A Cohomological Viewpoint on Elementary School Arithmetic

Daniel C. Isaksen

1. INTRODUCTION. From finite group theory to algebraic geometry to complex analysis, cohomological methods play a major role in modern mathematics. The subject has a long history throughout much of the twentieth century [**6**] and strongly influenced the development of modern mathematics. Mathematicians view such techniques as powerful but sophisticated tools applicable to a remarkably wide field of study, but they usually react with surprise to learn that the ubiquity of cohomology in mathematics extends even to arithmetic at the primary school level.

This article describes a cohomological viewpoint on the traditional method of manual addition of two multi-digit numbers. We explore extensions of groups and show how carrying is a particular example. Then we relate extensions to cohomological ideas. This leads to an addition rule for extensions.

The power of cohomology enlightens the study of extensions in ways that are slightly beyond the scope of this article. For example, the Schur-Zassenhaus Theorem [**15**, Th. IV.10.5] uses cohomology to describe groups built from two finite groups of relatively prime order. Also, see [**12**, Th. IV.4.1] for an explicit description of all p-groups with a cyclic subgroup of order p.

Group cohomology is a vast field of mathematics, and only the most elementary parts of the subject are presented here. Published accounts of this material occur mostly in sophisticated graduate level texts or in research papers; see [**1**], [**4**], [**5**], and [**9**]–[**16**], for example. The philosophy of this article differs from the philosophy of these accounts. The goal is to explain the mathematical phenomena in the most naive way possible, rather than to present the underlying ideas in a powerful or elegant way. The reader needs only a familiarity with the basic notions of finite group theory, such as homomorphisms and quotient groups.

Carrying arises as a serious mathematical issue in the work of Rota et al.; see [**2**], [**7**], and [**8**]. The authors construct the real numbers with the formal arithmetic of infinite binary expansions rather than with the usual topological construction. Floating-point arithmetic in computers is the motivation for this approach to the real number system. This idea is related to our discussion but not central to it.

2. CARRYING IS A COCYCLE. Our first task is to create a formal group theoretic framework for our study of arithmetic. Let us consider addition in the finite abelian group $\mathbb{Z}_{100}$. This serves as our model for addition of two multi-digit numbers. Of course, schoolchildren can add numbers larger than 99. However, most of the complexity of manual addition occurs already with two-digit numbers.

We warn the reader that we abuse the symbol $+$. It represents the addition operation in many different abelian groups. The sense of the symbol follows from the kind of elements being added.

Note that $\mathbb{Z}_{10}$ is a subgroup of $\mathbb{Z}_{100}$ consisting of the multiples of 10. We call this subgroup $\mathcal{T}$ for "tens." Also, $\mathbb{Z}_{10}$ is a quotient group $\mathbb{Z}_{100}/\mathcal{T}$ of $\mathbb{Z}_{100}$. We call this quotient group $\mathcal{O}$ for "ones." For any x in $\mathbb{Z}_{100}$, the coset $x + \mathcal{T}$ determines and is completely determined by the ones digit of x.

TABLE 1. Carrying is required exactly when the ones digits sum to 10 or more.

z	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	1
2	0	0	0	0	0	0	0	0	1	1
3	0	0	0	0	0	0	0	1	1	1
4	0	0	0	0	0	0	1	1	1	1
5	0	0	0	0	0	1	1	1	1	1
6	0	0	0	0	1	1	1	1	1	1
7	0	0	0	1	1	1	1	1	1	1
8	0	0	1	1	1	1	1	1	1	1
9	0	1	1	1	1	1	1	1	1	1

We introduce the following notation. An arbitrary element x of $\mathbb{Z}_{100}$ is represented by a symbol $[a][b]$, where a belongs to $\mathcal{T}$ and b belongs to $\mathcal{O}$. We think of a as the tens digit of x and b as the ones digit of x. For example, we write the number 53 as [5][3].

It is important to remember the difference between $\mathcal{T}$ and $\mathcal{O}$. Although they are isomorphic, their relationships to $\mathbb{Z}_{100}$ are very different.

