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Étale realization on the \mathbb{A}^1 -homotopy theory of schemes

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Abstract

We compare Friedlander's definition of the étale topological type for simplicial schemes to another definition involving realizations of pro-simplicial sets. This can be expressed as a notion of hypercover descent for étale homotopy. We use this result to construct a homotopy invariant functor from the category of simplicial presheaves on the étale site of schemes over S to the category of pro-spaces. After completing away from the characteristics of the residue fields of S , we get a functor from the Morel–Voevodsky \mathbb{A}^1 -homotopy category of schemes to the homotopy category of pro-spaces.

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1. Introduction

In the recent proof of the Milnor conjecture [26], a certain realization functor from the \mathbb{A}^1 -homotopy category of schemes over \mathbb{C} [20] to the ordinary homotopy category of spaces plays a useful role. The basic idea is to detect that a certain map in the stable \mathbb{A}^1 -homotopy category is not homotopy trivial by checking that its image in the ordinary stable homotopy category is not homotopy trivial.

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This analytic realization functor is defined by extending the notion of the underlying analytic space of a complex variety. As defined in [20, Section 3.3], it has two shortcomings. First, it is defined directly on the homotopy categories. It would be much preferable to have a functor on the point-set level that is homotopy invariant and therefore induces a functor on the homotopy categories. This problem was fixed in [6].

The second shortcoming is that analytic realization does not work over fields with positive characteristic. The goal of this paper is to use the étale topological type to avoid this problem. The étale topological type [1,9] is a substitute for the underlying analytic topology of a variety. In characteristic zero, the étale topological type $\text{Et } X$ of a variety X is the pro-finite completion of the underlying analytic space of X . In any characteristic, $\text{Et } X$ carries information about the étale cohomology of X and the algebraic fundamental group of X .

Using a model structure for \mathbb{A}^1 -homotopy theory slightly different than the one in [20], the étale topological type provides a functor from the category of simplicial presheaves on the Nisnevich site of smooth schemes over S to the category of pro-spaces. This functor is a left Quillen functor, which means that it automatically gives a functor on the homotopy categories.

The étale realization functor provides a calculational tool for \mathbb{A}^1 -homotopy theory over fields of positive characteristic. In future work, we hope to take Galois group actions into account to obtain a realization functor into a homotopy category of equivariant pro-spaces. However, the foundations for a suitable equivariant homotopy theory of pro-spaces have not yet been established. We also hope to stabilize our techniques to obtain a functor on stable \mathbb{A}^1 -homotopy theory. Although some progress on the foundations of the homotopy theory of pro-spectra has been made [3,15], it is not yet clear whether these theories are suitable for the current application.

The main tool for establishing the étale realization functor on \mathbb{A}^1 -homotopy theory is the étale hypercover descent theorem for the étale topological type (see Theorem 12). This theorem states that if $U \rightarrow X$ is an étale hypercover of X , then the natural map from the realization of the simplicial pro-space $n \mapsto \text{Et } U_n$ to $\text{Et } X$ is a weak equivalence of pro-spaces. Here the realization is internal to the category of pro-spaces.

This result is similar in spirit to [9, Proposition 8.1], but it differs in an important respect. In [9], the étale topological type is defined for simplicial schemes as well as ordinary schemes. In order to keep definitions straight, we shall write $\text{sEt } U$ for Friedlander's definition of the étale topological type of the simplicial scheme U . It is not obvious that $\text{sEt } U$ is weakly equivalent to the realization $\text{Re}(n \mapsto \text{Et } U_n)$.

The étale hypercover descent theorem is interesting for its own sake, even though our application is to \mathbb{A}^1 -homotopy theory. For example, it is closely related to [7]. Our work can probably be used to give a more conceptual proof of [7, Theorem 9], in which only the properties of the étale topological type are used (and not any special properties of étale K -theory). Descent theorems in general are an important step towards powerful calculational tools in algebraic geometry.

The étale hypercover descent theorem is stated in terms of the realization of a simplicial pro-space. Philosophically, we would prefer a statement involving the homotopy colimit of this simplicial pro-space. It is likely that the realization is in fact weakly equivalent to the homotopy colimit, but we have not been able to prove this. The trouble lies in our incomplete understanding of the homotopy theory of pro-spaces [13].

