

# Hamiltonian circuits in Cayley digraphs

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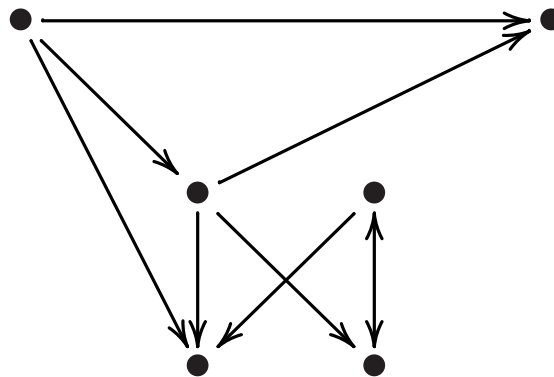
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## Digraphs

**Definition.** A **digraph** is a set  $V$  and a subset  $E$  of  $V \times V$ .

The elements of  $V$  are called **vertices**. We think of vertices as points.

The elements of  $E$  are called **edges**, or **directed edges**. We think of an edge  $(v, w)$  as an arrow from the vertex  $v$  to the vertex  $w$ .



## Cayley digraphs

**Definition.** Let  $G$  be a finite group, and let  $S$  be a subset of  $G$ . The **Cayley digraph**  $\text{Cay}(G; S)$  is the digraph whose vertices consist of the elements of  $G$ , and there is an edge from  $g$  to  $h$  if and only if  $g^{-1}h$  belongs to  $S$ .

There is an edge from  $g$  to  $h$  if and only if there exists an element  $s$  of  $S$  such that  $h = gs$ .

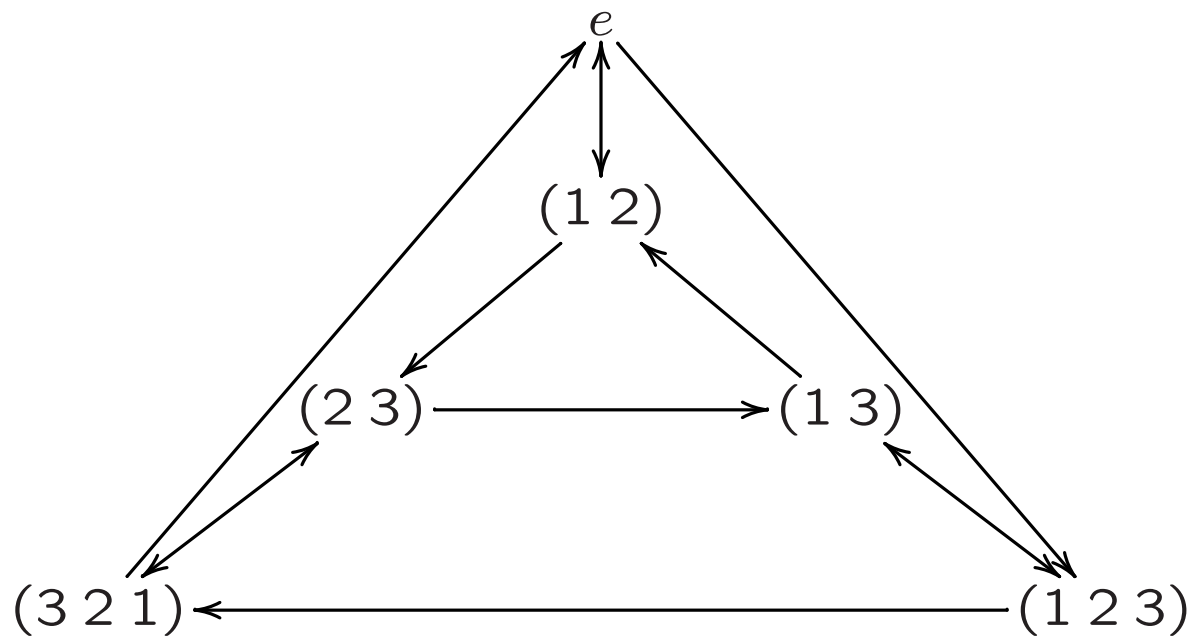
In other words, you can get from one vertex to another by right multiplication by elements of  $S$ .

