LARGE ANNIHILATORS IN CAYLEY-DICKSON ALGEBRAS

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Abstract. Cayley-Dickson algebras are non-associative $\mathbb{R}$-algebras that generalize the well-known algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. We study zero-divisors in these algebras. In particular, we show that the annihilator of any element of the 2^n-dimensional Cayley-Dickson algebra has dimension at most $2^n - 4n + 4$. Moreover, every multiple of 4 between 0 and this upper bound occurs as the dimension of some annihilator. Although a complete description of zero-divisors seems to be out of reach, we can describe precisely the elements whose annihilators have dimension $2^n - 4n + 4$.

1. Introduction

The Cayley-Dickson algebras are a sequence $A_0, A_1, \ldots$ of non-associative $\mathbb{R}$-algebras with involution. The first few are familiar: $A_0 = \mathbb{R}$, $A_1 = \mathbb{C}$, $A_2 = \mathbb{H}$ (the quaternions), and $A_3 = \mathbb{O}$ (the octonions). Each algebra A_n is constructed from the previous one A_{n-1} by a doubling procedure; unfortunately, this doubling procedure tends to destroy desirable algebra properties. For example, $\mathbb{R}$ is the only Cayley-Dickson algebra with trivial involution, $\mathbb{R}$ and $\mathbb{C}$ are the only commutative Cayley-Dickson algebras, and $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are the only associative ones.

Only the first four Cayley-Dickson algebras, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, are normed; equivalently, no A_n with $n \geq 4$ is alternative. When $n = 4$ two things happen. First of all, this weakening sequence of algebraic conditions stops: there is not an easily-phrased algebraic condition satisfied by A_n for $n \leq 4$ but not A_5, or A_n for $n \leq 5$ but not A_6, and so forth. Secondly, for all $n \geq 4$, the algebra A_n admits non-trivial zero-divisors. That is, there exist pairs of non-zero elements x and y in A_n such that $xy = 0$.

Historically, these two mysterious facts have conspired to discourage mathematicians from studying the higher Cayley-Dickson algebras. We instead take them as our point of departure: the locus of zero-divisors in A_4 is well-understood and both quite simple and interesting. In our view, the zero-divisors in the higher A_n offer some promise in two distinct ways. First, as n grows the locus of zero-divisors in A_n becomes more complicated, and this complexity serves as an analogue to the weakening sequence of algebraic criteria present in A_0, A_1, A_2, and A_3. Secondly, it is our hope that these loci of zero-divisors will prove to be geometrically interesting in their own right.

Accordingly, the goal of this article is to initiate a systematic study of the zero-divisors in the Cayley-Dickson algebras.

This research was conducted during the period the first author served as a Clay Mathematics Institute Research Fellow.

1.1. **Statement of results.** We now state the main results in more detail. If x belongs to A_n, then the **annihilator** of x is $\operatorname{Ann} x = \{y \in A_n : xy = 0\}$. Because A_n is noncommutative one should really call this the **right annihilator** of x, but it turns out that in Cayley-Dickson algebras the left and right annihilators of an element are always the same (see Corollary 9.3).

A theorem of Moreno [M1] says that the (real) dimension of $\operatorname{Ann} x$ is always a multiple of 4. The reader will find a different proof of this result in our Theorem 9.8. The first goal of this paper is to determine exactly which multiples of 4 can occur:

Theorem 1.2. *If x belongs to A_n, then $\dim(\operatorname{Ann} x) \leq 2^n - 4n + 4$. Moreover, if d is any multiple of 4 such that $0 \leq d \leq 2^n - 4n + 4$, then there exist elements of A_n whose annihilators have dimension d.*

So in A_4 there are annihilators of dimensions 0 and 4 only; in A_5 there are annihilators of dimensions $0, 4, 8, 12$, and 16; and in A_6 there are annihilators of dimensions $0, 4, 8, \ldots, 44$. Notice that the maximal dimension of the annihilators grows very quickly, the codimension in A_n being given by a linear function. When n is large, one has zero-divisors whose annihilator is 'almost' the whole algebra.

Also note that the above result even makes sense when $n \leq 3$, in which case it gives the well-known fact that these algebras have no zero-divisors.

The space of zero-divisors in A_n is closed under scalar multiplication, and so it forms a cone in the real vector space underlying A_n. It is therefore natural to focus on the norm 1 elements and look at the space

$$ZD(A_n) = \{x \in A_n : ||x|| = 1,\ \operatorname{Ann} x \neq 0\}.$$

We would eventually like to understand the topological properties of this space, although at the moment this seems to be a very complicated problem. It is natural, perhaps, to look at the subspaces

$$ZD_k(A_n) = \{x \in A_n : ||x|| = 1,\ \dim(\operatorname{Ann} x) = k\}.$$

These strata are also complicated, and unknown even in the case of A_5.

The dimension—and complexity—of the strata increases as k becomes small. So the 'simplest' stratum is $ZD_{2^n-4n+4}(A_n)$. With some effort we can describe this one completely:

Theorem 1.3. *When $n \geq 4$ the space $ZD_{2^n-4n+4}(A_n)$ is homeomorphic to a disjoint union of 2^{n-4} copies of the Stiefel variety $V_2(\mathbb{R}^7)$, i.e., the space of ordered pairs of orthonormal vectors in $\mathbb{R}^7$.*

Obtaining a description of $ZD_k(A_n)$ for smaller k is an intriguing open problem.

We close this introduction by describing one technique which is used repeatedly in the paper, and which will help explain Theorem 1.3. Every Cayley-Dickson algebra A_n contains a distinguished element i_n, and the 2-dimensional subspace $\langle 1, i_n \rangle$ is a subalgebra isomorphic to $\mathbb{C}$ that we denote by $\mathbb{C}_n$. It turns out that A_n is a vector space over $\mathbb{C}_n$. The special properties of i_n guarantee that multiplication in A_n behaves well with respect to $\mathbb{C}_n$; see Lemmas 5.7 and 5.8 for precise statements. This viewpoint on A_n underlies almost all of our main results.

For every a in A_n that is orthogonal to $\mathbb{C}_n$, we show in Theorem 10.2 that $(a, \pm i_n a)$ is a zero-divisor in A_{n+1}. Moreover, we completely determine its annihilator and find that it has dimension $2^n - 4 + \dim(\operatorname{Ann} a)$. So this gives a method for producing 'large' annihilators inside of A_{n+1}.

The space $ZD_4(A_4)$ is easily analyzed by hand and shown to be $V_2(\mathbb{R}^7)$. Applying the maps $a \mapsto (a, \pm i_4 a)$ gives two disjoint copies of $V_2(\mathbb{R}^7)$ inside of $ZD_{16}(A_5)$, and then applying $a \mapsto (a, \pm i_5 a)$ gives four disjoint copies inside of $ZD_{44}(A_6)$. This explains the spaces arising in Theorem 1.3. The proof of that theorem involves showing that when $n \geq 5$ *every* $(2^n - 4n + 4)$-dimensional annihilator in A_n is of the form $\mathrm{Ann}(a, \pm i_{n-1} a)$ for some a in A_{n-1}.

1.4. **Presentation of the paper.** It is our intention that this paper be the first in a series, and so we have been extremely careful in laying out the foundations. For this reason we have occasionally chosen to duplicate known results, either because they are not published or because the published proofs do not fit well into our approach. In particular, Guillermo Moreno has a number of nice results on zero-divisors in Cayley-Dickson algebras [M1], some of which we have chosen to reprove here in the interests of building an arsenal of techniques. We have made every effort to state clearly which theorems were already known to Moreno.

One benefit is that the text should be completely self-contained, except for undergraduate-level linear algebra and abstract algebra, and a small amount of topology.

The other guiding principle behind our expository style has been to avoid ad hoc proofs and results. We'll try to use the same basic ideas in all of the proofs. When a phenomenon occurs, we'll emphasize the underlying structure that makes it happen, rather than just record the technical consequences. We have not completely attained this goal, however. For instance, it breaks down in our proof of Theorem 14.6, where we arbitrarily manipulate linear quaternionic equations.

There are a variety of other papers in the literature on Cayley-Dickson algebras. Dickson's contribution was to construct the octonions (also sometimes called the Cayley numbers) as pairs of quaternions [Di]. It was actually Adrian Albert (a student of Dickson) who first iterated this construction to produce an infinite family of algebras [Al]. Since then, a number of articles have been written studying various algebraic properties of these algebras. We mention in particular [Ad], [Br], [ES], [KY], [M1], and [Sc]. Fred Cohen has suggested that an understanding of the zero-divisor loci of the Cayley-Dickson algebras might have useful applications in topology—see [Co] for more information.

1.5. **Acknowledgments.** The authors would like to thank Dan Christensen for his assistance with several computer calculations.

2. Cayley-Dickson algebras

We start with the inductive definition of Cayley-Dickson algebras. These are finite-dimensional $\mathbb{R}$-algebras equipped with a linear involution $(-)^*$ satisfying $(xy)^* = y^* x^*$. The few results of this section appear to have been known to every mathematician from the modern era who worked on Cayley-Dickson algebras.

Definition 2.1. *The algebra* $\boldsymbol{A_0}$ *is equal to* $\mathbb{R}$*, and the involution is the identity. The algebra* $\boldsymbol{A_n}$ *additively is* $A_{n-1} \times A_{n-1}$*. The multiplication is defined inductively by*

$$(a, b)(c, d) = (ac - d^* b, da + bc^*),$$

and the involution is also defined inductively by

$$(a, b)^* = (a^*, -b).$$

Note that A_n has dimension 2^n as an $\mathbb{R}$-vector space. We shall use the term "conjugation" to refer to the involution because it generalizes the usual conjugation on the complex numbers. The reader should verify inductively that conjugation is in fact an involution—that is, $\boldsymbol{(x^*)^* = x}$—and that it interacts with multiplication according to the formula $\boldsymbol{(xy)^* = y^*x^*}$.

One can see inductively that A_n contains a copy of $\mathbb{R}$ because the subalgebra $A_{n-1} \times 0$ of A_n is isomorphic to A_{n-1}. Moreover, since the inductive formulas defining multiplication and conjugation are $\mathbb{R}$-linear, we see that each A_n is an $\mathbb{R}$-algebra. In particular, the elements of $\mathbb{R}$ are central in A_n.

Example 2.2. The algebra A_1 is isomorphic to the complex numbers $\mathbb{C}$ with its usual conjugation. To see why, just check that $(0, 1)$ plays the role of i in $\mathbb{C}$.

Example 2.3. The algebra A_2 is isomorphic to the quaternions $\mathbb{H}$ with its usual conjugation. The elements $(i, 0)$, $(0, 1)$, and $(0, i)$ play the roles of the standard basis elements i, j, and k.

Example 2.4. The algebra A_3 is isomorphic to the octonions $\mathbb{O}$ with its usual conjugation.

Definition 2.5. *If x is any vector, then the **real part** $\boldsymbol{Re(x)}$ of x is defined inductively as follows. If x belongs to A_0, then $Re(x)$ equals x. If (a, b) belongs to A_n, then $Re(a, b) = Re(a)$. Also, the **imaginary part** $\boldsymbol{Im(x)}$ of x is equal to $x - Re(x)$.*

If x belongs to $\mathbb{R}$, then we say that x is **real**. If $\mathrm{Re}(x) = 0$, then we say that x is **imaginary**. Every vector can be uniquely written as the sum of a real vector and an imaginary vector.

The reader should check inductively that $\boldsymbol{2\mathrm{Re}(x) = x + x^*}$ and $\boldsymbol{2\mathrm{Im}(x) = x - x^*}$. One consequence is that $\boldsymbol{x^* = -x}$ **if** $\boldsymbol{x}$ **is imaginary**.

Lemma 2.6. *For all x and y, $Re(xy - yx) = 0$.*

Proof. If either x or y is real, then the formula is trivial. By linearity, we may assume that x and y are both imaginary, so $x^* = -x$ and $y^* = -y$. Therefore,

$$xy - yx = xy - y^*x^* = xy - (xy)^* = 2\mathrm{Im}(xy).$$

Thus, $xy - yx$ is imaginary, so its real part vanishes. □

Definition 2.7. *The **associator** of x, y, and z is $\boldsymbol{[x, y, z] = (xy)z - x(yz)}$.*

Lemma 2.8. *For all x, y, and z, $Re([x, y, z]) = 0$.*

Proof. By linearity, it suffices to assume that x, y, and z are of the form $(a, 0)$ or $(0, a)$. There are eight cases to check. Four of the cases are easy because both $(xy)z$ and $x(yz)$ are of the form $(0, a)$; thus their real parts are zero.

The other four cases have to be checked one at a time. The case $x = (a, 0)$, $y = (b, 0)$, $z = (c, 0)$ follows by induction. The other three cases are all very similar to one another; we give one example.

Let $x = (a, 0)$, $y = (0, b)$, and $z = (0, c)$. Then calculate that $[x, y, z] = (a(c^*b) - c^*(ba), 0)$. Now $\mathrm{Re}(a(c^*b) - c^*(ba))$ equals $\mathrm{Re}((c^*b)a - c^*(ba))$ by Lemma 2.6, and this last expression vanishes by induction. □

One might be tempted to conclude from Lemmas 2.6 and 2.8 that when computing the real part of any expression, one can completely ignore the order of the factors and the parentheses. This is not true. For example, we cannot conclude that $\mathrm{Re}(x(yz))$ equals $\mathrm{Re}(x(zy))$.

The following corollary says that Cayley-Dickson algebras are "flexible". This means that expressions of the form xyx are well-defined, even in the absence of associativity.