1.1. Organization

In some sense, the paper is organized backwards. We start with the application to \mathbb{A}^1 -homotopy theory, then discuss the étale hypercover descent theorem for the étale topological type, and finally we provide the details necessary for proving these theorems. The reason for this order is that a reader can learn about the main theorems of this paper without having to drag through the highly technical details of hypercovers, the étale topological type, and the homotopy theory of pro-spaces.

Section 2 begins with a review of simplicial presheaves and their homotopy theory. We assume familiarity with closed model structures. General references on this topic include [11], [12], or [22]. We conform to the conventions of [11] as closely as possible. See also [4] for more details on model structures as applied to simplicial presheaves. The first major result is that the étale realization functor is homotopy invariant on the local projective model structure for simplicial presheaves on the étale site. Specializing to the Nisnevich site of smooth schemes, étale realization is also homotopy invariant with respect to \mathbb{A}^1 -weak equivalences but only after completing away from the characteristics of the residue fields of the base scheme S . The reason for this completion is that $\text{Et } \mathbb{A}^1$ is non-trivial in positive characteristic.

Section 2 closes with a corollary concerning the behavior of the étale topological type on elementary distinguished squares. This result can be interpreted as excision for étale topological types.

Next, Section 3 gives the hypercover descent theorem for the étale topological type. This finishes the main thrust of the paper. The remaining sections are dedicated to developing language and machinery suitable for proving the étale hypercover descent theorem.

Section 4 introduces the language of simplicial schemes that is necessary to work with hypercovers. Section 3 describes rigid covers, which are also essential ingredients. Both of these sections build towards Section 6, which is dedicated to the study of hypercovers and rigid hypercovers. We redefine and clarify some of the constructions concerning the étale topological type that first appeared in [9].

Finally, Section 7 discusses some aspects of the homotopy theory of pro-spaces. See [1,24, Appendix; Exposé 1.8] for background on pro-categories. We use the homotopy theory of pro-spaces as developed in [13]. Some results from [14] on calculating colimits of pro-spaces are also necessary. An n -truncated realization functor for pro-spaces is important because the infinite colimits that are used to construct ordinary realizations are hard to handle in the category of pro-spaces.

1.2. Terminology

We make a few final remarks on terminology. We always mean simplicial sets [17] whenever we refer to spaces.

Some authors define an étale map to be any map $U \rightarrow X$ such that U is a (possibly infinite) disjoint union of schemes U^i and each map $U^i \rightarrow X$ is étale. We shall not follow this convention. For us, all étale covers will be finite unless explicitly stated otherwise. We will refer to *infinite étale covers* when we want to allow infinitely many pieces in an étale cover. This is an essential point in understanding the difference between a hypercover and a rigid hypercover (Section 6).

Throughout, we assume that the base scheme S is Noetherian. Since all of our schemes are of finite type over S , every scheme that we consider is Noetherian. This is a technical requirement for the machinery of étale topological types [9].

2. Étale realizations

We begin with a brief review of the construction of \mathbb{A}^1 -homotopy theory [20].

Let S be a Noetherian scheme. Consider the category Sm/S of schemes of finite type over S . We consider two Grothendieck topologies on this category. The étale topology has covers consisting of finite collections of étale maps that have surjective images. The Nisnevich topology [21] has covers consisting of finite collections of étale maps $\{U^a \rightarrow X\}$ that have surjective images and such that for every point x of X , there is a point u of some U^a such that the map $k(x) \rightarrow k(u)$ on residue fields is an isomorphism.

Let $\text{Spc}(S)$ be the category of simplicial presheaves on Sm/S . The notation stands for “spaces over S ”. This category has several model structures. Morel and Voevodsky start with the *Nisnevich local injective* model structure [16], in which the cofibrations are all monomorphisms and the weak equivalences are detected by Nisnevich sheaves of homotopy groups. They then formally invert the maps $X \times \mathbb{A}^1 \rightarrow X$ for every scheme X to obtain the \mathbb{A}^1 -*local injective* model structure.

For our purposes, we need a slightly different model structure. We start with the *Nisnevich local projective* model structure, in which the weak equivalences are again detected by Nisnevich sheaves of homotopy groups but the cofibrations are generated by maps of the form $\partial\Delta[n] \otimes X \rightarrow \Delta^n \otimes X$ for any scheme X . Then we formally invert the maps $X \times \mathbb{A}^1 \rightarrow X$ to obtain the \mathbb{A}^1 -*local projective* model structure. Both the \mathbb{A}^1 -local projective and \mathbb{A}^1 -local injective model structures have the same homotopy category. We choose to work with the projective version because it is easier to construct functors out of the projective version than out of the injective version.