**Arthur Cayley, 1821–1895**



## An example of a Cayley digraph

Let  $G = S_3$  be the symmetric group of order 6, and let  $S$  be the set  $\{(1\ 2), (1\ 2\ 3)\}$ .



## Product of two cyclic groups

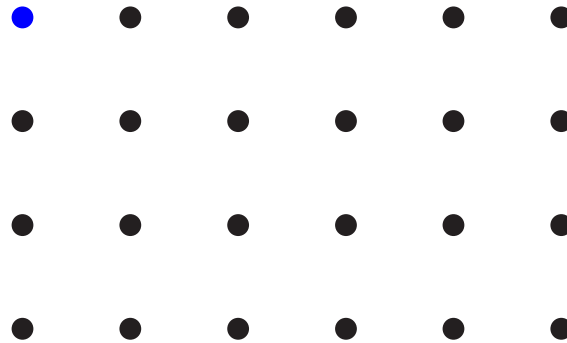
Let  $G = \mathbb{Z}_a \times \mathbb{Z}_b$ .

Elements of  $G$  are ordered pairs  $(x, y)$ , where  $x$  belongs to  $\mathbb{Z}_a$  and  $y$  belongs to  $\mathbb{Z}_b$ .

$x$  is an integer modulo  $a$ , and  $y$  is an integer modulo  $b$ .

(Strictly speaking, we should write  $\bar{x}$  and  $\bar{y}$ .)

## Product of two cyclic groups

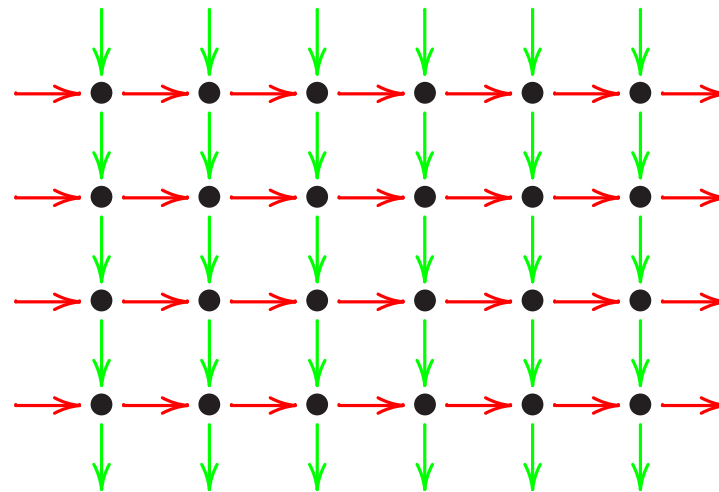


Place the identity element  $(0,0)$  in the upper left corner.

$x$ -coordinates increase to the right, and  $y$ -coordinates increase downward.

## Torus grids

Now add “horizontal edges”  $(x, y) \rightarrow (x + 1, y)$  and “vertical edges”  $(x, y) \rightarrow (x, y + 1)$ .



We have constructed the Cayley digraph of  $G = \mathbb{Z}_a \times \mathbb{Z}_b$  with respect to  $S = \{(1, 0), (0, 1)\}$ .

## Hamiltonian circuits

**Definition.** A **Hamiltonian circuit** in a digraph is an ordered sequence of edges

$$v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_n \rightarrow v_0$$

such that each vertex appears exactly once.