Given the groups $\mathcal{T}$ and $\mathcal{O}$ and their relationship to $\mathbb{Z}_{100}$, we can describe addition in $\mathbb{Z}_{100}$ in terms of the addition operations of $\mathcal{T}$ and $\mathcal{O}$. For any two elements $[a_1][b_1]$ and $[a_2][b_2]$ of $\mathbb{Z}_{100}$, their sum $[a_1][b_1] + [a_2][b_2]$ in $\mathbb{Z}_{100}$ can be written uniquely in the form $[a][b]$, with a and b functions of a_1, a_2, b_1, and b_2, but these functions are not given by entirely obvious expressions.

The element b of $\mathcal{O}$ depends only on the elements b_1 and b_2 of $\mathcal{O}$: the ones digit of a sum depends only on the ones digits of the summands. This is why we work from right to left when adding multi-digit numbers by hand. On the other hand, a depends on all four values a_1, a_2, b_1, and b_2 because adding two multi-digit numbers is not merely adding columns.

The formula for a is $a_1 + a_2 + z(b_1, b_2)$, where z is a function $\mathcal{O} \times \mathcal{O} \to \mathcal{T}$ expressing how the ones digits of the summands affect the tens digit of the sum. The addition symbols in this formula refer to addition in the group $\mathcal{T}$ because the elements a_1, a_2, and $z(b_1, b_2)$ belong to $\mathcal{T}$. Together with the group structures of $\mathcal{T}$ and $\mathcal{O}$, the function z determines completely the group structure of $\mathbb{Z}_{100}$.

The values of z are given in the chart in Table 1.

We can derive a critically important property of the function z using the associativity of addition. Starting with the equation

$$([a_1][b_1] + [a_2][b_2]) + [a_3][b_3] = [a_1][b_1] + ([a_2][b_2] + [a_3][b_3]),$$

consideration of the tens digit of the sum yields the equation

$$\begin{aligned} a_1 + a_2 + a_3 + z(b_1, b_2) + z(b_1 + b_2, b_3) \\ = a_1 + a_2 + a_3 + z(b_2, b_3) + z(b_1, b_2 + b_3). \end{aligned}$$

This implies that z satisfies the functional equation

$$z(b_2, b_3) - z(b_1 + b_2, b_3) + z(b_1, b_2 + b_3) - z(b_1, b_2) = 0. \tag{2.1}$$

Formula 2.1 is known as the *cocycle condition.*

Note that z also satisfies the formula

$$z(b, 0) = 0 = z(0, b) \tag{2.2}$$

for all b in $\mathcal{O}$. Formula 2.2 is known as the *normalization condition*. These formulas motivate the following important definition.

Definition 2.1. A function $z : \mathcal{O} \times \mathcal{O} \to \mathcal{T}$ is a *cocycle* if it satisfies the cocycle condition and the normalization condition.

3. OTHER GROUPS OF ORDER 100. In the previous section, we considered the standard carrying function and its relationship to addition in $\mathbb{Z}_{100}$. It is possible to build other abelian groups from the groups $\mathcal{T}$ and $\mathcal{O}$ and other functions $z : \mathcal{O} \times \mathcal{O} \to \mathcal{T}$. Using the function z described in the previous section, we can construct another function $2z : \mathcal{O} \times \mathcal{O} \to \mathcal{T}$.

Consider the set of symbols of the form $[a][b]$, where a belongs to $\mathcal{T}$ and b belongs to $\mathcal{O}$. We can use z to define a new addition operation on this set by the formula

$$[a_1][b_1] + [a_2][b_2] = [a_1 + a_2 + 2z(b_1, b_2)][b_1 + b_2].$$

Then $[0][0]$ is a zero element for this operation. Since $2z$ is a symmetric function, the operation is commutative. The operation is associative because $2z$ satisfies the cocycle condition of Formula 2.1.

We give some examples of addition formulas in this group:

$$[1][2] + [2][5] = [3][7]$$
$$[3][7] + [2][9] = [7][6]$$
$$[9][5] + [0][6] = [1][1].$$

Addition here is somewhat like ordinary addition, except that when carrying is necessary, we carry a 2 instead of a 1.