Following [5], there is another construction of the local projective model structure that is particularly useful for us. Start with the *objectwise projective* model structure,

in which the weak equivalences are objectwise weak equivalences and the cofibrations are the same as in the local projective model structure. Then we take the left Bousfield localization [11, Chapter 3] of this model structure at the set of two kinds of maps:

- (1) for every finite collection $\{X^a\}$ of schemes with disjoint union X , the map $\coprod X^a \rightarrow X$ from the coproduct (as presheaves) of the presheaves represented by each X^a to the presheaf represented by X , and
- (2) every Nisnevich hypercover $U \rightarrow X$ (see Definition 28).

This gives us the Nisnevich local projective model structure. In the language of [4], the \mathbb{A}^1 -local projective model structure is the universal model category on Sm/S subject to the two kinds of relations described above, plus the relations:

- (3) $X \times \mathbb{A}^1 \rightarrow X$ for every scheme X .

If we replace Nisnevich covers with étale covers, then we obtain the étale local injective and the étale local projective model structures on $\text{Spc}(S)$.

The étale topological type is a functor Et from schemes to pro-spaces. See Section 3 or [9] for the definition and properties of this functor. As described in [4], this functor can be extended in a canonical way to an *étale realization* functor, which we also denote Et , from simplicial presheaves to pro-spaces. The principle behind this extension is that Et is the unique functor such that $\text{Et } X$ is the étale topological type of X for every representable X and such that Et preserves colimits and simplicial structures. The following definition gives a concrete description of Et .

Definition 1. If X is a representable presheaf, then $\text{Et } X$ is the étale topological type of X . Next, if P is a discrete presheaf (i.e., each simplicial set $P(X)$ is zero dimensional), then P can be written as a colimit $\text{colim}_i X_i$ of representables and $\text{Et } P = \text{colim}_i \text{Et } X_i$. Finally, an arbitrary simplicial presheaf P can be written as the coequalizer of the diagram

$$\coprod_{[m] \rightarrow [n]} P_m \otimes \Delta^n \rightrightarrows \coprod_{[n]} P_n \otimes \Delta^n,$$

where each P_n is discrete. Define $\text{Et } P$ to be the coequalizer of the diagram

$$\coprod_{[m] \rightarrow [n]} \text{Et } P_m \otimes \Delta^n \rightrightarrows \coprod_{[n]} \text{Et } P_n \otimes \Delta^n.$$

Observe that if X is a simplicial scheme, then $\text{Et } X$ is equal to the realization of the simplicial pro-space $n \mapsto \text{Et } X_n$.

Theorem 2. *With respect to the étale local (or Nisnevich local) projective model structure on $\mathrm{Spc}(S)$ and the model structure on pro-simplicial sets given in [13], the functor Et is a left Quillen functor.*

Remark 3. The theorem is not true if we consider the local injective model structure on $\mathrm{Spc}(S)$. There are too many injective cofibrations.

Proof. By general nonsense from [4, Proposition 2.3], we need only show that Et takes relations (1) and (2) described above to weak equivalences of pro-spaces. Cofibrant replacements are no problem because the targets and sources of every map in question are already projective cofibrant. To show that U is projective cofibrant for every hypercover U , use Proposition 33 to conclude that U is a split simplicial scheme.

For relations of type (1), note that Et commutes with coproducts of schemes [9, Proposition 5.2]. For relations of type (2), see Theorem 12. \square

The point of the previous theorem is that Et induces a homotopy invariant derived functor $L\mathrm{Et}$.

Corollary 4. *The functor $L\mathrm{Et}$ induces a functor from the étale local (or Nisnevich local) homotopy category of simplicial presheaves to the homotopy category of pro-spaces. Moreover, $L\mathrm{Et} X$ is the usual étale topological type $\mathrm{Et} X$ for every scheme X in Sm/S .*

Proof. The first claim follows from the formal machinery of Quillen adjoint functors [11, Section 8.5]. The last claim follows from general nonsense and the fact that every representable presheaf is projective cofibrant. \square

In order for étale realization to be \mathbb{A}^1 -homotopy invariant, it is necessary to complete away from the characteristics of the residues fields of S . We next describe a model for \mathbb{Z}/p -completion of pro-spaces. This is very similar to the \mathbb{Z}/p -completion described in [19], except that we prefer to work with the category of pro-simplicial sets rather than the category of simplicial pro-finite sets. See [14] for the subtle distinctions between these categories.