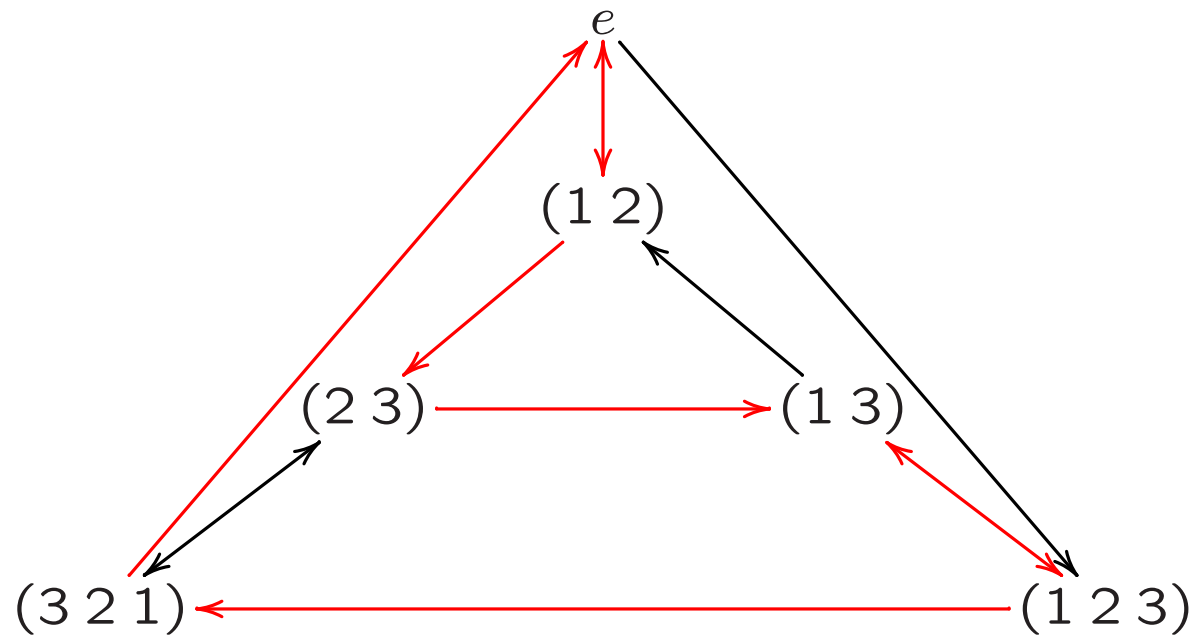
A Hamiltonian circuit is just a closed loop that visits each vertex exactly once.

It's a famously difficult question to determine whether an arbitrary digraph has a Hamiltonian circuit.

**Sir William Rowan Hamilton, 1805–1865**



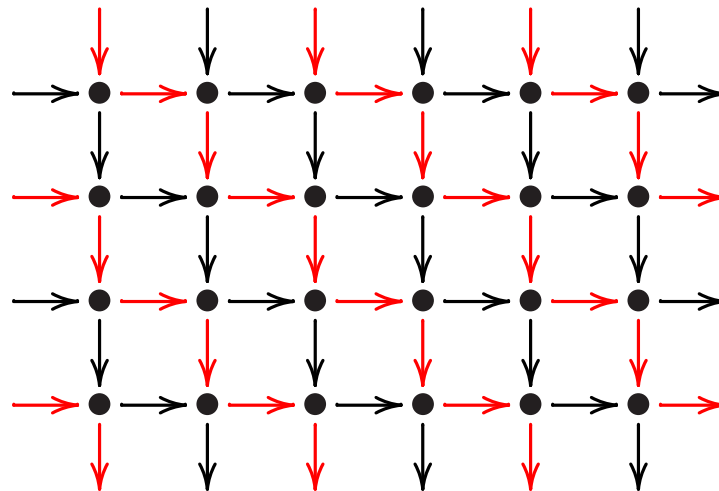
## Hamiltonian circuits of Cayley digraphs



This digraph has a Hamiltonian circuit.

## Hamiltonian circuits on torus grids

How about  $\mathbb{Z}_4 \times \mathbb{Z}_6$ ?

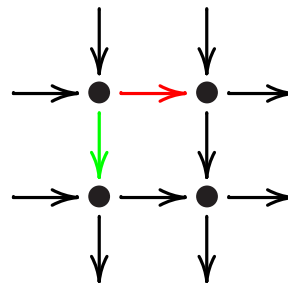


This digraph happens to have a Hamiltonian circuit.

## Edge forcing

**Definition.** A Hamiltonian circuit on a torus grid **travels horizontally** at a vertex  $(x, y)$  if the edge  $(x, y) \rightarrow (x+1, y)$  belongs to the circuit. It **travels vertically** at  $(x, y)$  if the edge  $(x, y) \rightarrow (x, y+1)$  belongs to the circuit.