Corollary 2.9. *For all x and y in A_n, $[x, y, x]$ vanishes.*

Proof. It suffices to prove the identity

$$x(yz) - (xy)z + z(yx) - (zy)x = 0$$

because we can set $z = x/2$ to recover the original formula.

If any one of x, y, and z is real, then the formula is trivial. By linearity, we may assume that x, y, and z are imaginary. Using that $x^* = -x$, $y^* = -y$, and $z^* = -z$, the above expression is its own conjugate. Therefore, the imaginary part of the expression is zero.

On the other hand, the real part of the expression is also zero by Lemma 2.8. □

Definition 2.10. *The **standard basis** of A_n is defined inductively as follows. The standard basis for A_0 consists of the element 1. The standard basis of A_n consists of elements of the form $(x, 0)$ or $(0, x)$, where x belongs to the standard basis of A_{n-1}.*

One readily checks by induction that the above does indeed give an $\mathbb{R}$-vector space basis of A_n.

3. Real inner product

The definition and basic properties of the real inner product described in this section were laid out in [M1].

Definition 3.1. *The **real inner product** $\langle x, y\rangle$ of two elements x and y in A_n is equal to $\mathrm{Re}(xy^*)$.*

Proposition 3.2. *The function $\langle -, -\rangle$ is a symmetric positive-definite inner product on A_n.*

Proof. The function $\langle -, -\rangle$ is $\mathbb{R}$-bilinear. For symmetry, $\mathrm{Re}(xy^*)$ equals $\mathrm{Re}(yx^*)$ because conjugation does not change the real part of a vector.

For positive-definiteness, compute that

$$\mathrm{Re}((a, b)(a, b)^*) = \mathrm{Re}(aa^* + b^*b).$$

By induction, $\mathrm{Re}(aa^*) + \mathrm{Re}(bb^*) \geq 0$, and $\mathrm{Re}(aa^*) + \mathrm{Re}(bb^*) = 0$ if and only if $a = b = 0$. □

As usual, we set $||x|| = \sqrt{\langle x, x\rangle} = \sqrt{\mathrm{Re}(xx^*)}$. This makes sense because of positive-definiteness.

Recall from Definition 2.10 that we can identify A_n with $\mathbb{R}^{2^n}$ (as $\mathbb{R}$-vector spaces). It is easy to check by induction that under this identification, the inner product of Definition 3.1 corresponds to the standard inner product on $\mathbb{R}^{2^n}$. Therefore, the standard basis of A_n is in fact an orthonormal basis.

Definition 3.3. *For any vector x, let $\boldsymbol{L_x}$ be the linear endomorphism of A_n given by left multiplication by x, and let $\boldsymbol{R_x}$ be the linear endomorphism of A_n given by right multiplication by x.*

Lemma 3.4. *The maps L_x and L_{x^*} are adjoint in the sense that $\langle L_x y, z\rangle$ equals $\langle y, L_{x^*} z\rangle$. The maps R_x and R_{x^*} are also adjoint.*

Proof. For the first claim, we want to show that $\mathrm{Re}((xy)z^*)$ equals $\mathrm{Re}(y(z^*x))$. This follows from Lemma 2.6, Lemma 2.8, and then Lemma 2.6 again.

The proof for R_x is similar. □

Corollary 3.5. *Suppose that x is imaginary. The map L_x is antisymmetric in the sense that $\langle L_x y, z\rangle = -\langle y, L_x z\rangle$. The map R_x is also antisymmetric.*

Proof. This follows immediately from Lemma 3.4 since $x^* = -x$. □

Note that L_x need not be an isometry, even if x has norm 1. For example, if x is a zero-divisor then there exists a non-zero y such that $xy = 0$. Then $\langle y, y\rangle$ is not zero, but $\langle L_x y, L_x y\rangle$ is zero.

Lemma 3.6. *For any x, $xx^* = x^*x = ||x||^2$.*

Proof. First note that xx^* is real because it is its own conjugate. Then xx^* equals $\mathrm{Re}(xx^*)$, which by definition is $||x||^2$. Finally, note that x^*x equals $\mathrm{Re}(x^*x)$; now apply Lemma 2.6 to show that x^*x equals xx^*. □

One consequence is that $\boldsymbol{x^2 = -||x||^2}$ **if $\boldsymbol{x}$ is imaginary**. Thus, the square of an imaginary vector is zero if and only if the original vector is zero.

Lemma 3.7. *Two imaginary vectors x and y anti-commute if and only if they are orthogonal.*

Proof. Since $x^* = -x$ and $y^* = -y$, the conjugate of xy is yx. Therefore, x and y anti-commute if and only if xy is imaginary, i.e., $\mathrm{Re}(xy)$ is zero. Finally, $\mathrm{Re}(xy)$ equals $\langle x, y^*\rangle = -\langle x, y\rangle$. □

Throughout the text, if V is a vector space with inner product and W is a subspace, we denote by $W^\perp$ the orthogonal complement of W in V.

Lemma 3.8. *Suppose B is an $\mathbb{R}$-subalgebra of A_n. If b is in B and x is in $B^\perp$, then bx and xb are in $B^\perp$.*

Proof. If b lies in B, then b^* also lies in B because $b^* = -b + 2\mathrm{Re}(b)$ and $\mathrm{Re}(b)$ lies in B. We need to show that $\langle a, bx\rangle$ equals zero for every a in B. This equals $\langle b^*a, x\rangle$ by Lemma 3.4, which is zero because b^*a belongs to B.

A similar argument shows that xb is also in $B^\perp$. □

The algebras A_0, A_1, A_2, and A_3 are normed in the sense that $||xy|| = ||x|| \cdot ||y||$. However, when $n \geq 4$ the algebra A_n is not normed—the presence of zero-divisors prevents this.

Exercise 3.9. If $a, x \in A_n$ and $[a, a, x] = 0$, prove using Lemmas 2.6 and 2.8 that $\langle ax, ay\rangle = ||a||^2\langle x, y\rangle$ for any $y \in A_n$. Since $[a, a, x]$ always vanishes in A_3 (see Lemma 4.3), this shows that A_3 is normed.

4. Alternators

Basic statements of alternativity, including our Lemma 4.4, were established by Schafer [Sc]. Moreno was able to completely classify the alternative elements in every A_n [M2].

Definition 4.1. *For x in A_n, let* **Alt** $\boldsymbol{x}$ *be the linear subspace of all y such that $[x,x,y]$ vanishes.*

We call this space the "alternator" of x.

Definition 4.2. *An element x of A_n is* ***alternative*** *if* $\mathrm{Alt}(x)$ *equals A_n, i.e., if all expressions $[x,x,y]$ vanish.*

In A_0, A_1, and A_2, all elements are alternative; this follows immediately from the fact that these algebras are associative. Even though A_3 is not associative, it turns out that every vector in A_3 is still alternative.

Lemma 4.3. *Every element of A_3 is alternative.*

Proof. Simply compute that $[(a,b),(a,b),(x,y)] = 0$ for all quaternions a, b, x, and y. One needs that $a + a^*$ and bb^* are both real and therefore central. □

Lemma 4.4. *Every standard basis vector (see Definition 2.10) is alternative.*

Proof. The proof is by induction. Direct computation shows that the associators $[(x,0),(x,0),(a,b)]$ and $[(0,x),(0,x),(a,b)]$ both vanish whenever x is alternative in A_{n-1}. For the second equation, the calculation is simplified by writing x as the sum of a real and an imaginary vector. □

Remark 4.5. It is natural to ask what are the possible dimensions of $\mathrm{Alt}\, x$, for x in A_n. Although we will not need this in the present paper, we can show that $\dim(\mathrm{Alt}\, x)$ is always a multiple of 4 (compare Theorem 9.8). In A_4, the only two possible dimensions are 8 and 16. In A_5, computer calculations show that there are examples of x with $\dim(\mathrm{Alt}\, x)$ equal to 4, 8, 12, 16, 24, and 32. We do not know if dimensions 20 and 28 can occur.

Later we will use the following definition frequently when considering specific examples.

Definition 4.6. *Two vectors a and b in A_n are a* ***quaternionic pair*** *if there is an injective algebra map $\phi : \mathbb{H} \to A_n$ such that $\phi(i) = a$ and $\phi(j) = b$.*

Remark 4.7. Vectors a and b are a quaternionic pair if and only if they are orthogonal imaginary unit vectors such that $[a,b,b]$ and $[a,a,b]$ both vanish. This is because $\mathbb{H}$ is the free associative $\mathbb{R}$-algebra on i and j subject to the relations $i^2 = j^2 = -1$ and $ij = -ji$. Lemma 3.6 tells us that a^2 and b^2 are both equal to -1 if and only if they are both imaginary unit vectors, and Lemma 3.7 tells us that the imaginary unit vectors a and b anti-commute if and only if they are orthogonal. Finally, in the presence of flexibility of A_n, the vanishing of $[a,a,b]$ and $[b,b,a]$ is equivalent to associativity of the algebra generated by a and b.

When a and b are a quaternionic pair, we write $\mathbb{H}\langle \boldsymbol{a}, \boldsymbol{b}\rangle$ for the subalgebra of A_n that they generate. Additively, $\mathbb{H}\langle a,b\rangle$ has an orthonormal basis consisting of 1, a, b, and ab.

Example 4.8. Let a and b be any two distinct imaginary standard basis elements of A_n. Then a and b are a quaternionic pair. See Lemma 4.4 for the fact that $[a, a, b]$ and $[b, b, a]$ both vanish.

5. Complex structure

Definition 5.1. *Let $\boldsymbol{i_n}$ be the element $(0, 1)$ of A_n. Let $\mathbb{C}_{\boldsymbol{n}}$ be the subalgebra of A_n additively generated by 1 and i_n.*

The notation suggests that i_n is the nth analogue of the square root of -1 in $\mathbb{C}$. Note that $\mathbb{C}_n$ is isomorphic to the complex numbers, where i_n plays the role of i. Our first goal is to show that A_n is a complex vector space, where the $\mathbb{C}$-action is given by left multiplication by elements of $\mathbb{C}_n$.

Lemma 5.2. *For any x in A_n, $[i_n, i_n, x] = 0$.*

Proof. This follows by direct computation with the definition of multiplication (see also Lemma 4.4). □

Proposition 5.3. *Additively, A_n is a $\mathbb{C}_n$-vector space, where the $\mathbb{C}_n$-action on A_n is given by left multiplication.*

Proof. The only thing to check is that if α and β belong to $\mathbb{C}_n$ and x belongs to A_n, then $\alpha(\beta x)$ equals $(\alpha\beta)x$. This follows immediately from Lemma 5.2. □

From now on, we will often view A_n not just as an $\mathbb{R}$-vector space but also as a $\mathbb{C}_n$-vector space.

Lemma 5.4. *Let $\phi\colon \mathbb{C}_n^\perp \to A_{n+1}$ be the map that takes a to $(a, i_n a)$. Then one has $\phi((p + qi_n)a) = (p + qi_{n+1})\phi(a)$ for all p and q in $\mathbb{R}$. That is to say, ϕ is complex-linear. The same is true for the map that takes a to $(a, -i_n a)$.*

Proof. Let α belong to $\mathbb{C}_n$, and let a belong to $\mathbb{C}_n^\perp$. Write α as $p+qi_n$. We want to show that $(\alpha a, i_n(\alpha a))$ equals $(p, q)(a, i_n a)$ as elements of A_{n+1}. This follows from direct computation, using that $a^* = -a$ and that a and i_n anti-commute. □

The element i_n has some special properties not enjoyed by a typical imaginary unit vector.

Lemma 5.5. *For all x in A_n, the associators $[x, x, i_n]$ and $[i_n, x, x]$ both vanish.*

Proof. To show that $[x, x, i_n]$ vanishes, directly compute with the inductive definition of multiplication. One needs to use that expressions of the form $a + a^*$ are central because they are real. Also, one needs to know that aa^* and a^*a are equal (see Lemma 3.6).

The argument for $[i_n, x, x]$ is similar. □

In fact, the property expressed in Lemma 5.5 determines i_n uniquely, up to a sign. Namely, if x is an imaginary unit vector in A_n such that $[x, y, y] = 0$ for all y, then $x = i_n$ or $x = -i_n$ [ES, Lemma 1.2]. We also remark that there is an automorphism of A_n which fixes every element of A_{n-1} and sends i_n to $-i_n$.

Lemma 5.6. *If a is a unit vector in $\mathbb{C}_n^\perp$, then a and i_n are a quaternionic pair (see Definition 4.6).*

Proof. Since a and i_n are orthonormal imaginary vectors, it suffices to show that $[i_n, a, a] = [i_n, i_n, a] = 0$. We have already checked these in Lemmas 5.2 and 5.5. □

The $\mathbb{C}_n$-vector space A_n is not a $\mathbb{C}_n$-algebra. However, we have the following two partial results along these lines. These lemmas are the key to computing with i_n. They allow one to do essentially any desired manipulation with expressions involving i_n.

Lemma 5.7. *Suppose that x belongs to $\mathbb{C}_n^\perp$, and let α belong to $\mathbb{C}_n$. For all y, $(yx)\alpha = (y\alpha^*)x$ and $\alpha(xy) = x(\alpha^* y)$.*

Proof. By linearity, we may assume that α equals 1 or i_n. The first case is easy. For the second case, compute with the inductive definition of multiplication. □

Lemma 5.8. *If x and y anti-commute and α belongs to $\mathbb{C}_n$, then $(\alpha x)y = -(\alpha y)x$ and $y(x\alpha) = -x(y\alpha)$.*

Proof. By linearity, we may assume that α is either 1 or i_n. The first case is obvious.