Theorem 5. *There is a model structure on the category of pro-spaces in which the weak equivalences are the maps inducing cohomology with coefficients in \mathbb{Z}/p .*

Proof. The proof is entirely analogous to the proof of the main theorem of [3]. We colocalize with respect to the objects $K(\mathbb{Z}/p, n)$ for all $n \geq 0$. More precisely, a pro-map $X \rightarrow Y$ is a weak equivalence if the induced map $\mathrm{Map}(Y, K(\mathbb{Z}/p, n)) \rightarrow \mathrm{Map}(X, K(\mathbb{Z}/p, n))$ is a weak equivalence of simplicial sets for every $n \geq 0$. Pro-categories have sufficiently good properties that this kind of colocalization always exists [3]. \square

Now let p be a fixed prime that does not occur as the characteristic of any residue field of S .

Theorem 6. *With respect to the \mathbb{A}^1 -local projective model structure on $\mathrm{Spc}(S)$ and the \mathbb{Z}/p -cohomological model structure on pro-simplicial sets described in Theorem 5, Et is a left Quillen functor.*

As for Theorem 2, this theorem is not true when considering the \mathbb{A}^1 -local injective model structure on $\mathrm{Spc}(S)$. There are too many injective cofibrations.

Proof. The argument is basically the same as in the proof of Theorem 2. The only significantly different part is in showing that

$$\mathrm{Et}(X \times \mathbb{A}^1) \rightarrow \mathrm{Et} X$$

is a \mathbb{Z}/p -cohomological weak equivalence for every scheme X in Sm/S . We need to show that this map induces an isomorphism in cohomology with coefficients in \mathbb{Z}/p . In order to understand these cohomology maps, Friedlander [9, Proposition 5.9] allows us to consider the map on étale cohomology induced by the projection

$$X \times \mathbb{A}^1 \rightarrow X.$$

By Milne [18, Corollary VI.4.20], the projection induces an isomorphism in étale cohomology. \square

The next corollary follows from Theorem 6 in the same way that Corollary 4 follows from Theorem 2.

Corollary 7. *The left derived functor $L\mathrm{Et}$ induces a functor from the \mathbb{A}^1 -homotopy category to the \mathbb{Z}/p -cohomological homotopy category of pro-spaces.*

The \mathbb{Z}/p -completion of a pro-space X is a fibrant replacement \hat{X} with respect to the \mathbb{Z}/p -cohomology model structure. This functor has the important property that a map $X \rightarrow Y$ is a \mathbb{Z}/p -cohomology isomorphism if and only if the induced map $\hat{X} \rightarrow \hat{Y}$ on \mathbb{Z}/p -completions is a weak equivalence of pro-spaces in the sense of [13]. Let $\widehat{\mathrm{Et}}$ be the functor from $\mathrm{Spc}(S)$ to pro-spaces that takes F to the \mathbb{Z}/p -completion of $\mathrm{Et} F$. Corollary 7 means that this functor takes \mathbb{A}^1 -local weak equivalences to weak equivalences of pro-spaces in the sense of [13].

2.1. Excision for the étale topological type

This section gives an interesting corollary about étale topological types and elementary distinguished squares. Recall that an *elementary distinguished*

square [20, Definition 3.1.3] is a diagram

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

of smooth schemes over S in which i is an open inclusion, p is étale, and $p : p^{-1}(X - U) \rightarrow X - U$ is an isomorphism (where the schemes $p^{-1}(X - U)$ and $X - U$ are given the reduced structure). The relevance of such squares is that the maps i and p form a Nisnevich cover of X .

One interpretation of [2, Lemma 4.1] says the following. Instead of localizing at all the hypercovers to obtain local model structures, one can localize at the maps from the homotopy pushout of the diagram

$$U \leftarrow U \times_X V \rightarrow V$$

into X , for every elementary distinguished square as in the previous paragraph. This leads immediately to the following excision theorem for étale topological types.

Theorem 8. *Given an elementary distinguished square of smooth schemes over S as in diagram 1, the square*

$$\begin{array}{ccc} \text{Et}(U \times_X V) & \longrightarrow & \text{Et}V \\ \downarrow & & \downarrow \\ \text{Et}U & \longrightarrow & \text{Et}X \end{array}$$

is a homotopy pushout square of pro-spaces.