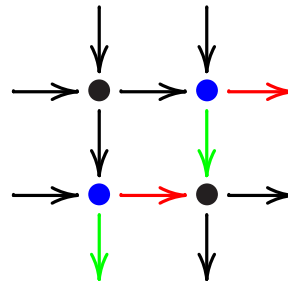
At any vertex  $(x, y)$ , a Hamiltonian circuit must travel **horizontally** or **vertically**.



## Edge forcing

**Lemma.** A Hamiltonian circuit travels horizontally at the vertex  $(x, y)$  if and only if it travels horizontally at the vertex  $(x+1, y-1)$ .

**Proof.** What happens at the vertex  $(x+1, y)$ ?



**Note.** A Hamiltonian circuit travels vertically at  $(x, y)$  if and only if it travels vertically at  $(x+1, y-1)$ .

## Cosets

**Definition.** A **coset** is a non-empty subset  $S$  of  $\mathbb{Z}_a \times \mathbb{Z}_b$  such that  $(x, y)$  belongs to  $S$  if and only if  $(x + 1, y - 1)$  belongs to  $S$ .

Strictly speaking, these are cosets of the subgroup  $H$  generated by  $(1, -1)$ .

A little elementary number theory shows that the size of any coset is  $\text{lcm}(a, b)$ , and the number of cosets is  $\frac{ab}{\text{lcm}(a, b)}$ .

**Corollary.** Constructing a Hamiltonian circuit amounts to choosing a direction of travel for each coset.

## Torus knots

**Definition.** A **torus knot** is a closed loop embedded in a torus.

**Definition.** A torus knot has **winding number**  $(m, n)$  if it winds  $m$  times “horizontally” and  $n$  times “vertically” around the torus.

**Theorem.** Positive integers  $m$  and  $n$  are relatively prime if and only if  $(m, n)$  is the winding number of some torus knot.

## Hamiltonicity of torus grids

**Theorem.** [Curran]  $\mathbb{Z}_a \times \mathbb{Z}_b$  has a Hamiltonian circuit if and only if there exist relatively prime positive integers  $m$  and  $n$  such that  $ma + nb = ab$ .

**Proof.** Suppose that a Hamiltonian circuit  $C$  exists.

It has  $ab$  edges.

Let  $(m, n)$  be the winding number of  $C$ , so  $m$  and  $n$  are relatively prime.

There must be  $ma$  horizontal edges in  $C$ , and there must be  $nb$  vertical edges in  $C$ .

Thus  $ab = ma + nb$ .

## Hamiltonicity of torus grids

Now suppose that there exist relatively prime  $m$  and  $n$  such that  $ma + nb = ab$ .

Since  $ma = b(n - a)$ , both  $a$  and  $b$  divide  $ma$ . Similarly, both  $a$  and  $b$  divide  $nb$ .

Construct  $C$  by choosing  $\frac{ma}{\text{lcm}(a,b)}$  cosets to travel horizontally, and by choosing  $\frac{nb}{\text{lcm}(a,b)}$  cosets to travel vertically.

## Hamiltonicity of torus grids

This works because

$$\frac{ma}{\text{lcm}(a, b)} + \frac{nb}{\text{lcm}(a, b)} = \frac{ab}{\text{lcm}(a, b)}$$

is the number of cosets in  $\mathbb{Z}_a \times \mathbb{Z}_b$ .

You have to worry about whether  $C$  is connected. But it is connected because it has “total” winding number  $(m, n)$ , and  $m$  and  $n$  are relatively prime.

**Exercise.** Count the distinct Hamiltonian circuits on  $\mathbb{Z}_a \times \mathbb{Z}_b$ .

## Example

Consider  $\mathbb{Z}_6 \times \mathbb{Z}_4$ , so  $a = 6$  and  $b = 4$ .