For the second case, $[i_n, x+y, x+y] = 0$ by Lemma 5.5. Expand this by linearity to obtain

$$[i_n, x, x] + [i_n, x, y] + [i_n, y, x] + [i_n, y, y] = 0.$$

The first and fourth terms are zero. Expand the other two terms to obtain

$$(i_n x)y - i_n(xy) + (i_n y)x - i_n(yx) = 0.$$

The second and fourth terms cancel because $xy = -yx$; the remaining two terms give the first desired identity.

The second identity can be obtained by conjugating the first identity. □

6. Hermitian inner product

This Hermitian inner product was first considered by Moreno [M3].

Definition 6.1. *Let x and y belong to A_n. The **Hermitian inner product $\langle x, y\rangle_H$** of x and y is the orthogonal projection of xy^* onto $\mathbb{C}_n$.*

Remark 6.2. One can check that $\langle x, y\rangle_H$ equals $\langle x, y\rangle - i_n\langle i_n x, y\rangle$. This follows from the definition of projection once one has checked that $\langle i_n, xy^*\rangle = -\langle i_n x, y\rangle$, and this last identity is an easy consequence of Lemma 3.4.

Proposition 6.3. *Definition 6.1 satisfies the usual properties of a Hermitian inner product.*

Proof. First, the inner product is additive in both variables.

Second, we show that $\langle \alpha x, y\rangle_H$ equals $\alpha\langle x, y\rangle_H$ for all α in $\mathbb{C}_n$. By linearity, we may assume that α is either 1 or i_n. The formula is obvious if α is 1, so we may assume that α equals i_n. In this case, $\langle i_n x, y\rangle_H$ equals $\langle i_n x, y\rangle - i_n\langle -x, y\rangle$ by Remark 6.2 and the fact that $i_n(i_n x)$ equals $-x$. Now this expression equals $i_n(\langle x, y\rangle - i_n\langle i_n x, y\rangle)$, which is equal to $i_n\langle x, y\rangle_H$ as desired.

Next we show that $\langle x, y\rangle_H$ and $\langle y, x\rangle_H$ are conjugate. This follows from the fact that xy^* and yx^* are conjugates; therefore, their projections onto $\mathbb{C}_n$ are also conjugate.

Finally, we show that the inner product is positive-definite. For any x, xx^* is real. Therefore, $\langle x, x\rangle_H$ equals xx^*, which equals $||x||^2$. □

Recall from Definition 3.3 that L_x is the linear map $A_n \to A_n$ given by left multiplication by x.

Lemma 6.4. *If x belongs to $\mathbb{C}_n^{\perp}$, then L_x is conjugate-linear in the sense that $L_x(y+z) = L_x(y) + L_x(z)$ and $L_x(\alpha y) = \alpha^* L_x(y)$ for α in $\mathbb{C}_n$.*

Proof. This is simply a restatement of Lemma 5.7. □

Lemma 6.5. *If x belongs to $\mathbb{C}_n^{\perp}$, then L_x is anti-Hermitian in the sense that $\langle L_x y, z\rangle_H = -\langle y, L_x z\rangle_H^*$.*

Proof. Start with $\langle xy, z\rangle_H$, which equals $\langle xy, z\rangle - i_n\langle i_n(xy), z\rangle$ by Remark 6.2. By Lemma 5.7, this equals $\langle xy, z\rangle + i_n\langle x(i_n y), z\rangle$. Now using Corollary 3.5, this expression equals $-\langle y, xz\rangle - i_n\langle i_n y, xz\rangle$. This is the negative conjugate of $\langle y, xz\rangle_H$, as desired. □

We record the following two results about conjugate-linear anti-Hermitian maps for later use.

Lemma 6.6. *Suppose that V is an odd-dimensional $\mathbb{C}$-vector space with a nondegenerate Hermitian inner product. If L is a conjugate-linear anti-Hermitian endomorphism of V, then L is singular.*

Proof. Choose a basis for V, and identify the elements of V with column vectors. Because L is conjugate-linear, there exists a complex matrix A such that $Lx = Ax^*$. The Hermitian inner product on V is given by $\langle x, y\rangle = x^T H y^*$ for some Hermitian matrix H. This means that H^T equals H^*. Also, because the inner product is nondegenerate, the matrix H is invertible.

The inner product $\langle Lx, y\rangle$ is equal to $(x^*)^T A^T H y^*$. On the other hand, $-\langle x, Ly\rangle^*$ is equal to $-(x^*)^T H^* A y^*$. Since L is anti-Hermitian, these two expressions are equal for all x and y. This means that $A^T H$ equals $-H^* A$, which is equal to $-(A^T H)^T$ because H^* equals H^T.

Write $B = A^T H$. Then $B = -B^T$. Since H is invertible, A is singular if and only if B is singular. If n is the dimension of V, then

$$\det(B) = \det(-B^T) = (-1)^n \det(B^T) = (-1)^n \det(B).$$

When n is odd, this implies that $\det B$ is zero. □

Lemma 6.7. *Suppose that V is a $\mathbb{C}$-vector space with a nondegenerate Hermitian inner product, and let L be a conjugate-linear anti-Hermitian endomorphism of V. Then the $\mathbb{C}$-codimension of $\ker L$ in V is even.*

Proof. Let $K = \ker L$. Consider the space W of all vectors y such that $\langle y, z\rangle_H$ equals zero for all z in K. The dimension of W is the same as the codimension of K because W is the orthogonal complement of K. Thus, we want to show that W is even-dimensional.

First we will show that L restricts to a map from W to itself. Suppose that w belongs to W; then $\langle w, z\rangle_H$ equals zero for all z in K. Now $\langle Lw, z\rangle_H$ equals $-\langle w, Lz\rangle_H^*$. For z in K, $Lz = 0$ by definition. Therefore, $\langle Lw, z\rangle_H$ equals zero for all z in K. This implies that Lw belongs to W.

Now the restriction of L to W is still conjugate-linear and anti-Hermitian. Moreover, it is also non-singular because we have ensured that W does not meet the kernel of L. Therefore, Lemma 6.6 tells us that the dimension of W must be even. □

7. Automorphisms of A_3

In this section we summarize some facts about automorphisms of A_3. These will be used to simplify some calculations later on. These rather old results are due to Élie Cartan [Ca]. See [ES] for a generalization of this result to all A_n.

Note that A_3 is generated as an $\mathbb{R}$-algebra by the three elements i_1, i_2, and i_3. In order to keep the notation more readable, in this section we will refer to these three elements as i, j, and t respectively.

To construct an algebra map $\phi : A_3 \to A_3$, one just needs to specify $x = \phi(i)$, $y = \phi(j)$, and $z = \phi(t)$. We know that x and y are a quaternionic pair. This implies that x and y are orthogonal imaginary unit vectors. We also know that z anti-commutes with x, y, and xy. This means that z must be an imaginary unit vector that is orthogonal to x, y, and xy.

It turns out that these conditions on x, y, and z are sufficient to guarantee that ϕ is an $\mathbb{R}$-algebra automorphism. The proof is a straight-forward computation. We summarize the last few paragraphs in the following theorem.

Theorem 7.1. *There is a one-to-one correspondence between algebra automorphisms $A_3 \to A_3$ and ordered triples (x, y, z) of pairwise orthogonal imaginary unit vectors in A_3 such that z is also orthogonal to xy.*

Using this concrete description of maps from A_3 to A_3, we can describe the automorphism group of A_3. It is a 14-dimensional Lie group G_2 that belongs to a fiber bundle

$$S^3 \to G_2 \to V_2(\mathbb{R}^7),$$

where $V_2(\mathbb{R}^7)$ is the Stiefel manifold of ordered pairs of orthonormal vectors in $\mathbb{R}^7$.

By Theorem 7.1, an automorphism of A_3 corresponds to a triple (x, y, z) of imaginary vectors of norm 1 that are pairwise orthogonal and such that z is orthogonal to xy. The map $G_2 \to V_2(\mathbb{R}^7)$ takes (x, y, z) to (x, y). Note that x and y belong to $\mathbb{R}^7$ because they are imaginary. By assumption, they are orthogonal unit vectors. This shows that (x, y) always belongs to $V_2(\mathbb{R}^7)$.

To compute the fiber of p, we need to describe the space of imaginary vectors z of norm 1 that are orthogonal to x, y, and xy. The orthogonality condition leaves a 4-dimensional subspace of possibilities for z. The norm condition guarantees that z belongs to a 3-dimensional sphere.

Observe, in particular, that the automorphism group of A_3 acts transitively on the set of imaginary unit vectors of A_3.

8. Associators

Ultimately, we are interested in understanding the annihilators of various vectors; i.e., given x, we want to describe all y such that $xy = 0$. It turns out that in order to do this, we will need to understand the spaces of associators and anti-associators (see Proposition 11.1 for the connection).

Definition 8.1. *For any pair of vectors x and y, let $\boldsymbol{A_{x,y}}$ be the linear endomorphism of A_n given by the formula $A_{x,y}(z) = [x, z, y]$. Also, let $\boldsymbol{A'_{x,y}}$ be the linear endomorphism of A_n given by the formula $A'_{x,y}(z) = (xz)y + x(zy)$.*

The A stands for "associator", of course. Another way to denote $A_{x,y}$ is as $R_y L_x - L_x R_y$. Similarly, $A'_{x,y} = R_y L_x + L_x R_y$.

Lemma 8.2. *If x and y are imaginary, then $A_{x,y}$ is antisymmetric (in the sense that $\langle A_{x,y}z, w\rangle$ equals $-\langle z, A_{x,y}w\rangle$) and $A'_{x,y}$ is symmetric.*

Proof. Use that $A_{x,y} = R_y L_x - L_x R_y$ and $A'_{x,y} = R_y L_x + L_x R_y$, together with the fact from Corollary 3.5 that L_x and R_y are both antisymmetric. □

Definition 8.3. *Given two vectors a and b in A_n, the* ***associator*** $\mathbf{Ass[a,b]}$ *is the kernel of $A_{a,b}$. The* ***anti-associator*** $\mathbf{Ass'[a,b]}$ *is the kernel of $A'_{a,b}$.*

Throughout this section, we will typically assume that a and b are a quaternionic pair. Recall that this means that a and b are orthogonal imaginary unit vectors such that the subalgebra $\mathbb{H}\langle a,b\rangle$ generated by a and b is isomorphic to the quaternions (see Definition 4.6).

Lemma 8.4. *Let a and b be a quaternionic pair, and let L belong to the subalgebra of all $\mathbb{R}$-linear endomorphisms of A_n that is generated by L_a, L_b, R_a, and R_b. Then the kernel of L splits as*

$$\ker L = \big(\ker L \cap \mathbb{H}\langle a,b\rangle\big) \oplus \big(\ker L \cap \mathbb{H}\langle a,b\rangle^{\perp}\big).$$

Proof. The desired splitting follows from the fact that L restricts to endomorphisms of $\mathbb{H}\langle a,b\rangle$ and $\mathbb{H}\langle a,b\rangle^{\perp}$. To see why this is true, note that L_a, L_b, R_a, and R_b restrict to endomorphisms of $\mathbb{H}\langle a,b\rangle$; therefore, L also restricts to an endomorphism of $\mathbb{H}\langle a,b\rangle$.

On the other hand, Lemma 3.8 indicates that L_a, L_b, R_a, and R_b restrict to an endomorphism of $\mathbb{H}\langle a,b\rangle^{\perp}$, so L does also. □

Lemma 8.4 can be applied to the maps $A_{a,b}$ and $A'_{a,b}$ to obtain splittings of $\mathrm{Ass}[a,b]$ and $\mathrm{Ass}'[a,b]$. Below, we will also need to apply it to other maps.

Lemma 8.5. *Let a and b be a quaternionic pair in A_3 (i.e., a and b are any orthonormal pair of imaginary vectors). Then $\mathrm{Ass}[a,b]$ equals $\mathbb{H}\langle a,b\rangle$ and $\mathrm{Ass}'[a,b]$ equals $\mathbb{H}\langle a,b\rangle^{\perp}$.*

Proof. Up to automorphism, it suffices to let $a = i$ and $b = j$, so $\mathbb{H}\langle a,b\rangle$ equals $\mathbb{H}$ and $\mathbb{H}\langle a,b\rangle^{\perp}$ equals $t\mathbb{H}$. Since $\mathbb{H}$ is an associative subalgebra, it is contained in $\mathrm{Ass}[i,j]$.

Now let h be an arbitrary element of $\mathbb{H}$, and compute that $A_{i,j}(th) = 2t(ijh)$. This shows that $\mathrm{Ass}[i,j]$ intersects $\mathbb{H}^{\perp}$ trivially. With Lemma 8.4, this proves the first claim.

For the second claim, $\mathbb{H}$ meets $\mathrm{Ass}'[i,j]$ trivially since $\mathbb{H}$ is an associative subalgebra. Now, for any h in $\mathbb{H}$, compute that $A'_{i,j}(th) = 0$. This shows that $\mathbb{H}^{\perp}$ is contained in $\mathrm{Ass}'[i,j]$. With Lemma 8.4, this proves the second claim. □

Starting with the previous lemma, we will compute the dimensions of various associators and anti-associators inductively. First we need some lengthy technical computations.