Proof. By Corollary 4, it suffices to show that the square

$$\begin{array}{ccc} \text{LEt}(U \times_X V) & \longrightarrow & \text{LEt}V \\ \downarrow & & \downarrow \\ \text{LEt}U & \longrightarrow & \text{LEt}X \end{array}$$

is a homotopy pushout square. Let P be the homotopy pushout of the diagram

$$U \leftarrow U \times_X V \rightarrow V.$$

From the paragraph preceding this theorem, we know that $P \rightarrow X$ is a local weak equivalence of presheaves. The functor LEt preserves weak equivalences by Theorem 2, so $\text{LEt} P \rightarrow \text{LEt} X$ is also a weak equivalence. Left derived functors commute with homotopy colimits, so the homotopy pushout of the diagram

$$\text{LEt} U \leftarrow \text{LEt}(U \times_X V) \rightarrow \text{LEt} V$$

is weakly equivalent to $\text{LEt} P$. \square

The previous theorem agrees with the cohomological excision theorem of [18, III.1.27], at least with locally constant coefficients, because the étale cohomology of a scheme is isomorphic to the singular cohomology of its étale topological type.

3. Hypercover descent for the étale topological type

This section reviews the definition of the étale topological type functor, which appeared throughout the previous section. The key result is the hypercover descent theorem as stated in Theorem 12.

For a scheme X , recall the cofiltered category $\text{HRR}(X)$ of rigid hypercovers of X . See Section 6 for more details on rigid hypercovers. Each object U of $\text{HRR}(X)$ is a simplicial scheme over X . Applying the component functor π to U gives a simplicial set. Thus we have a functor from $\text{HRR}(X)$ to simplicial sets. Since $\text{HRR}(X)$ is cofiltered, we regard this functor as a pro-space $\text{Et } X$; this is Friedlander's notion of the étale topological type of a scheme.

Given a scheme map $f : X \rightarrow Y$, rigid pullback as described in Definition 34 gives a functor $f^* : \text{HRR}(Y) \rightarrow \text{HRR}(X)$. If U is a rigid hypercover of Y , then there is a canonical rigid hypercover map $f^*U \rightarrow U$. These maps induce a map $\text{Et } X \rightarrow \text{Et } Y$ of pro-spaces. This map is *strict* in the sense that it is given by a natural transformation (of functors from $\text{HRR}(Y)$ to spaces) from the functor $\text{Et } Y$ to the functor $(\text{Et } X) \circ f^*$. The strictness of this map is critical for the proof of Proposition 10.

If X is a pointed and connected scheme, then $\text{Et } X$ is a pointed and connected pro-space [9, Proposition 5.2]. In this case, the pro-groups $\pi_i \text{Et } X$ determine the homotopy type of $\text{Et } X$ in the sense of the homotopy theory of pro-spaces from [13] because we do not have to worry about choosing basepoints. The étale topological type commutes with coproducts [9, Proposition 5.2], so the study of arbitrary schemes reduces easily to the study of pointed and connected schemes by considering one component at a time and choosing an arbitrary basepoint for each component.

When X is a simplicial scheme, we can again use the cofiltered category $\text{HRR}(X)$ of rigid hypercovers of X to form a pro-space. Each object U of $\text{HRR}(X)$ is a bisimplicial scheme over X . Applying the component functor π to U yields a bisimplicial set, and its realization is an ordinary simplicial set. This establishes a functor from $\text{HRR}(X)$ to simplicial sets. We regard it as a pro-space $\text{sEt } X$; this is Friedlander's notion of the étale topological type of a simplicial scheme.

Recall the diagonal functor that takes a bisimplicial set T to its diagonal simplicial set $n \mapsto T_{n,n}$. This functor was used instead of realization in [9]. However, the diagonal of a simplicial space is the same as its realization [23, p. 94], so our definition is the same.

When X is a scheme, note that $\text{Et } X$ is equal to $\text{sEt}(cX)$, where cX is the constant simplicial scheme with value X . This follows from Lemma 32.

Similarly to the case of ordinary schemes, a map $f : X \rightarrow Y$ of simplicial schemes gives rise to a strict map of pro-spaces $\text{sEt } X \rightarrow \text{sEt } Y$.

It is important to distinguish between $\text{sEt } X$ and $\text{Et } X$. As described in the previous paragraph, $\text{sEt } X$ is Friedlander’s étale topological type. On the other hand, $\text{Et } X$ is constructed by considering X to be a simplicial presheaf and then applying the étale realization functor $\text{Et } X$ of the previous section. More explicitly, $\text{Et } X$ is constructed by first considering the simplicial pro-space $n \mapsto \text{Et } X_n$ and then taking the realization of this simplicial object to obtain a pro-space.