This addition operation determines an abelian group E of order 100. From the standard classification of finite abelian groups [**3**, Ch. 11], E must be isomorphic to one of the four groups $\mathbb{Z}_{100}$, $\mathbb{Z}_{50} \oplus \mathbb{Z}_2$, $\mathbb{Z}_{20} \oplus \mathbb{Z}_5$, or $\mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$. These groups can be distinguished by the presence or absence of elements of order 4 and order 25. In the group E, the element $[0][4]$ has order 25, just as it does in the group $\mathbb{Z}_{100}$. However, unlike the group $\mathbb{Z}_{100}$, E has no elements of order 4. It follows that E is isomorphic to $\mathbb{Z}_{50} \oplus \mathbb{Z}_2$.

The reader is invited to determine which groups of order 100 are built from the other multiples of the standard carrying function. (Hint: Two of the groups occur only once, and the other two occur four times. The answer appears in Section 7.)

4. GENERAL COCYCLES. In the previous two sections, we considered the problem of constructing abelian groups of order 100 from the groups $\mathcal{T}$, $\mathcal{O}$, and a function $z : \mathcal{O} \times \mathcal{O} \to \mathcal{T}$, where z is a multiple of the standard carrying function. This method constructs all four abelian groups of order 100. It is natural to ask which other functions $z : \mathcal{O} \times \mathcal{O} \to \mathcal{T}$ can be used to construct groups of order 100.

We saw in the previous section that not every function produces an addition operation. In particular, the function must satisfy the cocycle condition of Formula 2.1 since addition must be associative.

Definition 4.1. An *extension* of the group $\mathcal{O}$ by the group $\mathcal{T}$ is an abelian group E such that $\mathcal{T}$ is a subgroup of E and the quotient group $E/\mathcal{T}$ is $\mathcal{O}$.

We make some remarks in Section 8 about the assumption that E is abelian.

Now consider an arbitrary extension E of $\mathcal{O}$ by $\mathcal{T}$. Assume that the elements of E are symbols of the form $[a][b]$, where a belongs to $\mathcal{T}$ and b belongs to $\mathcal{O}$. In order to understand how elements of E correspond to such symbols, first recall that the group $\mathcal{T}$ is a subgroup of E. Write the elements of this subgroup in the form $[a][0]$, where a belongs to $\mathcal{T}$. In E,

$$[a_1][0] + [a_2][0] = [a_1 + a_2][0],$$

and $[0][0]$ represents the zero element of E.

The extension E comes with a projection $p : E \to \mathcal{O}$. For each nonzero b in $\mathcal{O}$, choose an element x of E such that $p(x) = b$. Write $[0][b]$ for the element x.

Finally, let $[a][b]$ be the element $[a][0] + [0][b]$. The value of p on any symbol $[a][b]$ is just b.

Lemma 4.2. *Every element of E can be written uniquely in the form* $[a][b]$.

Proof. Let x be an element of E. Let $b = p(x)$, and let a be given by the formula

$$[a][0] = x - [0][b].$$

This is possible since $x - [0][b]$ belongs to the kernel of p, and this kernel is the subgroup $\mathcal{T}$. Thus every element of E can be written in the form $[a][b]$. A counting argument shows that this expression is unique. ∎

From now on, we assume that all extensions consist of elements of the form $[a][b]$. As we saw in Section 2, general addition in E need not be given by the formula

$$[a_1][b_1] + [a_2][b_2] = [a_1 + a_2][b_1 + b_2],$$

although this formula is true when E is isomorphic to the product group $\mathcal{T} \times \mathcal{O}$. Nevertheless, $[a_1][b_1] + [a_2][b_2]$ is equal to $[a][b]$ for some a and b. By applying p, we find that b is equal to $b_1 + b_2$. In order to understand completely the addition in E, we must describe the value of a more explicitly.

Definition 4.3. Given an extension E of $\mathcal{O}$ by $\mathcal{T}$, the *associated cocycle* of E is the function $z : \mathcal{O} \times \mathcal{O} \to \mathcal{T}$ given by the formula

$$[0][b_1] + [0][b_2] = [z(b_1, b_2)][b_1 + b_2].$$

Note that z determines the addition operation for E because

$$\begin{aligned}[a_1][b_1] + [a_2][b_2] &= [a_1][0] + [a_2][0] + [0][b_1] + [0][b_2] \\ &= [a_1 + a_2 + z(b_1, b_2)][b_1 + b_2].\end{aligned}$$

The terminology of Definition 4.3 is sensible because associated cocycles obey the cocycle and normalization conditions. The cocycle condition follows from the associativity law in E, and the normalization condition follows from the identity law

$$[a][b] + [0][0] = [a][b] = [0][0] + [a][b].$$

We have shown that every extension E yields a cocycle $z : \mathcal{O} \times \mathcal{O} \to \mathcal{T}$ that completely determines E. We next demonstrate that every cocycle determines a group structure on the set of symbols of the form $[a][b]$.