Lemma 8.6. *If b is imaginary, then*

(1) $A_{(a,0),(b,0)}(x,y) = \big((ax)b - a(xb), (yb)a - (ya)b\big)$.
(2) $A_{(a,0),(0,b)}(x,y) = \big(b(ya) - a(by), b(ax) - (bx)a\big)$.
(3) $A'_{(a,0),(b,0)}(x,y) = \big((ax)b + a(xb), -(yb)a - (ya)b\big)$.
(4) $A'_{(a,0),(0,b)}(x,y) = \big(b(ya) + a(by), b(ax) + (bx)a\big)$.

Proof. Compute using the inductive definition of multiplication. □

Lemma 8.7. *Let a and b be a quaternionic pair in A_n. Then we have*

$$\mathrm{Ass}[(a,0),(b,0)] = \mathrm{Ass}[a,b] \times (\mathrm{Ass}[a,b] \cap \mathbb{H}\langle a,b\rangle^{\perp}).$$

In particular, the dimension of $\mathrm{Ass}[(a,0),(b,0)]$ *is* $2\dim \mathrm{Ass}[a,b] - 4$.

Proof. According to part (1) of Lemma 8.6, we need to find all x and y such that $(ax)b - a(xb) = 0$ and $(yb)a - (ya)b = 0$. The solution space of the first equation is $\mathrm{Ass}[a,b]$.

We now have to find the solutions to the second equation. By Lemma 8.4 applied to $R_aR_b - R_bR_a$, the solution space splits as the direct sum of its intersections with $\mathbb{H}\langle a,b\rangle$ and $\mathbb{H}\langle a,b\rangle^{\perp}$. Some quaternionic arithmetic indicates that the solution space intersects $\mathbb{H}\langle a,b\rangle$ trivially.

Now we may assume that y belongs to $\mathbb{H}\langle a,b\rangle^{\perp}$. This implies by Lemma 3.8 that yb is also in $\mathbb{H}\langle a,b\rangle^{\perp}$. Using that orthogonal imaginary vectors anti-commute, we compute that $(yb)a-(ya)b = (ay)b-a(yb)$. Thus, the solution space of the second equation is the intersection of $\mathrm{Ass}[a,b]$ with $\mathbb{H}\langle a,b\rangle^{\perp}$. This space has dimension $\dim \mathrm{Ass}[a,b] - 4$ because $\mathbb{H}\langle a,b\rangle$ is contained in $\mathrm{Ass}[a,b]$. □

Lemma 8.8. *Let a and b be a quaternionic pair in A_n. Then we have*

$$\mathrm{Ass}[(a,0),(0,b)] = (\mathrm{Ass}'[a,b] \oplus \mathbb{R} \oplus \mathbb{R}a) \times (\mathrm{Ass}'[a,b] \oplus \mathbb{R}b \oplus \mathbb{R}ab).$$

In particular, the dimension of $\mathrm{Ass}[(a,0),(0,b)]$ *is equal to* $2\dim \mathrm{Ass}'[a,b] + 4$.

Proof. According to part (2) of Lemma 8.6, we need to find all x and y such that $b(ya) - a(by) = 0$ and $b(ax) - (bx)a = 0$. Let K_1 be the solution space of the first equation, and let K_2 be the solution space of the second equation.

To find K_1 and K_2, first note that they split as the direct sums of their intersections with $\mathbb{H}\langle a,b\rangle$ and $\mathbb{H}\langle a,b\rangle^{\perp}$ because of Lemma 8.4 applied to $L_bR_a - L_aL_b$ and $L_bL_a - R_aL_b$. Some quaternionic arithmetic indicates that the intersection of $\mathbb{H}\langle a,b\rangle$ with K_1 is the 2-dimensional subspace generated by b and ab. Also, the intersection of $\mathbb{H}\langle a,b\rangle$ with K_2 is the 2-dimensional subspace generated by 1 and a.

Now we may assume that x and y belong to $\mathbb{H}\langle a,b\rangle^{\perp}$. By Lemma 3.8, ya, ax, and bx are also in $\mathbb{H}\langle a,b\rangle^{\perp}$. Using that orthogonal imaginary vectors anti-commute, we compute that $b(ya)-a(by) = (ay)b+a(yb)$ and $b(ax)-(bx)a = -(ax)b-a(xb)$.

Thus, the intersection of K_1 with $\mathbb{H}\langle a,b\rangle^{\perp}$ equals $\mathrm{Ass}'[a,b]$ since $\mathrm{Ass}'[a,b]$ is contained in $\mathbb{H}\langle a,b\rangle^{\perp}$. The same is true for K_2. It follows that $\dim \mathrm{Ass}[(a,0),(0,b)]$ equals

$$2 + \dim \mathrm{Ass}'[a,b] + 2 + \dim \mathrm{Ass}'[a,b].$$

□

Lemma 8.9. *Let a and b be a quaternionic pair. Then* $\mathrm{Ass}'[(a,0),(b,0)]$ *equals*

$$\mathrm{Ass}'[a,b] \times (\mathrm{Ass}'[a,b] \oplus \mathbb{H}\langle a,b\rangle).$$

In particular, the dimension of $\mathrm{Ass}'[(a,0),(b,0)]$ *is equal to* $2\dim \mathrm{Ass}'[a,b] + 4$.

Proof. According to part (3) of Lemma 8.6, we need to find all x and y such that $(ax)b + a(xb) = 0$ and $-(yb)a - (ya)b = 0$. The solution space of the first equation is $\mathrm{Ass}'[a,b]$.

Let K denote the solution space of the second equation. By Lemma 8.4 applied to $-R_aR_b - R_bR_a$, K splits as the direct sum of its intersections with $\mathbb{H}\langle a,b\rangle$ and

$\mathbb{H}\langle a,b\rangle^{\perp}$. Some quaternionic arithmetic demonstrates that $\mathbb{H}\langle a,b\rangle$ is contained in K.

Now we may assume that y belongs to $\mathbb{H}\langle a,b\rangle^{\perp}$. This implies that yb also lies in $\mathbb{H}\langle a,b\rangle^{\perp}$. Using that orthogonal imaginary vectors anti-commute, we compute that $-(yb)a-(ya)b=(ay)b+a(yb)$. Thus, the intersection of K with $\mathbb{H}\langle a,b\rangle^{\perp}$ equals $\mathrm{Ass}'[a,b]$. It follows that $\dim \mathrm{Ass}'[(a,0),(b,0)]$ equals

$$\dim \mathrm{Ass}'[a,b]+4+\dim \mathrm{Ass}'[a,b].$$

□

Lemma 8.10. *Let a and b be a quaternionic pair. Then $\mathrm{Ass}'[(a,0),(0,b)]$ equals*

$$((\mathrm{Ass}[a,b]\cap \mathbb{H}\langle a,b\rangle^{\perp})\oplus \mathbb{R}b\oplus \mathbb{R}ab)\times((\mathrm{Ass}[a,b]\cap \mathbb{H}\langle a,b\rangle^{\perp})\oplus \mathbb{R}\oplus \mathbb{R}a).$$

In particular, the dimension of $\mathrm{Ass}'[(a,0),(0,b)]$ is equal to $2\dim \mathrm{Ass}[a,b]-4$.

Proof. According to part (4) of Lemma 8.6, we need to find all x and y such that $b(ya)+a(by)=0$ and $b(ax)+(bx)a=0$. Let K_1 denote the solution space of the first equation, and K_2 the solution space of the second equation.

To find K_1 and K_2, first note that by Lemma 8.4 applied to $L_bR_a+L_aL_b$ and $L_bL_a+R_aL_b$, they split as the direct sums of their intersections with $\mathbb{H}\langle a,b\rangle$ and $\mathbb{H}\langle a,b\rangle^{\perp}$. Quaternionic arithmetic easily shows that the intersection of $\mathbb{H}\langle a,b\rangle$ with K_1 is the 2-dimensional subspace generated by 1 and a. Also, the intersection of $\mathbb{H}\langle a,b\rangle$ with K_2 is the 2-dimensional subspace generated by b and ab.

Now we may assume that x and y belong to $\mathbb{H}\langle a,b\rangle^{\perp}$. This implies that ax, bx, and ya are also elements of $\mathbb{H}\langle a,b\rangle^{\perp}$. Using that orthogonal imaginary vectors anti-commute, we compute that $b(ya)+a(by)=(ay)b-a(yb)$ and $b(ax)+(bx)a=-(ax)b+a(xb)$.

Thus, the intersection of K_1 with $\mathbb{H}\langle a,b\rangle^{\perp}$ equals the intersection of $\mathrm{Ass}[a,b]$ with $\mathbb{H}\langle a,b\rangle^{\perp}$. The same is true for K_2. It follows that $\dim \mathrm{Ass}[(a,0),(0,b)]$ equals

$$2+(\dim \mathrm{Ass}'[a,b]-4)+2+(\dim \mathrm{Ass}'[a,b]-4).$$

□

Now we are ready to construct associators and anti-associators of various prescribed dimensions.

Proposition 8.11. *Let $n\geq 3$, and let $d\leq 2^{n-1}$ be congruent to 4 modulo 8. Then there exist imaginary standard basis vectors a and b in A_n such that $\dim \mathrm{Ass}[a,b]$ equals d. There also exist imaginary standard basis vectors a' and b' in A_n such that $\dim \mathrm{Ass}'[a',b']$ equals d.*

Proof. The proof is by induction. The base case is $n=3$, which is demonstrated in Lemma 8.5.

Any two distinct imaginary standard basis vectors a and b are of course orthonormal and imaginary. Moreover, $[a,a,b]=[b,b,a]=0$ by Lemma 4.4. Therefore, a and b form a quaternionic pair.

Suppose that the lemma is true for $n-1$. Note that d equals 4 or 12 modulo 16. We will split the proof into these two cases.

First suppose that d equals 4 modulo 16. Write $d=16k+4$ with $k<2^{n-5}$. Choose imaginary standard basis vectors a and b in A_{n-1} such that $\dim \mathrm{Ass}[a,b]$ equals $8k+4$; this is possible by induction because $8k+4\leq 2^{n-2}$. Note that $(a,0)$, $(b,0)$, and $(0,b)$ are imaginary standard basis vectors of A_n. Then Lemma 8.7

says that $\dim \mathrm{Ass}[(a,0),(b,0)]$ equals $2(8k+4)-4$, which is the same as d. Also, Lemma 8.10 tells us that $\dim \mathrm{Ass}'[(a,0),(0,b)] = d$.

Now suppose that d equals 12 modulo 16. Write $d = 16k+12$ with $k < 2^{n-5}$. Choose imaginary standard basis vectors a and b in A_{n-1} such that $\dim \mathrm{Ass}'[a,b]$ equals $8k+4$; this is possible by induction because $8k+4 \leq 2^{n-2}$. Again, $(a,0)$, $(b,0)$, and $(0,b)$ are imaginary standard basis vectors. Then Lemma 8.8 tells us that $\dim \mathrm{Ass}[(a,0),(0,b)]$ equals $2(8k+4)+4$, which is just d. Also, Lemma 8.9 says that $\dim \mathrm{Ass}'[(a,0),(b,0)]$ equals d. □

9. General properties of zero-divisors

The first few basic results of this section have been known for around a decade to most workers in the subject, certainly including Moreno [M1] and Khalil-Yiu [KY].

Definition 9.1. *A* ***zero-divisor*** *is a non-zero vector x such that $xy = 0$ for some non-zero vector y. The* ***annihilator*** $\mathbf{Ann}(x)$ *of a vector x in A_n is the kernel of L_x.*

Note that $\mathrm{Ann}(x)$ is non-zero if and only if x is zero-divisor. It turns out (by Corollary 9.3 below) that $\mathrm{Ann}(x)$ also equals the kernel of R_x, so we don't have to talk about 'left' and 'right' annihilators.

Lemma 9.2. *The following equations are equivalent:*

(1) $xy = 0$.
(2) $x^*y = 0$.
(3) $xy^* = 0$.

Proof. For the equivalence between (1) and (2), we need to show that if $xy = 0$, then x^*y is also zero. Compute $||x^*y||^2 = \mathrm{Re}((x^*y)(x^*y)^*) = \mathrm{Re}((x^*y)(y^*x))$. Using Lemmas 2.8 and 2.6, this is equal to $\mathrm{Re}((x(x^*y))y^*)$.

Now a straightforward calculation shows that $x(x^*y)$ always equals $x^*(xy)$, which is zero because $xy = 0$. Thus x^*y is a vector of norm zero and hence is zero.

The same argument shows that (1) and (3) are equivalent. □

Corollary 9.3. *For any x and y, we have $xy = 0$ if and only if $yx = 0$.*

Proof. Suppose that $xy = 0$. By conjugation, $y^*x^* = 0$. Now Lemma 9.2 (the equivalence of (2) and (3)) implies that $yx = 0$. □

Lemma 9.4. *Every zero-divisor in A_n is imaginary.*

Proof. Suppose that $xy = 0$ for some non-zero x and y. Then x^*y also equals zero by Lemma 9.2, so $(x+x^*)y = 0$. Since $x + x^*$ is real and y is non-zero, this shows that $x + x^*$ is zero. In other words, x is imaginary. □

Lemma 9.5. *Every zero-divisor in A_n is orthogonal to $\mathbb{C}_n$. So for any non-zero x, $\mathrm{Ann}(x)$ is orthogonal to $\mathbb{C}_n$.*

Proof. The second statement follows directly from the first. Suppose that (a,b) is a zero-divisor. By Lemma 9.4, a is imaginary; this means that (a,b) is orthogonal to 1.