We would like to compare $\text{sEt } X$ with $\text{Et } X$. In general they are not isomorphic. Nevertheless, we shall prove that the natural map $\text{Re}(n \mapsto \text{Et } X_n) \rightarrow \text{sEt } X$ is a weak equivalence of pro-spaces.

In order to avoid the infinite colimits that are used in constructing realizations, we introduce n -truncated realizations. For any simplicial scheme X , let $\text{sEt}_n X$ be the pro-space given by the functor $\text{Re}_n \circ \pi$ from $\text{HRR}(X)$ to spaces, where Re_n is the n -truncated realization functor (see Section 7). In other words, we take a bisimplicial scheme U in $\text{HRR}(X)$, consider the simplicial space πU , and then take the n -truncated realization of this simplicial space to obtain a simplicial set.

In general, $\text{sEt}_n X$ is not equivalent to $\text{sEt } X$, but the next proposition tells us that the pro-spaces $\text{sEt}_n X$ are close enough to $\text{sEt } X$ to determine its homotopy type.

Proposition 9. *Suppose that X is a pointed simplicial scheme. The pro-map $\pi_i \text{sEt}_n X \rightarrow \pi_i \text{sEt } X$ is an isomorphism of pro-groups whenever $i < n$.*

Proof. This follows immediately from Corollary 47 applied to each bisimplicial set πU , where U is any rigid hypercover of X . \square

Although $\text{sEt } X$ and $\text{Et } X$ are not the same, their n -truncated versions are in fact isomorphic.

Proposition 10. *The pro-space $\text{sEt}_n X$ is isomorphic to the pro-space*

$$\text{Re}_n(m \mapsto \text{Et } X_m).$$

Proof. For simplicity of notation, let Y be the pro-space $\text{Re}_n(m \mapsto \text{Et } X_m)$. As described in Remarks 43 and 45, Y is a colimit of a diagram of strict maps such that the diagram has no loops and each object is the source of only finitely many arrows. Moreover, each of the categories $\text{HRR}(X_m)$ has finite limits because of the existence of rigid limits (see Section 6.2). This allows us to apply the method of [14, Section 3.1] to compute Y . The index set K for Y is the product category

$$\text{HRR}(X_0) \times \text{HRR}(X_1) \times \cdots \times \text{HRR}(X_n).$$

For each $V = (V_0, V_1, \dots, V_n)$ in K , the space Y_V is the coequalizer of the diagram

$$\coprod_{\substack{\phi : [m] \rightarrow [k] \\ m, k \leq n}} \pi(V_{k, \cdot} \overset{R}{\times} \phi^* V_{m, \cdot}) \otimes \Delta[m] \rightrightarrows \coprod_{m \leq n} \pi(V_{m, \cdot}) \otimes \Delta[m].$$

In this diagram, the upper map is induced by the maps $\phi_* : \Delta[m] \rightarrow \Delta[k]$ and the projections $V_{k,\cdot} \times^R \phi^* V_{m,\cdot} \rightarrow V_{k,\cdot}$, while the lower map is induced by the maps

$$V_{k,\cdot} \times^R \phi^* V_{m,\cdot} \rightarrow \phi^* V_{m,\cdot} \rightarrow V_{m,\cdot}.$$

The forgetful functor $\text{HRR}(X) \rightarrow K$ is cofinal by Proposition 41. Therefore, we might as well assume that $\text{HRR}(X)$ is the indexing category for Y . If V is a rigid hypercover of X , then Y_V is the coequalizer of the diagram

$$\coprod_{\substack{\phi : [m] \rightarrow [k] \\ m, k \leq n}} \pi(V_{k,\cdot} \times^R \phi^* V_{m,\cdot}) \otimes \Delta[m] \rightrightarrows \coprod_{m \leq n} \pi(V_{m,\cdot}) \otimes \Delta[m].$$

For every $\phi : [m] \rightarrow [k]$, the rigid hypercover map $V_{k,\cdot} \rightarrow V_{m,\cdot}$ gives us a map $V_{k,\cdot} \rightarrow \phi^* V_{m,\cdot}$. Since $\text{HRR}(X_k)$ is actually a directed set, this means that $V_{k,\cdot} \times^R \phi^* V_{m,\cdot}$ is isomorphic to $V_{k,\cdot}$. It follows that Y_V is isomorphic to the coequalizer of the diagram

$$\coprod_{\substack{\phi : [m] \rightarrow [k] \\ m, k \leq n}} \pi(V_{k,\cdot}) \otimes \Delta[m] \rightrightarrows \coprod_{m \leq n} \pi(V_{m,\cdot}) \otimes \Delta[m].$$

In other words, Y_V is $\text{Re}_n(m \mapsto \pi V_{m,\cdot})$. This is precisely the definition of $\text{sEt}_n X$.