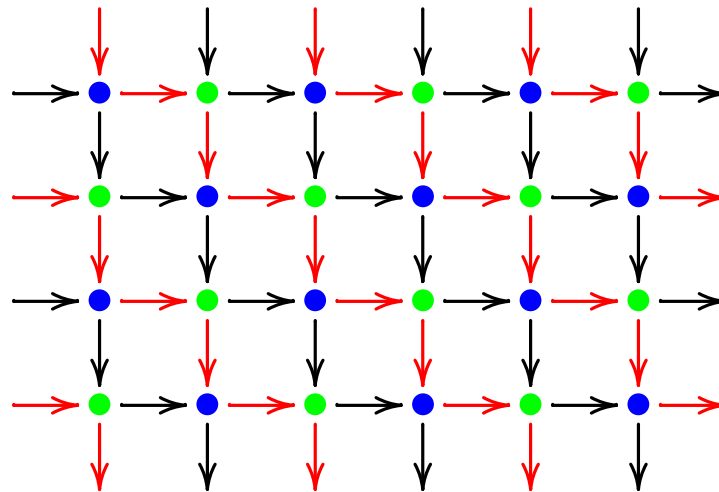
If  $m = 2$  and  $n = 3$ , then  $ma + nb = ab$ .

The size of each coset is  $12 = \text{lcm}(a, b)$ , and the number of cosets is  $2 = \frac{24}{12} = \frac{ab}{\text{lcm}(a, b)}$ .

$$\frac{ma}{\text{lcm}(a, b)} = \frac{12}{12} = 1, \text{ and } \frac{nb}{\text{lcm}(a, b)} = \frac{12}{12} = 1.$$

## Example

One coset should travel **horizontally**, and one coset should travel **vertically**.



## Counterexample

Consider  $\mathbb{Z}_6 \times \mathbb{Z}_3$ , so  $a = 6$  and  $b = 3$ .

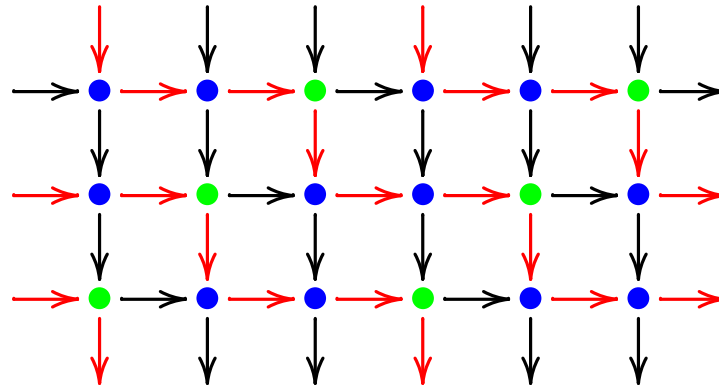
If  $m = 2$  and  $n = 2$ , then  $ma + nb = ab$ .

The size of each coset is  $6 = \text{lcm}(a, b)$ , and the number of cosets is  $3 = \frac{18}{6} = \frac{ab}{\text{lcm}(a, b)}$ .

$$\frac{ma}{\text{lcm}(a, b)} = \frac{12}{6} = 2, \text{ and } \frac{nb}{\text{lcm}(a, b)} = \frac{6}{6} = 1.$$

## Counterexample

Two cosets should travel **horizontally**, and one coset should travel **vertically**.



But  $m$  and  $n$  are not relatively prime, and the circuit is not connected.

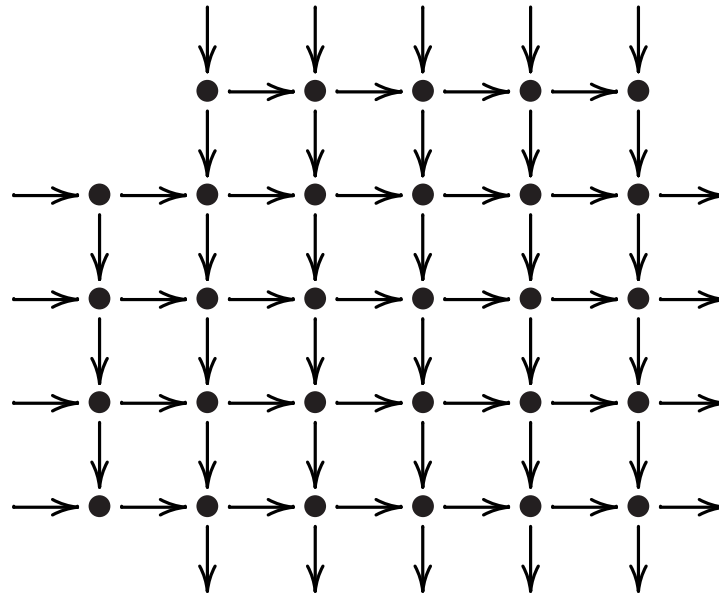
## Competing theorems

**Theorem.** [Curran]  $\mathbb{Z}_a \times \mathbb{Z}_b$  has a Hamiltonian circuit if and only if there exist relatively prime positive integers  $m$  and  $n$  such that  $ma + nb = ab$ .