Now $ac - d^*b = 0$ and $da + bc^* = 0$ for some non-zero (c,d). Compute that $(-d^*,c)(b,a)$ equals $(-d^*b - a^*c, -ad^* + cb^*)$. Using that $a^* = -a$, this equals

$(ac - d^*b, (da + bc^*)^*)$, which is zero. Therefore, (b, a) is also a zero-divisor. By Lemma 9.4, b is imaginary. Thus (a, b) is orthogonal to i_n. □

Lemma 9.6. *For any non-zero x in A_n, $\mathrm{Ann}(x)$ is a $\mathbb{C}_n$-vector space.*

Proof. All we have to do is show that $\ker L_x$ is closed under left multiplication by an element α of $\mathbb{C}_n$. By the previous lemma, we may assume x is orthogonal to $\mathbb{C}_n$.

Lemma 6.4 says that L_x is conjugate-linear. If $L_x(y) = 0$, then $L_x(\alpha y) = \alpha^* L_x(y) = 0$. Therefore, if y belongs to $\ker L_x$, then so does αy. □

The previous lemma implies that the real dimension of $\mathrm{Ann}(x)$ is always a multiple of 2. Soon we will show that the real dimension of $\mathrm{Ann}(x)$ is in fact a multiple of 4.

Lemma 9.7. *For any non-zero x in A_n and any non-zero α in $\mathbb{C}_n$, $\mathrm{Ann}(x)$ equals $\mathrm{Ann}(\alpha x)$.*

Proof. We need to show that $xy = 0$ if and only if $(\alpha x)y = 0$.

Suppose that $xy = 0$. We have that x belongs to $\mathrm{Ann}(y)$, which means that αx belongs to $\mathrm{Ann}(y)$ by Lemma 9.6. It follows that $(\alpha x)y = 0$.

On the other hand, suppose that $(\alpha x)y = 0$. We have that αx belongs to $\mathrm{Ann}(y)$, which means that $x = \alpha^* \alpha x / ||\alpha||^2$ belongs to $\mathrm{Ann}(y)$ by Lemma 9.6. It follows that $xy = 0$. □

The following result was originally proven by Moreno—see [M1, Cor. 1.17].

Theorem 9.8. *Let $n \geq 2$. For any x in A_n, the real dimension of $\mathrm{Ann}(x)$ is a multiple of 4.*

Proof. If x is zero, then $\mathrm{Ann}(x)$ equals A_n, so it has dimension 2^n. By the assumption on n, this is a multiple of 4. Likewise, if x is not a zero-divisor, then $\mathrm{Ann}(x) = 0$.

If x is a zero-divisor, then Lemma 9.5 says that x belongs to $\mathbb{C}_n^\perp$. Under these conditions, Lemmas 6.4 and 6.5 imply that the map L_x is conjugate-linear and anti-Hermitian. Lemma 6.7 implies that the complex codimension of the kernel of L_x is even. Since the complex dimension of A_n is 2^{n-1}, this implies that the complex dimension of the kernel of L_x is also even.

Thus, the kernel is an even-dimensional $\mathbb{C}_n$-vector space, so its real dimension is a multiple of 4. □

Lemma 9.9. *Let a and b belong to A_{n-1}. The dimension of $\mathrm{Ann}(a, b)$ is at most $2^{n-1} - 2 + \dim(\mathrm{Ann}(a) \cap \mathrm{Ann}(b))$.*

Proof. Recall that Lemma 9.5 tells us that $\mathrm{Ann}(a, b)$ is a subspace of the $(2^n - 2)$-dimensional space $\mathbb{C}_n^\perp$. Let W be the subspace of $\mathbb{C}_n^\perp$ consisting of all vectors of the form $(c, 0)$ with c imaginary in A_{n-1}. This is a $(2^{n-1} - 1)$-dimensional subspace of $\mathbb{C}_n^\perp$.

Let us investigate the intersection $\mathrm{Ann}(a, b) \cap W$, which consists of vectors $(c, 0)$ such that $(a, b)(c, 0) = 0$. This means that ac and bc^* are zero. In other words, $\mathrm{Ann}(a, b) \cap W$ is equal to $(\mathrm{Ann}(a) \cap \mathrm{Ann}(b)) \times 0$.

Now W and $\mathrm{Ann}(a, b)$ are both subspaces of $\mathbb{C}_n^\perp$. Therefore,

$$\dim W + \dim \mathrm{Ann}(a, b) \leq \dim \mathbb{C}_n^\perp + \dim(W \cap \mathrm{Ann}(a, b)).$$

Plugging in what we know, we get

$$2^{n-1} - 1 + \dim \mathrm{Ann}(a, b) \leq 2^n - 2 + \dim(\mathrm{Ann}(a) \cap \mathrm{Ann}(b))$$

Now just simplify the inequality to obtain

$$\dim \mathrm{Ann}(a, b) \leq 2^{n-1} - 1 + \dim(\mathrm{Ann}(a) \cap \mathrm{Ann}(b)).$$

Finally, $\mathrm{Ann}(a, b)$ and $\mathrm{Ann}(a) \cap \mathrm{Ann}(b)$ are both complex vector spaces (see Lemma 9.6), so their real dimensions are even. □

Proposition 9.10. *Let $n \geq 2$. For any non-zero x in A_n, the real dimension of $\mathrm{Ann}(x)$ is at most $2^n - 4n + 4$.*

Proof. The proof is by induction. The base cases are $n = 2$ and $n = 3$, which say that A_2 and A_3 have no zero-divisors.

Assume for induction that the proposition is true for $n-1$. Let $x = (a, b)$, where a and b belong to A_{n-1}. By the induction assumption, we know that $\dim(\mathrm{Ann}(a) \cap \mathrm{Ann}(b)) \leq 2^{n-1} - 4(n-1) + 4$. Now Lemma 9.9 implies that $\dim \mathrm{Ann}(x) \leq 2^{n-1} - 2 + 2^{n-1} - 4(n-1) + 4$, which simplifies to the inequality $\dim \mathrm{Ann}(x) \leq 2^n - 4n + 6$. Finally, recall from Theorem 9.8 that $\dim \mathrm{Ann}(x)$ is a multiple of 4. □

The above result shows that $\mathrm{Ann}(x)$ has dimension at most 4 in A_4—i.e., that every zero-divisor in A_4 has a 4-dimensional annihilator. However, as n increases the proposition seems to become very weak: it gives a linear lower bound on the codimension of each annihilator, while the dimension of A_n grows exponentially. For example, it says that in A_6 the kernel of L_x has dimension at most 44. This, in conjunction with the fairly naive method used to prove Proposition 9.10, makes it seem rather surprising that the upper bound it establishes is in fact sharp. We will show that there exists an element x of A_n such that the real dimension of $\mathrm{Ann}(x)$ is equal to $2^n - 4n + 4$. Moreover, we will show in Theorem 13.2 that all smaller dimensions (that are multiples of 4) also occur.

We close this section with two simple lemmas that will be needed later.

Lemma 9.11. *Let a and b be imaginary elements of A_{n-1}. The following three subsets of A_n are identical:*

(i) $\mathrm{Ann}(a, b)$.
(ii) The set of all (x, y) such that x and y are imaginary, $ax = -yb$, and $bx = ya$.
(iii) The set of all (x, y) such that x and y are imaginary, $ax = -yb$, and $xb = ay$.

Proof. The equivalence of the first two subsets comes from just writing out $(a, b)(x, y) = (0, 0)$ with the definition of multiplication. Also, recall from Lemma 9.5 that x and y have to be imaginary, so $x^* = -x$ and $y^* = -y$.

For the third set, conjugate the equation $bx = ya$, using that a, b, x, and y are all imaginary. □

Lemma 9.12. *Suppose that B is a subalgebra of A_n containing a vector x. Then $\mathrm{Ann}(x)$ decomposes as*

$$(\mathrm{Ann}(x) \cap B) \oplus (\mathrm{Ann}(x) \cap B^{\perp}).$$

Proof. Let y belong to $\mathrm{Ann}(x)$, and write $y = y_1 + y_2$, where y_1 belongs to B and y_2 belongs to $B^{\perp}$. All we have to do is show that y_1 and y_2 also belong to $\mathrm{Ann}(x)$.

Since B is a subalgebra, xy_1 belongs to B. Recall from Lemma 3.8 that xy_2 belongs to $B^{\perp}$. Now xy_1 and xy_2 are orthogonal vectors whose sum is zero, so they must both be zero. □

10. Constructions of zero-divisors

We now begin the task of producing zero-divisors whose annihilators have various dimensions.

Theorem 10.1. *Let a be a vector in $\mathbb{C}_n^\perp$, and let α and β be elements of $\mathbb{C}_n$ such that $\alpha^2 + \beta^2$ is not zero. Then $\mathrm{Ann}(\alpha a, \beta a)$ is equal to $\mathrm{Ann}(a) \times \mathrm{Ann}(a)$; in particular, the dimension of $\mathrm{Ann}(\alpha a, \beta a)$ is $2 \dim \mathrm{Ann}(a)$.*

Proof. First note that $\mathrm{Ann}(a) \times \mathrm{Ann}(a)$ is contained in $\mathrm{Ann}(\alpha a, \beta a)$, using that $\mathrm{Ann}(a) \subseteq \mathrm{Ann}(\alpha a)$ (and similarly for β), with equality unless $\alpha = 0$. So we will prove the subset in the other direction.

Without loss of generality, we may rescale a and assume that it is a unit vector. Recall that a and i_n are a quaternionic pair; they generate a subalgebra $\mathbb{H}\langle a, i_n\rangle$ of A_n isomorphic to the quaternions with additive basis consisting of 1, a, i_n, and $i_n a$.

Consider the subalgebra $B = \mathbb{H}\langle a, i_n\rangle \times \mathbb{H}\langle a, i_n\rangle$ of A_{n+1}. Note that B is isomorphic to the octonions, and $(\alpha a, \beta a)$ belongs to B. Since the octonions have no zero-divisors, Lemma 9.12 implies that $\mathrm{Ann}(\alpha a, \beta a)$ is contained in $B^\perp$.

Our goal is to find all x and y satisfying the equation $(\alpha a, \beta a)(x, y) = (0, 0)$. The previous paragraph says that x and y must belong to $\mathbb{H}\langle a, i_n\rangle^\perp$. In particular, x and y are orthogonal to αa and βa.

The equation $(\alpha a, \beta a)(x, y) = (0, 0)$ is equivalent to the pair of equations $(\alpha a)x + y(\beta a) = 0$ and $y(\alpha a) - (\beta a)x = 0$, since a, x, and y are all imaginary.

We'll work with the first equation first. Since αa and x anti-commute, we get $-x(\alpha a) + y(\beta a) = 0$. Lemma 5.7 implies that αa equals $a\alpha^*$ and βa equals $a\beta^*$, so we obtain $-x(a\alpha^*) + y(a\beta^*) = 0$. Next use Lemma 5.8 to get $a(x\alpha^*) - a(y\beta^*) = 0$. Finally, use Lemma 5.7 again and factor to obtain $a(\alpha x - \beta y) = 0$. Therefore, the first equation is equivalent to the condition that $\alpha x - \beta y$ belongs to $\mathrm{Ann}(a)$.

For the second equation, use similar arguments to get $-a(\beta x + \alpha y) = 0$. Therefore, the second equation is equivalent to the condition that $\beta x + \alpha y$ belongs to $\mathrm{Ann}(a)$.

Since $\alpha^2 + \beta^2$ is not zero, it follows that x and y both belong to $\mathrm{Ann}(a)$, as desired. □

The previous theorem handles a large class of zero-divisors of the form $(\alpha a, \beta a)$, where α and β belong to $\mathbb{C}_n$ while a belongs to $\mathbb{C}_n^\perp$. However, it does not include the situation where $\alpha^2 + \beta^2 = 0$, i.e., when β equals $i_n\alpha$ or $-i_n\alpha$. The following theorem takes care of these remaining cases.

Theorem 10.2. *If a is a vector in $\mathbb{C}_n^\perp$, then $\mathrm{Ann}(a, i_n a)$ is equal to*

$$\{(x, i_n x) : x \in \mathrm{Ann}(a)\} \oplus \{(y, -i_n y) : y \in \mathbb{H}\langle a, i_n\rangle^\perp\},$$

and $\mathrm{Ann}(a, -i_n a)$ is equal to

$$\{(x, -i_n x) : x \in \mathrm{Ann}(a)\} \oplus \{(y, i_n y) : y \in \mathbb{H}\langle a, i_n\rangle^\perp\},$$

In particular, the dimensions of $\mathrm{Ann}(a, i_n a)$ and of $\mathrm{Ann}(a, -i_n a)$ are both equal to $2^n - 4 + \dim \mathrm{Ann}(a)$.

Proof. We prove the theorem for $\mathrm{Ann}(a, i_n a)$; the proof for $\mathrm{Ann}(a, -i_n a)$ is identical (or one can use the automorphism of A_n which fixes A_{n-1} pointwise and interchanges i_n and $-i_n$).

By direct computation, one can verify that $(x, i_n x)$ belongs to $\mathrm{Ann}(a, i_n a)$ when $ax = 0$ and that $(y, -i_n y)$ belongs to $\mathrm{Ann}(a, i_n a)$ when y belongs to $\mathbb{H}\langle a, i_n\rangle^{\perp}$.

Now suppose that (z, w) belongs to $\mathrm{Ann}(a, i_n a)$. Write (z, w) in the form $(x, i_n x) + (y, -i_n y)$, where x equals $(z - i_n w)/2$ and y equals $(z + i_n w)/2$. We want to show that x belongs to $\mathrm{Ann}(a)$ and that y belongs to $\mathbb{H}\langle a, i_n\rangle^{\perp}$.