The next theorem describes the étale topological type of a simplicial scheme X in terms of the étale topological types of each scheme X_n and realizations of pro-spaces.

Theorem 11. *For any simplicial scheme X , the natural map*

$$\text{Re}(n \mapsto \text{Et } X_n) \rightarrow \text{sEt } X$$

is a weak equivalence in the category of pro-spaces.

Proof. As in [9, Proposition 5.2], we can write X as a disjoint union of simplicial schemes X^a , where each X^a is connected in the sense that the simplicial set $n \mapsto \pi X_n^a$ is connected. Since Et , sEt , and realization all commute with disjoint unions, it suffices to assume that X is connected. We choose any basepoint in X_0 .

Now both $\text{sEt } X$ and $\text{Re}(n \mapsto \text{Et } X_n)$ are pointed connected pro-spaces. By Isaksen [13, Corollary 7.5], it suffices to show that the natural map $\text{Re}(n \mapsto \text{Et } X_n) \rightarrow \text{sEt } X$ induces an isomorphism of pro-homotopy groups in all dimensions. By Corollary 49 and Proposition 9, we may as well consider the map $\text{Re}_m(n \mapsto \text{Et } X_n) \rightarrow \text{sEt}_m X$ to study the homotopy groups in dimension less than m . This map induces an isomorphism on pro-homotopy groups by Proposition 10. Since m was arbitrary, the map $\pi_i \text{Re}(n \mapsto \text{Et } X_n) \rightarrow \pi_i \text{sEt } X$ is a pro-isomorphism for all i . \square

We come to the key ingredient for the proof of Theorem 2. The following result is a hypercover descent theorem for the étale topological type.

Theorem 12. *Let U be a hypercover of a scheme X . Then the natural map*

$$\text{Re}(n \mapsto \text{Et } U_n) \rightarrow \text{Et } X$$

is a weak equivalence of pro-spaces.

Proof. By Theorem 11, the map

$$\text{Re}(n \mapsto \text{Et } U_n) \rightarrow \text{sEt } U$$

is a weak equivalence. By Friedlander [9, Proposition 8.1], the map $\text{sEt } U \rightarrow \text{Et } X$ is a weak equivalence. Thus, the composition of these two maps is also a weak equivalence. \square

4. Simplicial schemes

The point of this section is to study simplicial schemes and to make some useful constructions concerning them.

4.1. Finite limits of schemes

We first study how finite limits interact with étale maps and separated maps. The results here are not particularly striking, but they do not appear in the standard literature [8,10,18,25].

Proposition 13. *Let $f : U \rightarrow X$ be a map of finite diagrams of schemes such that the map $f^a : U^a \rightarrow X^a$ is étale (resp., separated) for every a . Then the map $\lim f : \lim U \rightarrow \lim X$ is étale (resp., separated).*

Proof. Every finite limit can be expressed in terms of finite products and fiber products, so it suffices to consider a diagram of schemes

$$\begin{array}{ccccc} U & \longrightarrow & V & \longleftarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longleftarrow & Z \end{array}$$

such that the three vertical maps are étale (resp., separated). We want to show that the induced map

$$U \times_V W \rightarrow X \times_Y Z$$

is also étale (*resp.*, separated). We prove the lemma for étale maps. The proof for separated maps is identical. See [8, Proposition I.5.3.1] for the necessary properties of separated maps.

Recall that base changes preserve étale maps [18, Proposition I.3.3(c)]. Let f be the map in question. Factor f as

$$U \times_V W \rightarrow U \times_Y W \rightarrow X \times_Y W \rightarrow X \times_Y Z.$$

The second and third maps are étale because they are base changes of $U \rightarrow X$ and $W \rightarrow Z$, respectively. It remains to show that the first map is also étale. The diagram

$$\begin{array}{ccc} U \times_V W & \longrightarrow & V \\ \downarrow & & \downarrow \Delta \\ U \times_Y W & \longrightarrow & V \times_Y V \end{array}$$

is a pullback square, where Δ is the diagonal map. It suffices to observe that Δ is étale [18, Proposition I.3.5]. \square

4.2. Simplicial schemes

We work in the category of schemes or more generally in the category of schemes over a fixed base scheme S ; these two cases are actually the same since the category of schemes has a terminal object $\text{Spec } \mathbb{Z}$.