Proposition 4.4. *Let $z : \mathcal{O} \times \mathcal{O} \to \mathcal{T}$ be a cocycle. Then the set of symbols of the form $[a][b]$ with a in $\mathcal{T}$ and b in $\mathcal{O}$ is an abelian group under the addition operation*

$$[a_1][b_1] + [a_2][b_2] = [a_1 + a_2 + z(b_1, b_2)][b_1 + b_2].$$

Proof. The normalization condition guarantees that addition satisfies the identity law, while associativity follows from the cocycle condition on z. It is not obvious that the addition is commutative. However, direct calculations with combinations of the cocycle condition imply that

$$z(b_1, b_2) = z(b_2, b_1)$$

for all b_1 and b_2 in $\mathcal{O}$. It follows that the addition is commutative. ∎

5. COBOUNDARIES. In the previous section, we saw that every extension E gives a cocycle. Also, every cocycle can be used to build an extension E. We would like to know whether this correspondence is one-to-one. In fact, it is not. We explore this phenomenon in this section.

Definition 5.1. A group homomorphism $\phi : E \to E'$ of extensions is an *isomorphism of extensions* if ϕ restricts to the identity map on $\mathcal{T}$ and the induced map on quotients $\overline{\phi} : \mathcal{O} \to \mathcal{O}$ is the identity map on $\mathcal{O}$.

An isomorphism of extensions must satisfy the following two properties. First, $\phi([a][0]) = [a][0]$ for every a in $\mathcal{T}$. Second, for every $[a][b]$ in E, $\phi([a][b]) = [a_0][b]$ for some a_0 in $\mathcal{T}$.

Proposition 5.2. *Suppose that E and E' are two extensions with associated cocycles z and z' (see Definition 4.3) such that for every b_1 and b_2 in $\mathcal{O}$*

$$z(b_1, b_2) - z'(b_1, b_2) = h(b_2) - h(b_1 + b_2) + h(b_1),$$

where h is some function $\mathcal{O} \to \mathcal{T}$ satisfying $h(0) = 0$. Then E and E' are isomorphic as extensions.

Proof. Define a function $\phi : E \to E'$ by $\phi([a][b]) = [a + h(b)][b]$. From Definition 5.1, it is immediate that ϕ restricts to the identity map on $\mathcal{T}$ and that ϕ induces the identity map on $\mathcal{O}$.

Observe that $\phi([0][0]) = [0][0]$, so ϕ preserves the identity element. Finally, we must check that ϕ preserves addition. From the definition of ϕ, it follows that

$$\begin{aligned} \phi([a_1][b_1] + [a_2][b_2]) &= \phi([a_1 + a_2 + z(b_1, b_2)][b_1 + b_2]) \\ &= [a_1 + a_2 + z(b_1, b_2) + h(b_1 + b_2)][b_1 + b_2]. \end{aligned}$$

On the other hand,

$$\begin{aligned} \phi([a_1][b_1]) + \phi([a_2][b_2]) &= [a_1 + h(b_1)][b_1] + [a_2 + h(b_2)][b_2] \\ &= [a_1 + a_2 + h(b_1) + h(b_2) + z'(b_1, b_2)][b_1 + b_2]. \end{aligned}$$

These two expressions are equal precisely because of the hypothesis of the proposition. ∎

The previous proposition motivates the following definition.

Definition 5.3. Given a function $h : \mathcal{O} \to \mathcal{T}$ such that $h(0) = 0$, the *coboundary of h* is the function $\delta h : \mathcal{O} \times \mathcal{O} \to \mathcal{T}$ given by the formula

$$\delta h(b_1, b_2) = h(b_2) - h(b_1 + b_2) + h(b_1).$$

A *coboundary* is any function $\mathcal{O} \times \mathcal{O} \to \mathcal{T}$ that arises in this way from such a function h.

