - T D_{k'_1} D_{k'_2}(x_1, u_2; y_1, y_2) \right]
\]

\[
- \left[T D_{k'_1} D_{k'_2}(x_1, x_2; y_1, y_2) - T D_{k'_1} D_{k'_2}(x_1, x_2; y_1, y_2) \right] \right\} \, d\mu_1(u_1) \, d\mu_2(u_2).
\]
Noting that $2^{-k_1 d_1} D_{k_1}$ and $2^{-k_2 d_2} D_{k_2}$ are bump functions, respectively, associated to $B(y_1, C2^{-k_1'})$ and $B(y_2, C2^{-k_2'})$ with an absolute constant, by the property (iv) of Definition 5.1, we obtain

\[
\left| D_{k_1} D_{k_2} T D_{k_1'} D_{k_2'} (x_1, x_2; y_1, y_2) \right| \\
\leq C 2^{k_1' (d_1 + n_1)} 2^{k_2' (d_2 + n_2)} \int_{X_1 \times X_2} |D_{k_1} (x_1, u_1) D_{k_2} (x_2, u_2)| \\
\times \rho_1 (u_1, x_1)^{n_1} \rho_2 (u_2, x_2)^{n_2} d\mu_1 (u_1) d\mu_2 (u_2) \\
\leq C 2^{-(k_1 - k_1') n_1} 2^{-(k_2 - k_2') n_2} 2^{k_1' d_1} 2^{k_2' d_2},
\]

which is a desired estimate.

**Case 3.** $\rho_1 (x_1, y_1) < 2^{l_1 - k_1'}$ and $\rho_2 (x_2, y_2) \geq 2^{l_2 - k_2'}$. In this case, by (5.12) and property (ii) of Definition 5.1, we can write

\[
D_{k_1} D_{k_2} T D_{k_1'} D_{k_2'} (x_1, x_2; y_1, y_2) \\
= 2^{k_1' d_1} \int_{X_1 \times X_2} \int_{X_1 \times X_2} D_{k_1} (x_1, u_1) D_{k_2} (x_2, u_2) \\
\times \left\{ \begin{array}{c}
K^{-k_1' d_1} D_{k_1'} u_1 (u_2, z_2) - K^{-k_1' d_1} D_{k_1'} x_1 (u_2, z_2) \\
- K^{-k_1' d_1} D_{k_1'} u_1 (x_2, z_2) + K^{-k_1' d_1} D_{k_1'} x_1 (x_2, z_2) \end{array} \right\} \\
\times D_{k_1'} (z_1, y_1) D_{k_2'} (z_2, y_2) d\mu_1 (u_1) d\mu_2 (u_2) d\mu_1 (z_1) d\mu_2 (z_2).
\]

Choose $l_2 \in \mathbb{N}$ large enough such that (5.13) holds. Then this choice and the property (ii) yield that

\[
\left| D_{k_1} D_{k_2} T D_{k_1'} D_{k_2'} (x_1, x_2; y_1, y_2) \right| \\
\leq C 2^{k_1' (d_1 + n_1)} \int_{X_1 \times X_2} \int_{X_1 \times X_2} |D_{k_1} (x_1, u_1) D_{k_2} (x_2, u_2) D_{k_1'} (z_1, y_1) D_{k_2'} (z_2, y_2)| \\
\times \rho_1 (u_1, x_1)^{n_1} \rho_2 (u_2, x_2)^{n_2} d\mu_1 (u_1) d\mu_2 (u_2) d\mu_1 (z_1) d\mu_2 (z_2) \\
\leq C 2^{-(k_1 - k_1') n_1} 2^{-(k_2 - k_2') n_2} 2^{k_1' d_1} 2^{k_2' d_2} \rho_2 (x_2, y_2)^{d_2 + n_2},
\]

which is also a desired estimate.

**Case 4.** $\rho_1 (x_1, y_1) \geq 2^{l_1 - k_1'}$ and $\rho_2 (x_2, y_2) < 2^{l_2 - k_2'}$. The proof for this case is similar to Case 3. We omit the details.
Using (5.11), Lemma 4.1, Remark 4.1 and the Fefferman-Stein vector-valued inequality and some computation similar to the proof of Theorem 4.1, we can verify

$$\|Tf\|_{H^p(X_1 \times X_2)} \leq C\|f\|_{H^p(X_1 \times X_2)}.$$

This completes the proof of Theorem 5.2.
References


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