**Theorem.** [Trotter-Erdős]  $\mathbb{Z}_a \times \mathbb{Z}_b$  has a Hamiltonian circuit if and only if there exist positive integers  $m$  and  $n$  such that  $m + n = \gcd(a, b)$  and  $\gcd(a, m) = 1 = \gcd(b, n)$ .

## Hypohamiltonicity

Now let's delete the vertex  $(0, 0)$  from  $\mathbb{Z}_a \times \mathbb{Z}_b$ .

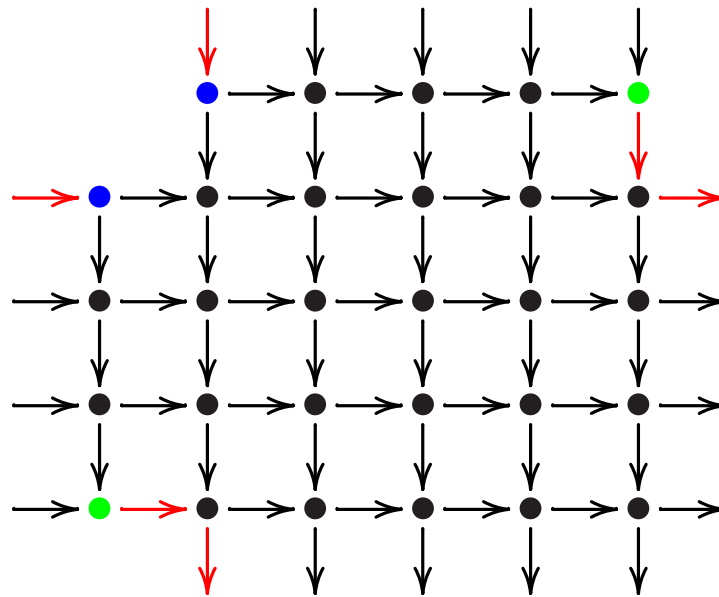


Also delete any hanging edges.

## Hypohamiltonicity

Two **vertices** have only one in-edge, and two **vertices** have only one out-edge.

This forces certain **edges** to be in the Hamiltonian circuit.





## Hypohamiltonicity

**Theorem.** [Penn-Witte]  $\mathbb{Z}_a \times \mathbb{Z}_b - \{(0,0)\}$  has a Hamiltonian circuit if and only if there exist relatively prime positive integers  $m$  and  $n$  such that  $ma + nb = ab - 1$ .

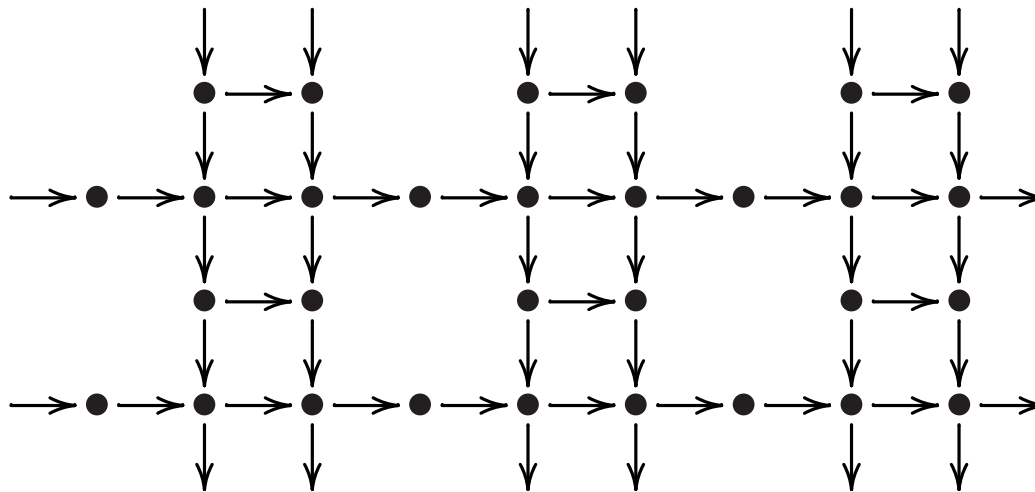
**Example.** Let  $a = 6$  and  $b = 5$ . If  $m = 4$  and  $n = 1$ , then  $ma + nb = ab - 1$ .