The terminology of cocycles and coboundaries comes from topology. In the definition of the singular cohomology of a space X, the coboundaries arise as the geometric boundaries of subspaces of X. Cocycles arise as subspaces of X that have no loose edges. For example, a closed loop is a 1-cocycle but a line segment is not.

Proposition 5.2 can be reinterpreted to mean that two cocycles give the same extension if their difference is a coboundary. The converse is the next proposition.

Proposition 5.4. *Suppose that $\phi : E \to E'$ is an isomorphism of extensions of $\mathcal{O}$ by $\mathcal{T}$. Then the difference $z - z'$ of the associated cocycles is a coboundary.*

Proof. Let $h : \mathcal{O} \to \mathcal{T}$ be defined by the formula

$$\phi([0][b]) = [h(b)][b].$$

Then

$$\begin{aligned}\phi([0][b_1] + [0][b_2]) &= \phi([z(b_1, b_2)][b_1 + b_2]) \\ &= \phi([z(b_1, b_2)][0]) + \phi([0][b_1 + b_2]) \\ &= [z(b_1, b_2)][0] + [h(b_1 + b_2)][b_1 + b_2] \\ &= [z(b_1, b_2) + h(b_1 + b_2)][b_1 + b_2].\end{aligned}$$

On the other hand,

$$\begin{aligned}\phi([0][b_1]) + \phi([0][b_2]) &= [h(b_1)][b_1] + [h(b_2)][b_2] \\ &= [h(b_1) + h(b_2) + z'(b_1, b_2)][b_1 + b_2].\end{aligned}$$

By comparing the first terms,

$$z(b_1, b_2) - z'(b_1, b_2) = h(b_1) - h(b_1 + b_2) + h(b_2). \qquad \blacksquare$$

Propositions 5.2 and 5.4 imply immediately the following theorem.

Theorem 5.5. *Two extensions are isomorphic if and only if their associated cocycles differ by a coboundary.*

6. COHOMOLOGY. In this section, we express the conclusion of Theorem 5.5 in terms of cohomology groups.

Definition 6.1. Let $Z(\mathcal{O}; \mathcal{T})$ be the set of cocycles (see Definition 2.1).

The Z stands for *Zykel* (German for *cycle*).

Definition 6.2. Let $B(\mathcal{O}; \mathcal{T})$ be the set of coboundaries (see Definition 5.3).

Lemma 6.3. *The sets $Z(\mathcal{O}; \mathcal{T})$ and $B(\mathcal{O}; \mathcal{T})$ are abelian groups.*

Proof. Let z_1 and z_2 be cocycles. One checks that $z_1 - z_2$ is a cocycle. This shows that $Z(\mathcal{O}; \mathcal{T})$ is an abelian group.

Now consider two coboundaries δh_1 and δh_2. Then $\delta h_1 - \delta h_2 = \delta(h_1 - h_2)$, so $\delta h_1 - \delta h_2$ is a coboundary. This shows that $B(\mathcal{O}; \mathcal{T})$ is an abelian group. ∎

Lemma 6.4. *The group $B(\mathcal{O}; \mathcal{T})$ is a subgroup of $Z(\mathcal{O}; \mathcal{T})$.*

Proof. Consider a function $h : \mathcal{O} \to \mathcal{T}$ and its coboundary δh. The coboundary satisfies the normalization condition because $h(0) = 0$. We must show that δh also satisfies the cocycle condition. This follows from a straightforward calculation involving the formula for δh given in Definition 5.3. Note that the minus signs in the cocycle condition and in the definition of coboundaries play a critical role. ∎

Once we have Lemmas 6.3 and 6.4, it is natural to make the following definition.

Definition 6.5. The *cohomology group $H(\mathcal{O}; \mathcal{T})$* is the quotient group

$$\frac{Z(\mathcal{O}; \mathcal{T})}{B(\mathcal{O}; \mathcal{T})}.$$

Theorem 5.5 can now be restated as follows.