As in the proof of Theorem 10.1, we know that z and w must belong to $\mathbb{H}\langle a, i_n\rangle^{\perp}$. It follows from Lemma 3.8 that $i_n w$ is also in $\mathbb{H}\langle a, i_n\rangle^{\perp}$. This shows that y belongs to $\mathbb{H}\langle a, i_n\rangle^{\perp}$.

We also know from the proof of Theorem 10.1 that $z - i_n w$ belongs to $\mathrm{Ann}(a)$. That is, $x \in \mathrm{Ann}(a)$. □

11. Anti-associators and zero-divisors

In this section, our goal is to describe $\mathrm{Ann}(a, b)$ when a and b are a quaternionic pair of alternative vectors. Our description will be in terms of the anti-associator $\mathrm{Ass}'[a, b]$. There appears to be a connection between this result and some statements in [M2]; Moreno was the first person to study zero-divisors which are pairs of alternative vectors.

Proposition 11.1. *Let a and b be a quaternionic pair of alternative vectors. Then* $\mathrm{Ann}(a, b)$ *is equal to*

$$\{(x, (ax)b) : x \in \mathrm{Ass}'[a, b]\}.$$

In particular, the dimension of $\mathrm{Ann}(a, b)$ *equals* $\dim \mathrm{Ass}'[a, b]$.

Proof. Suppose that $(a, b)(x, y) = (0, 0)$. Since we know x and y must be imaginary, this equation is equivalent to the two equations $ax = -yb$ and $xb = ay$ (see Lemma 9.11). Multiply the first equation by b on the right to obtain $y = (ax)b$, and multiply the second equation by a on the left to obtain $y = -a(xb)$. □

Corollary 11.2. *Let a and b be orthogonal imaginary vectors in A_3 such that $||a|| = ||b|| \neq 0$. Then we have*

$$\mathrm{Ann}(a, b) = \{(x, -(ab)x/||ab||) : x \in \mathbb{H}\langle a, b\rangle^{\perp}\}.$$

Proof. By rescaling (a, b), we may assume that a and b are unit vectors.

Note that a and b are automatically alternative because every vector in A_3 is alternative. So a and b are automatically a quaternionic pair. Therefore, Proposition 11.1 applies. We have replaced $\mathrm{Ass}'[a, b]$ with $\mathbb{H}\langle a, b\rangle^{\perp}$ with the help of Lemma 8.5.

Now we just have to do some octonionic arithmetic and observe that $(ax)b$ equals $-(ab)x$; here we need that x is orthogonal to $\mathbb{H}\langle a, b\rangle$, and we are using Lemma 8.5. □

12. Zero-divisors in A_4

The algebras A_0, A_1, A_2, and A_3 are all normed algebras. Therefore, they have no zero-divisors. However, A_n does have zero-divisors when $n \geq 4$. The purpose of this section is to thoroughly describe the pairs of non-zero vectors (x, y) in A_4 such that $xy = 0$.

The results of this have been known for nearly two decades, but their provenance is a bit complicated. The main ingredients can be found in [ES], but it does not seem that Eakin and Sathaye were aware of this. Cohen states the main result without proof in [Co], and also asserts that Paul Yiu had told him about a different, also unpublished proof. To our knowledge, the first complete published proofs of these results are [M1, Corollary 2.14] and [KY, Theorem 3.2.3].

Proposition 12.1. *A vector (a, b) in A_4 is a zero-divisor if and only if a and b are orthogonal imaginary vectors such that $||a|| = ||b||$.*

Proof. One direction is Corollary 11.2. For the other direction, suppose that (a, b) is a zero-divisor.

First of all, Lemma 9.5 says that a and b are both imaginary. Moreover, a and b must both be non-zero. For example, Theorem 10.1 says that the dimension of $\mathrm{Ann}(a, 0)$ equals $2 \dim \mathrm{Ann}(a)$. But $\mathrm{Ann}(a)$ is trivial because A_3 has no zero-divisors, so $\mathrm{Ann}(a, 0)$ is also trivial. The same argument applies to $\mathrm{Ann}(0, b)$.

There exist x and y such that $(a, b)(x, y) = (0, 0)$. Lemma 9.11 says that $ax = -yb$ and $bx = ya$. Now A_3 is a normed algebra, so $||a|| \cdot ||x|| = ||y|| \cdot ||b||$ and $||b|| \cdot ||x|| = ||y|| \cdot ||a||$. Using that a, b, x, and y are all non-zero (see the previous paragraph), it follows that $||a|| = ||b||$ (and also $||x|| = ||y||$).

It remains to show that a and b are orthogonal. If we take the equation $ax = -yb$, multiply by x on the right, and use that A_3 is alternative, we obtain $||x||^2 a = (yb)x$. Similarly, if we start with $bx = ya$, we obtain $||y||^2 a = -y(bx)$. In particular, $2||x||^2 a$ equals $A_{y,x} b$ (using that $||x|| = ||y||$).

This now allows us to compute:

$$\langle a, b \rangle = \frac{1}{2||x||^2} \langle A_{y,x} b, b \rangle = 0$$

where the last equality follows from the anti-symmetry of $A_{y,x}$ (see Lemma 8.2). Thus, a and b are orthogonal. □

Proposition 12.1 allows us to describe geometrically the space of all unit zero-divisors in A_4. It is homeomorphic to the Stiefel manifold $V_2(\mathbb{R}^7)$ of orthonormal pairs of vectors in $\mathbb{R}^7$. Here, $\mathbb{R}^7$ arises as the space of imaginary vectors in A_3.

One can see directly from Proposition 12.1 and Corollary 11.2 that if x is any vector in A_4, then $\mathrm{Ann}(x)$ has real dimension 0 or 4, depending on whether x is a zero-divisor or not. Compare this observation with Theorem 9.8 and Proposition 9.10 above.

Corollary 11.2 allows one to describe geometrically the space of all pairs of unit vectors in A_4 whose product is zero. It turns out to be homeomorphic to the 14-dimensional Lie group G_2 (see Section 7).

13. Existence of annihilators with various dimensions

In these last three sections we finally prove the main results stated in the introduction.

Proposition 13.1. *Let d be any non-negative integer less than 2^{n-1} that is congruent to 0 modulo 4. There exists a vector x in A_n such that the dimension of $\mathrm{Ann}(x)$ is equal to d.*

Proof. The proof is by induction. The base cases $n \leq 3$ are trivial.

First suppose that d is congruent to 0 modulo 8. Write $d = 8k$. By induction, we may find a vector y in A_{n-1} such that the dimension of $\mathrm{Ann}(y)$ is $4k$. Then the dimension of $\mathrm{Ann}(y, 0)$ is $8k$ by Theorem 10.1.

Now suppose that d is congruent to 4 modulo 8. By Proposition 8.11, we may choose imaginary standard basis vectors a and b such that $\dim \mathrm{Ass}'[a, b] = d$. Then Proposition 11.1 tells us that $\dim \mathrm{Ann}(a, b) = d$. □

Theorem 13.2. *Let $n \geq 1$. There exists a vector x of A_n such that $\mathrm{Ann}(x)$ has dimension d if and only if $0 \leq d \leq 2^n - 4n + 4$ and d is congruent to 0 modulo 4.*

Proof. One direction is a combination of Theorem 9.8 and Proposition 9.10.

For the other direction, the proof is by induction on n. The bases cases are $n \leq 3$, which require nothing. Suppose the result has been proved for A_{n-1}, where $n \geq 3$, and let d satisfy the given conditions. If $d < 2^{n-1}$, then Proposition 13.1 implies the existence of the desired x.

Now assume that $d \geq 2^{n-1}$. Use induction to choose an a in A_{n-1} such that $\dim \mathrm{Ann}(a) = d - 2^{n-1} + 4$. Note that $d - 2^{n-1} + 4 \leq 2^{n-1} - 4(n-1) + 4$ because $d \leq 2^n - 4n + 4$.

Let $x = (a, i_{n-1}a)$. Then Theorem 10.1 implies that $\dim \mathrm{Ann}(x) = d$. □

14. Top-dimensional annihilators in A_5

The first few Cayley-Dickson algebras have no non-trivial zero-divisors. The fourth one has zero-divisors, but these have been well-understood for some years now—largely because they are homogeneous in a variety of ways. One consequence of this homogeneity is the fact that each zero-divisor has a 4-dimensional annihilator.

We have now demonstrated that no analogous fact holds for A_n with $n \geq 5$. Indeed, we have shown exactly what dimensions of annihilators occur in A_n for all n; our results tell us that as n increases, the number of possibilities for the dimension of an annihilator in A_n grows exponentially.

This indicates that the analysis of the space of zero-divisors in A_n will be quite complicated, but it also gives us a hint as to how that analysis might be carried out. Namely, write $ZD(A_n) = \{x \in A_n : ||x|| = 1,\ \mathrm{Ann}(x) \neq 0\}$. We can partition this space into the subsets

$$ZD_k(A_n) = \{x \in A_n : ||x|| = 1,\ \dim \mathrm{Ann}(x) = k\}$$

where $k = 0, 4, \ldots, 2^n - 4n + 4$. This decomposition of A_n is a stratification in the sense that

$$\overline{ZD_k(A_n)} = \bigcup_{k' \geq k} ZD_{k'}(A_n),$$

where the union is disjoint. At present, it seems that the most accessible approach to the study of the zero-divisor locus in A_n is to analyze one $ZD_k(A_n)$ at a time. We conclude this article with the beginning of this program, namely a complete determination of $ZD_{2^n-4n+4}(A_n)$ for all n.

Definition 14.1. *Let $n \geq 2$. An element of A_n is a* ***top-dimensional zero-divisor*** *if its annihilator has dimension $2^n - 4n + 4$. Let T_n be the space of top-dimensional zero-divisors in A_n that have norm 1.*

Notice that T_n is nothing other than $ZD_{2^n-4n+4}(A_n)$. Proposition 9.10 tells us that annihilators of zero-divisors have dimension at most $2^n - 4n + 4$, so our terminology makes sense.

Note that T_2 and T_3 are homeomorphic to S^3 and S^7 respectively because every annihilator in A_2 or A_3 is zero-dimensional. We explained in Section 12 that T_4

is homeomorphic to the Stiefel manifold $V_2(\mathbb{R}^7)$ of orthonormal pairs of imaginary unit vectors in A_3.

In this section we will study T_5, which is the space of (necessarily imaginary) unit vectors x in A_5 such that $\text{Ann}(x)$ is 16-dimensional. This will serve as the base case of an induction carried out in the next section, where we describe T_n for all $n \geq 5$.

Lemma 14.2. *Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be zero-divisors in A_4. Then $\text{Ann}(a)$ and $\text{Ann}(b)$ intersect non-trivially if and only if $a_1a_2/||a_1a_2|| = b_1b_2/||b_1b_2||$.*

Proof. Note that by Proposition 12.1 we know a_1 and a_2 are orthogonal and imaginary, as are b_1 and b_2. From Corollary 11.2, the annihilators intersect if and only if there exists a non-zero x in $\mathbb{H}\langle a_1, a_2\rangle^\perp \cap \mathbb{H}\langle b_1, b_2\rangle^\perp$ such that $-(a_1a_2)x/||a_1a_2|| = -(b_1b_2)x/||b_1b_2||$. Thus, we have

$$(a_1a_2/||a_1a_2|| - b_1b_2/||b_1b_2||)\, x = 0.$$

Since the octonions have cancellation, this equation has a non-zero solution in x if and only if the left-hand factor is zero. □

Recall from Proposition 12.1 that if (a, b) is a zero-divisor in A_4, then $a/||a||$ and $b/||b||$ form a quaternionic pair in A_3.

Lemma 14.3. *Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be zero-divisors in A_4. Then $\text{Ann}(a)$ equals $\text{Ann}(b)$ if and only if $a_1a_2/||a_1a_2|| = b_1b_2/||b_1b_2||$ and $\mathbb{H}\langle a_1/||a_1||, a_2/||a_2||\rangle = \mathbb{H}\langle b_1/||b_1||, b_2/||b_2||\rangle$.*

Proof. First note that we are free to multiply a and b by real scalars. Since we already know $||a_1|| = ||a_2||$ and $||b_1|| = ||b_2||$ (by Proposition 12.1), we can assume that $||a_1|| = ||a_2|| = ||b_1|| = ||b_2|| = 1$. This will simplify the notation somewhat. Under this assumption, we must show that $\text{Ann}(a) = \text{Ann}(b)$ if and only if $a_1a_2 = b_1b_2$ and $\mathbb{H}\langle a_1, a_2\rangle = \mathbb{H}\langle b_1, b_2\rangle$.

First suppose that $\text{Ann}(a)$ and $\text{Ann}(b)$ are equal. It follows from Corollary 11.2 that $\mathbb{H}\langle a_1, a_2\rangle^\perp = \mathbb{H}\langle b_1, b_2\rangle^\perp$ and therefore $\mathbb{H}\langle a_1, a_2\rangle = \mathbb{H}\langle b_1, b_2\rangle$. Also, Lemma 14.2 says that $a_1a_2 = b_1b_2$.