**Note.** If  $\mathbb{Z}_a \times \mathbb{Z}_b - \{(0,0)\}$  has a Hamiltonian circuit, then it is unique.

## Deleting subgroups

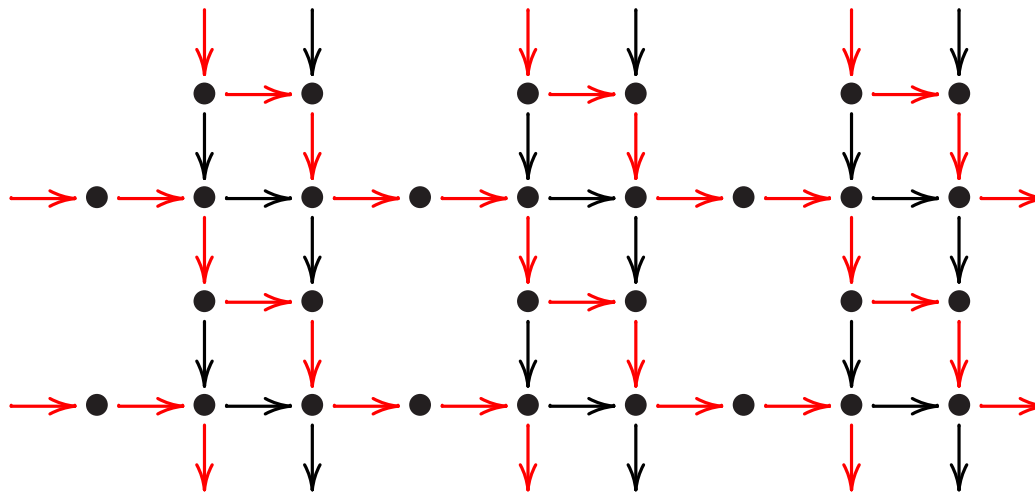
Now let's generalize further to  $\mathbb{Z}_a \times \mathbb{Z}_b - \mathbb{Z}_c \times \mathbb{Z}_d$ , where  $c$  divides  $a$  and  $d$  divides  $b$ .

**Example.**  $\mathbb{Z}_9 \times \mathbb{Z}_4 - \mathbb{Z}_3 \times \mathbb{Z}_2$ .



## Deleting subgroups

Lots of forcing.



## Deleting subgroups

**Theorem.** [Barone-Mauntel-Miller]  $\mathbb{Z}_a \times \mathbb{Z}_b - \mathbb{Z}_c \times \mathbb{Z}_d$  has a Hamiltonian circuit if and only if there exist positive integers  $m$  and  $n$  such that:

1.  $mA + nB = AB - 1$ , where  $A = \frac{a}{c}$  and  $B = \frac{b}{d}$ .

2.  $\gcd(dm, cn) = 1$ .

## Example

Consider  $\mathbb{Z}_9 \times \mathbb{Z}_4 - \mathbb{Z}_3 \times \mathbb{Z}_2$ , so  $a = 9$ ,  $b = 4$ ,  $c = 3$ , and  $d = 2$ .  
Then  $A = 3$  and  $B = 2$ .

If  $m = 1$  and  $n = 1$ , then  $mA + nB = AB - 1$ , and  $\gcd(dm, cn) = 1$ .  
This digraph has a Hamiltonian circuit.

## Further variations and generalizations

1. Hyperhamiltonicity [Gallian-Witte] [Jungreis] [Miller].
2. Delete an arbitrary subgroup [Wu].
3. Closed circuits of various lengths [Wilkinson].