Theorem 6.6. *The group $H(\mathcal{O}; \mathcal{T})$ is isomorphic to the set of isomorphism classes of extensions of $\mathcal{O}$ by $\mathcal{T}$.*

Proof. Definition 4.3 establishes a function from the set of extensions to the group $H(\mathcal{O}; \mathcal{T})$. Proposition 5.4 tells us that the function is well-defined. Proposition 5.2 tells us that the function is injective. Proposition 4.4 tells us that the function is surjective. ∎

Note that $H(\mathcal{O}; \mathcal{T})$ is an abelian group. According to Theorem 6.6, the set of isomorphism classes of extensions of $\mathcal{O}$ by $\mathcal{T}$ is also an abelian group. This means that there is a notion of addition of extensions. We now describe this addition explicitly.

Let E and E' be two extensions with associated cocycles z and z' and projections p and p' to $\mathcal{O}$. We construct another extension with associated cocycle $z + z'$.

First consider the product $P = E \times E'$. This group is not an extension of $\mathcal{O}$ by $\mathcal{T}$. Rather, it is an extension of $\mathcal{O} \times \mathcal{O}$ by $\mathcal{T} \times \mathcal{T}$. Let Q be the subgroup of P consisting of all pairs of the form $([a_1][b], [a_2][b])$, and let R be the subgroup of Q consisting of all pairs of the form $([a][0], [-a][0])$.

Definition 6.7. The *sum extension* of the extensions E and E' is the quotient group Q/R.

In order to justify Definition 6.7, we should prove that $E'' = Q/R$ is an extension of $\mathcal{O}$ by $\mathcal{T}$ with associated cocycle $z + z'$.

Lemma 6.8. *The group E'' is an extension of $\mathcal{O}$ by $\mathcal{T}$.*

Proof. Consider the surjective function $\tilde{p} : Q \to \mathcal{O}$ given by the formula

$$\tilde{p}([a_1][b], [a_2][b]) = b.$$

Since $\tilde{p}$ takes the subgroup R to 0, it induces a well-defined surjective homomorphism $p'' : E'' \to \mathcal{O}$. It suffices to show that the kernel of p'' is isomorphic to $\mathcal{T}$.

Let S be the subgroup of Q consisting of pairs of the form $([a_1][0], [a_2][0])$. Then the kernel of p'' is the group S/R. This group is isomorphic to $\mathcal{T}$. ■

Since E'' is an extension of $\mathcal{O}$ by $\mathcal{T}$, we can take the elements of E'' to be symbols of the form $[a][b]$, where a belongs to $\mathcal{T}$ and b belongs to $\mathcal{O}$. The element of Q/R represented by the pair $([a_1][b], [a_2][b])$ corresponds under this notation to the symbol $[a_1 + a_2][b]$.

Lemma 6.9. *The cocycle $z + z'$ is the associated cocycle of the extension E''.*

Proof. In Q,

$$\begin{aligned}([0][b_1], [0][b_1]) + ([0][b_2], [0][b_2]) &= ([0][b_1] + [0][b_2], [0][b_1] + [0][b_2]) \\ &= ([z(b_1, b_2)][b_1 + b_2], [z'(b_1, b_2)][b_1 + b_2]).\end{aligned}$$

Therefore, $[0][b_1] + [0][b_2] = [(z + z')(b_1, b_2)][b_1 + b_2]$ in E''. This is the defining equation for the associated cocycle of E''. ■

7. THE GROUP OF EXTENSIONS. Very often in application, the extra structure of an abelian group on the set of isomorphism classes of extensions is a tremendous aid in the study of such extensions. This section gives examples of interesting information that cohomological techniques provide about finite group theory.

We first consider two obvious extensions of $\mathcal{O}$ by T. One is, of course, the extension $\mathbb{Z}_{100}$. Another is the *trivial extension* $\mathcal{O} \times \mathcal{T}$; it corresponds to the zero element of the group $H(\mathcal{O}; \mathcal{T})$. The associated cocycle of the trivial extension is the constant function with value zero because

$$[0][b_1] + [0][b_2] = [0][b_1 + b_2].$$

All the extensions of $\mathcal{O}$ by $\mathcal{T}$ are abelian groups of order 100. We noted in Section 3 that all four isomorphism types of abelian groups of order 100 occur as extensions of $\mathbb{Z}_{10}$ by $\mathbb{Z}_{10}$: each of these four groups has a subgroup isomorphic to $\mathbb{Z}_{10}$ whose quotient is also isomorphic to $\mathbb{Z}_{10}$. The reader is invited to check this directly for each of the four groups.