Now suppose that $a_1a_2 = b_1b_2$ and $\mathbb{H}\langle a_1, a_2\rangle = \mathbb{H}\langle b_1, b_2\rangle$. It then follows from Corollary 11.2 that $\text{Ann}(a)$ and $\text{Ann}(b)$ are equal. □

Proposition 14.4. *Suppose that (a, b) is an element of A_5 such that a and b are zero-divisors in A_4 with $\text{Ann}(a) = \text{Ann}(b)$. Then $b = \alpha a$ for some $\alpha \in \mathbb{C}_4$. One has that $\dim \text{Ann}(a, b) = 16$ if and only if $\alpha = \pm i_4$, and $\dim \text{Ann}(a, b) = 8$ otherwise.*

Proof. Let $a = (a_1, a_2)$. After rescaling a and b, we may assume that $||a_1|| = ||a_2|| = 1$. Up to an automorphism of A_3, we may additionally assume that $a_1 = i$ and $a_2 = j$.

Lemma 14.3 implies that b_1 and b_2 belong to $\mathbb{H}$ and that $b_1b_2 = ||b_1b_2||k$. Since b_1 and b_2 must both be imaginary, as well as orthogonal, it follows that $b_1 = Pi - Qj$ and $b_2 = Qi + Pj$ for some real numbers P and Q.

Note that i_4a equals $(-j, i)$. Therefore, b equals $Pa + Qi_4a = (P + Qi_4)a$. Now apply Theorem 10.1 to find that $\text{Ann}(a, (P + Qi_4)a)$ is equal to $\text{Ann}\, a \times \text{Ann}\, a$ (which has dimension 8) if $P + Qi_4 \neq \pm i_4$. In case $P + Qi_4 = \pm i_4$, Theorem 10.2 applies. □

Lemma 14.5. *Suppose* (a, b) *is an element of* T_5*. Then* a *and* b *are zero-divisors in* A_4 *and* $\mathrm{Ann}(a)$ *and* $\mathrm{Ann}(b)$ *intersect non-trivially.*

Proof. By Lemma 9.9,

$$16 = \dim \mathrm{Ann}(a, b) \leq 16 - 2 + \dim(\mathrm{Ann}(a) \cap \mathrm{Ann}(b)).$$

From this it follows that $\mathrm{Ann}(a) \cap \mathrm{Ann}(b)$ is non-zero. □

Theorem 14.6. *Let* a *and* b *be elements of* A_4 *such that* $\mathrm{Ann}\, a \cap \mathrm{Ann}\, b \neq 0$ *and* $(a, b) \neq (0, 0)$*. Then*

$$\dim \mathrm{Ann}(a, b) = \begin{cases} 16 & \text{if } b = \pm i_4 a, \\ 12 & \text{if } a \text{ is orthogonal to } b,\ ||a|| = ||b||, \text{ and } b \neq \pm i_4 a, \\ 8 & \text{otherwise.} \end{cases}$$

Before proving this theorem we note the following immediate corollary:

Corollary 14.7. *The space* T_5 *is homeomorphic to* $V_2(\mathbb{R}^7) \amalg V_2(\mathbb{R}^7)$*.*

Proof. We will show that T_5 is the disjoint union of the spaces

$$X_+ = \{(a, i_4 a) : ||a|| = \tfrac{1}{\sqrt{2}},\ a \text{ is a zero-divisor in } A_4\}$$

and

$$X_- = \{(a, -i_4 a) : ||a|| = \tfrac{1}{\sqrt{2}}, a \text{ is a zero-divisor in } A_4\}.$$

As explained in Section 12, each of these spaces is homeomorphic to $V_2(\mathbb{R}^7)$.

First observe that both X_+ and X_- are contained in T_5 because of Theorem 10.2. Next we will show that X_+ and X_- are disjoint. If $(a, i_4 a) = (b, -i_4 b)$, then it follows that $2 i_4 a = 0$. Since i_4 is alternative, this implies that $a = 0$, which prohibits $(a, i_4 a)$ from belonging to X_+.

Finally, we must show that every element of T_5 is contained in X_+ or X_-. Suppose that (a, b) is an arbitrary element of T_5. By Lemma 14.5, a and b are zero-divisors in A_4 whose annihilators intersect nontrivially. Since $\dim \mathrm{Ann}(a, b) = 16$, we have by Theorem 14.6 that $b = \pm i_4 a$. □

Our final task in this section is the following:

Proof of Theorem 14.6. Let a and b be elements A_4 which are not both zero, and whose annihilators intersect nontrivially. If either a or b is zero then we are in the third case from the statement of the theorem, and the fact that the annihilator is 8-dimensional follows from Theorem 10.1. So we may as well assume both a and b are nonzero.

We can write $a = (a_1, a_2)$ and $b = (b_1, b_2)$, where a_1, a_2, b_1, and b_2 are all octonions. Since a and b are zero-divisors, Proposition 12.1 implies that a_1, a_2, b_1, and b_2 are all imaginary, that a_1 and a_2 are perpendicular, that b_1 and b_2 are perpendicular, and that $||a_1|| = ||a_2||$ and $||b_1|| = ||b_2||$. After scaling a and b by the same constant, we can assume $||a_1|| = ||a_2|| = 1$. Since a_1 and a_2 are orthogonal imaginary unit vectors, we know that $a_1 a_2$ is also an imaginary unit vector. Therefore, up to automorphism of A_3, we may assume that $a_1 a_2 = k$, which implies by Lemma 14.2 that $b_1 b_2$ is a scalar multiple of k. Note that a_1, a_2, b_1, and b_2 are all orthogonal to k since a_1 and a_2 are orthogonal to $a_1 a_2$, while b_1 and b_2 are orthogonal to $b_1 b_2$.

Consider the linear span V of $\{1, a_1, a_2, b_1, b_2, a_1a_2, b_1b_2\}$; its dimension is at most 6. Thus, up to automorphism, we may assume that i is orthogonal to V. Since V is invariant under left multiplication by k, j is orthogonal to V as well. We have now shown that a_1, a_2, b_1, and b_2 are all orthogonal to $\mathbb{H}$. Thus, once again up to automorphism, we may assume that $a_1 = t$. This implies $a_2 = kt$, as $a_1a_2 = k$.

Since b_1 is orthogonal to $\mathbb{H}$, we have that $b_1 = \alpha t$ for some element α of $\mathbb{H}$. It then follows that $b_2 = (\alpha k)t$ because b_1b_2 is a scalar multiple of k and because $||b_1|| = ||b_2||$. So we have $a = (t, kt)$ and $b = (\alpha t, (\alpha k)t)$. To relate our present situation to the three cases in the statement of the theorem, note that $b = \pm i_4 a$ if and only if $\alpha = \pm k$; a is orthogonal to b if and only if α is imaginary; and $||a|| = ||b||$ if and only if $||\alpha|| = 1$.

Consider now the subalgebra $\mathbb{H}$ of A_3; we may use it to form the algebra

$$\Omega = \mathbb{H} \times \mathbb{H} \subset A_4.$$

This in turn gives rise to a 16-dimensional subalgebra $\Omega \times \Omega$ of A_5. By assumption, (a, b) is an element of $\Omega^\perp \times \Omega^\perp$. Moreover, one easily sees that in addition to the automatic

$$(\Omega \times \Omega) \cdot (\Omega^\perp \times \Omega^\perp) \subset \Omega^\perp \times \Omega^\perp,$$

we have the formula

$$(\Omega^\perp \times \Omega^\perp) \cdot (\Omega^\perp \times \Omega^\perp) \subset \Omega \times \Omega.$$

Similarly to Lemma 9.12, this in turn implies that $\mathrm{Ann}(a, b)$ is isomorphic to

$$(\mathrm{Ann}(a, b) \cap (\Omega \times \Omega)) \oplus (\mathrm{Ann}(a, b) \cap (\Omega^\perp \times \Omega^\perp)).$$

In other words, we must solve the two equations

$$(t, kt, \alpha t, (\alpha k)t) \cdot (x, y, z, w) = (0, 0, 0, 0) \tag{14.8}$$

and

$$(t, kt, \alpha t, (\alpha k)t) \cdot (xt, yt, zt, wt) = (0, 0, 0, 0), \tag{14.9}$$

where x, y, z, and w belong to $\mathbb{H}$.

At this point, the proof becomes rather unpleasant. First of all, we can expand out equation (14.8). We do this by using the inductive definition of multiplication in Cayley-Dickson algebras twice, and obtain

$$\begin{aligned} x^* - ky^* - \alpha z^* + \alpha k w^* &= 0 \\ kx + y - \alpha k z - \alpha w &= 0 \\ \alpha x + \alpha k y^* + z + k w^* &= 0 \\ \alpha k x^* - \alpha y + k z^* - w &= 0. \end{aligned} \tag{S1}$$

When doing the same for equation (14.9) we get

$$\begin{aligned} -x^* + ky - \alpha^* z - w^* \alpha k &= 0 \\ x^* k - y - z^* \alpha k + \alpha^* w &= 0 \\ x^* \alpha - k \alpha^* y - z - w^* k &= 0 \\ -x^* \alpha k + \alpha^* y - z^* k + w &= 0. \end{aligned} \tag{S2}$$

Each of these systems corresponds to a system of 16 real equations in 16 unknowns, and a little thought shows that the coefficient matrices are negative transposes of each other (see the very end of the proof for more information about this). So the solution spaces in (14.8) and (14.9) have the same dimension.

We will concentrate on solving (14.8). We know that all the solutions have x and z imaginary by Lemma 9.5, because $((x, y), (z, w))$ will be a zero-divisor in A_5. We can take advantage of this fact to simplify the four quaternionic equations in (S1). We obtain the new system

$$\begin{aligned} -x - ky^* + \alpha z + \alpha k w^* &= 0 \\ kx + y - \alpha k z - \alpha w &= 0 \\ \alpha x + \alpha k y^* + z + k w^* &= 0 \\ -\alpha k x - \alpha y - kz - w &= 0 \\ \mathrm{Re}(x) = \mathrm{Re}(z) &= 0. \end{aligned}$$

By adding appropriate multiples of the first and third equations (and of the second and fourth equations), the system can be simplified to

$$\begin{aligned} -x - ky^* + \alpha z + \alpha k w^* &= 0 \\ kx + y - \alpha k z - \alpha w &= 0 \\ (\alpha^2 + 1)z + (\alpha^2 + 1)k w^* &= 0 \\ (\alpha^2 + 1)kz + (\alpha^2 + 1)w &= 0 \\ \mathrm{Re}(x) = \mathrm{Re}(z) &= 0. \end{aligned}$$

Case 1: $\boldsymbol{\alpha^2 + 1 \neq 0}$. In this case we may cancel α^2+1 from the last two equations (since $\mathbb{H}$ is a division algebra) and obtain $z + kw^* = 0 = kz + w$. Together with $\mathrm{Re}(z) = 0$, this is equivalent to $w \in \langle i, j\rangle$ and $z = kw$. Plugging this into the first two equations then gives $x + ky^* = 0 = kx + y$. Together with $\mathrm{Re}(x) = 0$, the same analysis shows that $y \in \langle i, j\rangle$ and $x = ky$. So we have a 4-dimensional solution space for (14.8).

Case 2: $\boldsymbol{\alpha^2 + 1 = 0}$ (equivalently, α is imaginary and has norm 1). In this case the third and fourth equations disappear. We use the second equation to solve for y, and plug this into the first equation. We get (remembering that α, x, and z are imaginary):

$$\begin{aligned} (kxk - x) + (\alpha z + kzk\alpha) + (\alpha k w^* + k w^* \alpha) &= 0 \\ \mathrm{Re}(x) = \mathrm{Re}(z) &= 0, \end{aligned}$$

and y is eliminated.

Note that if q is an imaginary quaternion of norm 1, then $x - qxq = 2\pi_{1,q}(x)$, where $\pi_{1,q}$ denotes orthogonal projection onto the subspace $\langle 1, q\rangle$ (it suffices to check this claim when $q = i$). The analysis now divides up into two more cases.

Subcase 1: $\boldsymbol{\alpha = \pm k}$. The first equation becomes $(kxk-x) \pm (kw^*k - w^*) = 0$. So we have $\pi_{1,k}(x) = \mp\pi_{1,k}(w^*)$ and $\mathrm{Re}(x) = \mathrm{Re}(z) = 0$. This has an 8-dimensional solution set: x and z can be any imaginary quaternions, and there are two degrees of freedom left in choosing w.

Subcase 2: $\boldsymbol{\alpha \neq \pm k}$. Write $\alpha = r\beta + sk$ where β is orthogonal to k, $||\beta|| = 1$, and $r, s \in \mathbb{R}$; so $r^2 + s^2 = 1$, and $r \neq 0$. Substituting into the first equation and re-arranging, we have

$$(kxk - x) + r[\beta z - (k\beta)(\beta z)(k\beta)] + r[\beta k w^* - \beta(\beta k w^*)\beta] + s(kw^*k - w^*) = 0$$

(remember that $\beta^2 = -1$). Dividing by 2, this becomes

$$-\pi_{1,k}(x) + r\pi_{1,\beta k}(\beta z) + r\pi_{1,\beta}(\beta k w^*) - s\pi_{1,k}(w^*) = 0.$$

But note that $\{1, k, \beta, \beta k\}$ is an orthonormal basis for $\mathbb{H}$, and so by separating out each component, the above equation can be distilled into:

$$\begin{aligned} -\pi_1 x + r\pi_1(\beta z) + r\pi_1(\beta k w^*) - s\pi_1(w^*) &= 0 \\ -\pi_k x - s\pi_k(w^*) &= 0 \\ r\pi_{\beta k}(\beta z) &= 0 \\ r\pi_\beta(\beta k w^*) &= 0. \end{aligned}$$

Note that $\pi_1(\beta z) = \beta\pi_\beta z$, $\pi_1(\beta k w^*) = \beta k \pi_{\beta k}(w^*)$, $\pi_{\beta k}(\beta z) = \beta\pi_k z$, and $\pi_\beta(\beta k w^*) = \beta k \pi_k(w^*)$. Using that $r\beta \neq 0$, together with $\mathrm{Re}(x) = \mathrm{Re}(z) = 0$, we are finally reduced to the equations

$$\begin{aligned} \pi_\beta z - k\pi_{\beta k} w + r^{-1} s\beta\pi_1 w &= 0 \\ -\pi_k x + s\pi_k w &= 0 \\ \pi_k z = \pi_k w = \mathrm{Re}(x) = \mathrm{Re}(z) &= 0. \end{aligned}$$

Since $\pi_k w = 0$, the second equation reduces to $\pi_k x = 0$. In the end, we have three degrees of freedom for w, two for x, and then one for z, yielding a six-dimensional solution space.