The situation becomes more complicated when we consider isomorphisms of extensions (see Definition 5.1) rather than merely isomorphisms of abelian groups. Some of the abelian groups of order 100 occur as extensions in more than one nonisomorphic way.

It is known that the cohomology group $H(\mathcal{O}; \mathcal{T})$ is isomorphic to $\mathbb{Z}_{10}$ [**15**, Th. IV.7.1]. The extension $\mathbb{Z}_{100}$ serves as a generator of this group. Let z be the "carrying" cocycle corresponding to this extension studied in Section 2. Descriptions of the addition operations determined by various explicit cocycles are accessible by direct computation. The cocycles $2z$, $4z$, $6z$, and $8z$ correspond to mutually nonisomorphic extensions that are isomorphic as abelian groups to $\mathbb{Z}_{50} \oplus \mathbb{Z}_2$. The cocycles z, $3z$, $7z$, and $9z$ correspond to mutually nonisomorphic extensions that are isomorphic as abelian groups to $\mathbb{Z}_{100}$. These four cocycles are the four generators of the cohomology group. The cocycle 0 represents the trivial extension $\mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$, and the cocycle $5z$ represents an extension that is isomorphic as an abelian group to $\mathbb{Z}_{20} \oplus \mathbb{Z}_5$.

One implication of this calculation is that there are precisely four nonisomorphic ways to think of the group $\mathbb{Z}_{100}$ as an extension of $\mathbb{Z}_{10}$ by $\mathbb{Z}_{10}$. The same is true for the group $\mathbb{Z}_{50} \oplus \mathbb{Z}_2$. Up to isomorphism, the groups $\mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$ and $\mathbb{Z}_{20} \oplus \mathbb{Z}_5$ have unique structures as extensions.

8. GENERAL COHOMOLOGY. We have been studying the *second* cohomology group of $\mathcal{O}$ with coefficients in $\mathcal{T}$. For every $n \geq 0$, there is an nth cohomology group $H^n(\mathcal{O}; \mathcal{T})$. These cohomology groups are powerful tools in group theory. The low dimensional groups ($n = 0, 1, 2$) have descriptions in naive group theoretic terms. We have already explained this in detail for the case $n = 2$. The higher cohomology groups can also be described in terms of a more complicated notion of extension; see **[4]** and **[5]**. The cohomology groups have connections to other aspects of group theory and also to fields such as algebraic topology.

We make a few comments about generalizing the ideas of this exposition to groups other than $\mathcal{O}$ and $\mathcal{T}$. We can substitute an arbitrary abelian group A for $\mathcal{T}$ and an arbitrary (possibly nonabelian) group G for $\mathcal{O}$. As in Definition 4.1, we can consider (possibly nonabelian) groups E that are extensions of G by A. However, we must make the additional assumption that A be a normal subgroup of E in order to make sense of the quotient E/A. We write the group operations of E and G multiplicatively since they are possibly nonabelian, but we continue to write the group operation of A additively since it is still abelian.

An extension E of G by A induces an action of the group G on A by the formula

$$g \cdot a = \tilde{g} a \tilde{g}^{-1},$$

where $\tilde{g}$ is any element of E belonging to the coset g. Although there is a choice in the value of $\tilde{g}$, this choice does not affect the value of $\tilde{g} a \tilde{g}^{-1}$ since A is abelian.

Choose an action of G on A. We consider only those extensions that induce this given action. When studying extensions of $\mathcal{O}$ by $\mathcal{T}$ in Section 4, we assumed that $\mathcal{O}$ acted trivially on $\mathcal{T}$. Any extension that induces the trivial action is necessarily an abelian group. We have the following generalization of Theorem 6.6.

Theorem 8.1. *Extensions that induce the given action of G on A are in one-to-one correspondence with elements of $H(G; A) = Z(G; A)/B(G; A)$.*

Here $Z(G; A)$ is the group of normalized functions $z : G \times G \to A$ satisfying the *twisted cocycle condition*

$$g_1 \cdot z(g_2, g_3) - z(g_1 g_2, g_3) + z(g_1, g_2 g_3) - z(g_1, g_2) = 0.$$

When the action of G on A is trivial, this formula reduces to the untwisted cocycle condition of Formula 2.1. Also, $B(G; A)$ is the group of functions $z : G \times G \to A$ that are the coboundaries of functions $G \to A$. The coboundary δh of a function $h : G \to A$ is given by the formula

$$\delta h(g_1, g_2) = g_1 \cdot h(g_2) - h(g_1 g_2) + h(g_1).$$

What happens when A is also nonabelian? Answering this question leads to the study of nonabelian cohomology, as in **[9]** and **[10]**. Unfortunately, the results in this subject are not nearly as nice as the results that we have discussed here.