We have now handled all of the cases necessary for the proof. We will add a few comments about the two systems (S1) and (S2). We claimed earlier that these gave 16×16 real matrices which are negative transposes of each other. To see why, let $C\colon \mathbb{H} \to \mathbb{H}$ denote the conjugation operator, so $C(q) = q^*$. We will identify C with a 4×4 matrix using the standard basis for $\mathbb{H}$, and where we have matrices acting on the left. In the same way we identify L_q and R_q with 4×4 matrices.

The system (S1) gives rise to a 16×16 real matrix which can be written in block form as

$$\begin{bmatrix} C & -L_k C & -L_\alpha C & L_\alpha L_k C \\ L_k & I & -L_\alpha L_k & -L_\alpha \\ L_\alpha & L_\alpha L_k C & I & L_k C \\ L_\alpha L_k C & -L_\alpha & L_k C & -I \end{bmatrix},$$

whereas the system (S2) gives the matrix

$$\begin{bmatrix} -C & L_k & -L_{\alpha^*} & -R_k R_\alpha C \\ R_k C & -I & -R_k R_\alpha C & L_{\alpha^*} \\ R_\alpha C & -L_k L_{\alpha^*} & -I & -R_k C \\ -R_k R_\alpha C & L_{\alpha^*} & -R_k C & I \end{bmatrix}.$$

Note that we have $(L_q)^T = L_{q^*}$ and $(R_q)^T = R_{q^*}$ by Lemma 3.4. Also note that $CR_q = L_{q^*}C$, by the formula $(xq)^* = q^* x^*$, and that C itself is diagonal (so $C^T = C$). Using these ideas, it follows that the two 16×16 matrices are indeed negative transposes of each other. □

15. Top-dimensional zero-divisors

The goal of this section is to completely determine the spaces T_n for all n. We have already computed some low-dimensional cases. Our approach will be by induction, and we'll start with several preliminary calculations.

Lemma 15.1. *Suppose that (a,b) is a top-dimensional zero-divisor in A_n. Then a and b are top-dimensional zero-divisors in A_{n-1}, and the dimension of* $\mathrm{Ann}(a)\cap\mathrm{Ann}(b)$ *is at least* $2^{n-1}-4n+6$.

Proof. The proof of the second claim follows from Lemma 9.9 and arithmetic.

For the first claim, note that the second part implies that the dimensions of $\mathrm{Ann}(a)$ and $\mathrm{Ann}(b)$ are at least $2^{n-1}-4n+6$. But these dimensions must be multiples of 4 and no bigger than $2^{n-1}-4n+8$ by Theorem 9.8 and Proposition 9.10, so they have to be equal to $2^{n-1}-4n+8$. □

Lemma 15.2. *Let a and b be non-zero vectors in $\mathbb{C}_{n-2}^{\perp}$, and let x denote the element $((a,i_{n-2}a),(b,-i_{n-2}b))$ of A_n. The dimension of* $\mathrm{Ann}(x)$ *is at most* $2^n-8n+20$.

Proof. Recall that Theorem 10.2 gives a complete description of $\mathrm{Ann}(a,i_{n-2}a)$ and $\mathrm{Ann}(b,-i_{n-2}b)$. It follows readily that $\mathrm{Ann}(a,i_{n-2}a)\cap\mathrm{Ann}(b,-i_{n-2}b)$ is equal to

$$\{(x,i_{n-2}x): x\in\mathrm{Ann}\,a\cap\mathbb{H}\langle b,i_{n-2}\rangle^{\perp}\}\oplus\{(y,-i_{n-2}y): y\in\mathrm{Ann}\,b\cap\mathbb{H}\langle a,i_{n-2}\rangle^{\perp}\}.$$

So this intersection is isomorphic to

$$(\mathrm{Ann}(a)\cap\mathbb{H}\langle b,i_{n-2}\rangle^{\perp})\oplus(\mathrm{Ann}(b)\cap\mathbb{H}\langle a,i_{n-2}\rangle^{\perp}),$$

which is a subspace of $\mathrm{Ann}(a)\oplus\mathrm{Ann}(b)$. This last space has dimension at most $2^{n-1}-8n+24$ by Proposition 9.10, so $\mathrm{Ann}(a,i_{n-2}a)\cap\mathrm{Ann}(b,-i_{n-2}b)$ has dimension at most $2^{n-1}-8n+24$.

Now Lemma 9.9 implies that $\mathrm{Ann}(x)$ has dimension at most $2^n-8n+22$. Finally, recall from Theorem 9.8 that the dimension of $\mathrm{Ann}(x)$ is a multiple of 4. □

A similar argument shows that the dimension of the annihilator of the element $((a,-i_{n-2}a),(b,i_{n-2}b))$ is also at most $2^n-8n+20$. In particular, when $n\geq 5$, elements of the form $((a,-i_{n-2}a),(b,i_{n-2}b))$ and $((a,i_{n-2}a),(b,-i_{n-2}b))$ are never top-dimensional zero-divisors of A_n.

Lemma 15.3. *Let a and b be non-zero vectors in $\mathbb{C}_n^{\perp}$. Then $\mathbb{H}\langle a/||a||,i_n\rangle=\mathbb{H}\langle b/||b||,i_n\rangle$ if and only if $b=\alpha a$ for some element α of $\mathbb{C}_n$.*

Proof. Without loss of generality, we may assume that a and b are unit vectors.

First suppose that $b=\alpha a$. Then b is an element of $\mathbb{H}\langle a,i_n\rangle$, so $\mathbb{H}\langle b,i_n\rangle$ is contained in $\mathbb{H}\langle a,i_n\rangle$. On the other hand, $a=\alpha^*b/||\alpha||^2$, so a is an element of $\mathbb{H}\langle b,i_n\rangle$ and $\mathbb{H}\langle a,i_n\rangle$ is contained in $\mathbb{H}\langle b,i_n\rangle$.

Now suppose that $\mathbb{H}\langle a,i_n\rangle=\mathbb{H}\langle b,i_n\rangle$. Then b is a linear combination of 1, i_n, a, and i_na. However, since b is orthogonal to $\mathbb{C}_n$, b is in fact a linear combination of a and i_na. This is equivalent to saying that $b=\alpha a$ for some α in $\mathbb{C}_n$. □

Lemma 15.4. *Let a and b be non-zero elements of $\mathbb{C}_n^{\perp}$. The following three conditions are equivalent:*

(1) $\mathrm{Ann}(a,i_na)$ *and* $\mathrm{Ann}(b,i_nb)$ *are equal.*
(2) $\mathrm{Ann}(a,-i_na)$ *and* $\mathrm{Ann}(b,-i_nb)$ *are equal.*
(3) $b=\alpha a$ *for some* α *in* $\mathbb{C}_n$.

Proof. First assume either condition (1) or condition (2). By Theorem 10.2, we know that $\mathbb{H}\langle a,i_n\rangle^{\perp}$ and $\mathbb{H}\langle b,i_n\rangle^{\perp}$ are equal, so Lemma 15.3 applies.

Now assume that $b=\alpha a$. By Lemma 5.4, (b,i_nb) equals $\alpha'(a,i_na)$ for some α' in $\mathbb{C}_{n+1}$. Similarly, $(b,-i_nb)=\alpha'(a,-i_na)$. Lemma 9.7 therefore tells us that condition (1) and condition (2) are both true. □

Lemma 15.5. *Let $a = (a', i_{n-1}a')$ and $b = (b', i_{n-1}b')$ be non-zero elements of $\mathbb{C}_n^\perp$ such that a' and b' are in $\mathbb{C}_{n-1}^\perp$. If $\mathrm{Ann}(a, i_n a) \cap \mathrm{Ann}(b, i_n b)$ has codimension at most 2 in $\mathrm{Ann}(a, i_n a)$, then $b = \alpha a$ for some α in $\mathbb{C}_n$.*

A similar result holds when $a = (a', -i_{n-1}a')$ and $b = (b', -i_{n-1}b')$.

Proof. By Theorem 10.2, we know that $\mathrm{Ann}(a, i_n a) \cap \mathrm{Ann}(b, i_n b)$ is isomorphic to $(\mathrm{Ann}(a) \cap \mathrm{Ann}(b)) \oplus (\mathbb{H}\langle a, i_n\rangle^\perp \cap \mathbb{H}\langle b, i_n\rangle^\perp)$, and $\mathrm{Ann}(a, i_n a)$ is isomorphic to $\mathrm{Ann}(a) \oplus \mathbb{H}\langle a, i_n\rangle^\perp$. Therefore, the space $(\mathrm{Ann}(a) \cap \mathrm{Ann}(b)) \oplus (\mathbb{H}\langle a, i_n\rangle^\perp \cap \mathbb{H}\langle b, i_n\rangle^\perp)$ has codimension at most 2 in $\mathrm{Ann}(a) \oplus \mathbb{H}\langle a, i_n\rangle^\perp$. All of the spaces $\mathrm{Ann}(a)$, $\mathrm{Ann}(b)$, $\mathbb{H}\langle a, i_n\rangle^\perp$, and $\mathbb{H}\langle b, i_n\rangle^\perp$ are $\mathbb{C}_n$-vector spaces. This means that either $\mathbb{H}\langle a, i_n\rangle^\perp = \mathbb{H}\langle b, i_n\rangle^\perp$ or $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ (or both). In the first case, Lemma 15.3 applies.

In the second case, Lemma 15.4 applies, so $b' = \alpha' a'$ for some α' in $\mathbb{C}_{n-1}$. Lemma 5.4 shows that $b = \alpha a$ for some α in $\mathbb{C}_n$. □

Proposition 15.6. *Let $n \geq 5$. If x is a top-dimensional zero-divisor of A_n, then x is of the form $(a, i_n a)$ or $(a, -i_n a)$ for some top-dimensional zero-divisor a of A_{n-1}.*

Proof. The proof is by induction. The base case $n = 5$ is proved in Corollary 14.7. Notice that although the induction establishes that $T_{n+1} = T_n \coprod T_n$ and it happens to be the case that $T_5 = T_4 \coprod T_4$, one cannot take $n = 4$ as the base. This is because the inductive step uses as a hypothesis the way that T_n is built as two copies of T_{n-1}, and it is of course not the case that T_4 is isomorphic to a disjoint union of two copies of T_3. Aesthetically this state of affairs is unfortunate, because it forces us to calculate T_5 directly in Corollary 14.7, whose proof is rather unilluminating.

Assume for induction that the proposition is true for $n - 1$. Suppose that x is a top-dimensional zero-divisor in A_n and write $x = (a, b)$ for some elements a and b of A_{n-1}. Lemma 15.1 says that a and b are top-dimensional zero-divisors of A_{n-1}, so the induction assumption says that a is of the form $(a', i_{n-2}a')$ or $(a', -i_{n-2}a')$ and b is of the form $(b', i_{n-2}b')$ or $(b', -i_{n-2}b')$.

But Lemma 15.2 implies that either:

(1) $a = (a', i_{n-2}a')$ and $b = (b', i_{n-2}b')$, or
(2) $a = (a', -i_{n-2}a')$ and $b = (b', -i_{n-2}b')$.

In either case, Lemma 15.1 says that the codimension of $\mathrm{Ann}(a) \cap \mathrm{Ann}(b)$ in $\mathrm{Ann}(a)$ is at most 2, so Lemma 15.5 applies. It follows that $b = \alpha a$ for some α in $\mathbb{C}_{n-1}$.

Now we know that x equals $(a, \alpha a)$. Theorem 10.1 (and the fact that x is a top-dimensional zero-divisor) implies that α must be i_{n-1} or $-i_{n-1}$. Thus x equals $(a, i_{n-1}a)$ or $(a, -i_{n-1}a)$. □

Theorem 15.7. *Let $n \geq 4$. The space T_n is homeomorphic to 2^{n-4} disjoint copies of $V_2(\mathbb{R}^7)$.*

Proof. The proof is by induction. The base case $n = 4$ was dealt with in Section 12.

Assume for induction that the theorem has been proved for $n - 1$. Proposition 15.6 implies that T_n is homeomorphic to two disjoint copies of T_{n-1}. Note that $(a, i_{n-1}a)$ and $(b, -i_{n-1}b)$ are never equal if and a and b are non-zero—this follows since i_{n-1} is alternative, and therefore is not a zero-divisor. □

Remark 15.8. The above result prompts the following question. Let c be a non-negative multiple of 4, and let $T_c(A_n) = ZD_{2^n-4n+4-c}(A_n)$. Does there exist a positive integer N such that for all $n > N$ the space $T_c(A_n)$ is a disjoint union of two copies of $T_c(A_{n-1})$? The above theorem is the case $c = 0$.

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