9. CONCLUSION. One measure of the power of an idea is its ability to enlighten the most fundamental concepts. We have seen that cohomology is linked to fundamental concepts of decimal arithmetic. Mathematicians know well that cohomology is a deeply significant idea because of its importance to research mathematics, but it is also significant for its importance to elementary mathematics.

ACKNOWLEDGEMENT. The author thanks James Dolan and Tim Chow for observing that carrying is a cocycle.

REFERENCES

1. S. Eilenberg and S. Mac Lane, Cohomology theory in abstract groups I, *Ann. of Math.* (2) **48** (1947) 51–78.
2. F. Faltin, N. Metropolis, B. Ross, and G.-C. Rota, The real numbers as a wreath product, *Advances in Math.* **16** (1975) 278–304.
3. J. A. Gallian, *Contemporary Abstract Algebra*, 3rd ed., D. C. Heath, Lexington, MA, 1994.
4. K. W. Gruenberg, A new treatment of group extensions, *Math. Z.* **102** (1967) 340–350.
5. D. F. Holt, An interpretation of the cohomology groups $H^n(G, M)$, *J. Algebra* **60** (1979) 307–320.
6. S. Mac Lane, Origins of the cohomology of groups, *Enseign. Math.* (2) **24** (1978) 1–29.
7. N. Metropolis, G.-C. Rota, and S. Tanny, Significance arithmetic: the carrying algorithm, *J. Combinatorial Theory Ser. A* **14** (1973) 386–421.
8. N. Metropolis and G.-C. Rota, Significance arithmetic—on the algebra of binary strings, in *Studies in Numerical Analysis* (papers in honour of Cornelius Lanczos on the occasion of his 80th birthday), Academic Press, London-New York, 1974, pp. 241–251.
9. J.-P. Serre, *Cohomologie Galoisienne*, Lecture Notes in Mathematics, vol. 5, Springer-Verlag, Berlin-Heidelberg-New York, 1964.
10. J.-P. Serre, *Local Fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979.

General textbooks on group cohomology:

11. A. Babakhanian, *Cohomological Methods in Group Theory*, Pure and Applied Mathematics, vol. 11, M. Dekker, New York, 1972.
12. K. S. Brown, *Cohomology of Groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1982.
13. K. W. Gruenberg, *Cohomological Topics in Group Theory*, Lecture Notes in Mathematics, vol. 143, Springer-Verlag, Berlin-New York, 1970.
14. S. Lang, *Topics in Cohomology of Groups*, Lecture Notes in Mathematics, vol. 1625, Springer-Verlag, Berlin, 1996.
15. S. Mac Lane, *Homology*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
16. U. Stammbach, *Homology in Group Theory*, Lecture Notes in Mathematics, vol. 359, Springer-Verlag, Berlin-New York, 1973.

DANIEL C. ISAKSEN first learned to add at Round Meadow Elementary School in Calabasas, California. After obtaining his B.A. in Mathematics at the University of California at Berkeley, he learned how to think about addition in terms of cohomology while pursuing graduate studies in algebraic topology at the University of Chicago. He wrote this article while he was employed in the Mathematics Faculty of Universität Bielefeld in Germany. Isaksen is a Visiting Assistant Professor and NSF Postdoctoral Research Fellow at the University of Notre Dame.
University of Notre Dame, Notre Dame, IN 46556
isaksen.1@nd.edu

References

[1] **Cohomology Theory in Abstract Groups. I**
Samuel Eilenberg; Saunders MacLane
The Annals of Mathematics, 2nd Ser., Vol. 48, No. 1. (Jan., 1947), pp. 51-78.
Stable URL:
http://links.jstor.org/sici?sici=0003-486X%28194701%292%3A48%3A1%3C51%3ACTIAGI%3E2.0.CO%3B2-3