Let Δ be the category whose objects are the non-empty ordered sets $[n] = \{0 < 1 < 2 < \dots < n\}$ and whose morphisms are the weakly monotonic maps. This is the usual indexing category for simplicial objects. Let Δ_+ be the category Δ with an initial object $[-1]$ adjoined. The opposite of Δ_+ is the usual indexing category for augmented simplicial objects. Let $\Delta_{\leq n}$ be the full subcategory of Δ on the objects $[m]$ for $m \leq n$. Note that $\Delta_{\leq n}$ is a finite category.

Definition 14. A *simplicial scheme* is a functor from Δ^{op} to schemes. An *n-truncated simplicial scheme* is a functor from $\Delta_{\leq n}^{\text{op}}$ to schemes. An *augmented simplicial scheme* is a functor from Δ_+^{op} to schemes. A *bisimplicial scheme* is a functor from $(\Delta \times \Delta)^{\text{op}}$ to schemes. An *augmented bisimplicial scheme* is a functor from $(\Delta \times \Delta_+)^{\text{op}}$ to schemes.

Note that augmented bisimplicial schemes are augmented in only one direction. Augmented bisimplicial schemes are perhaps more correctly but awkwardly called simplicial augmented simplicial schemes.

For every scheme X , let cX be the constant simplicial scheme with value X .

Recall the n th latching object $L_n X$ of a simplicial object X [11, Definition 15.2.5]. It is a certain finite colimit of the objects X_m for $0 \leq m \leq n - 1$. Beware that $L_n X$ does not necessarily exist for every simplicial scheme X because the category of schemes is not cocomplete.

Definition 15. A simplicial scheme X is *split* if $L_n X$ exists for every $n \geq 0$ and the canonical map $L_n X \rightarrow X_n$ is the inclusion of a direct summand. If X is split, let NX_n be the subscheme of X_n such that $X_n = L_n X \amalg NX_n$.

The idea is that NX_n is the non-degenerate part of X_n and that X_n splits into a direct sum of its degenerate part and its non-degenerate part. Note that NX_n is well defined because the category of schemes is locally connected [1, Section 9].

4.3. Skeleta and coskeleta

Definition 16. If X is a simplicial scheme, then the n -skeleton $\text{sk}_n X$ is the n -truncated simplicial scheme given by restriction of X along the inclusion $\Delta_{\leq n}^{\text{op}} \rightarrow \Delta^{\text{op}}$.

There is another possible definition of $\text{sk}_n X$, at least when X is split up to dimension n . Namely, we could consider the simplicial scheme given in dimension m by

$$\text{colim}_{\substack{\phi : [m] \rightarrow [k] \\ k \leq n}} X_k.$$

In general, this does not exist because the necessary colimits may not exist in the category of schemes. However, it does exist when X is split up to dimension n . In this case, $(\text{sk}_n X)_m$ is a disjoint union of one copy of NX_k for each surjective map $[m] \rightarrow [k]$ with $k \leq n$. In the end, it does not really matter which construction we consider, so we will not worry about the ambiguous notation.

Similarly, for a simplicial set X , there are two possible definitions of $\text{sk}_n X$, one an n -truncated simplicial set and the other a simplicial set that is degenerate above dimension n . Again, it is not very important which construction we use, especially since both exist for every simplicial set.

Definition 17. The n th coskeleton functor cosk_n from n -truncated simplicial schemes to simplicial schemes is right adjoint to the functor sk_n .

We abuse notation and write $\text{cosk}_n X$ instead of $\text{cosk}_n(\text{sk}_n X)$ for a simplicial scheme X . To avoid confusion, we write cosk_n^S for the n th coskeleton functor in the category of schemes over S . By convention, $\text{cosk}_{-1} X$ is the constant simplicial scheme $c\text{Spec } \mathbb{Z}$. More generally, $\text{cosk}_{-1}^S X$ is the constant simplicial scheme cS . This convention makes our definition of hypercovers in Section 6 more concise.

Each object $(\text{cosk}_n X)_m$ is a finite limit of the objects X_k for $k \leq n$. Also, $(\text{cosk}_n X)_m$ is isomorphic to X_m when $m \leq n$. In other words, $\text{cosk}_n X$ and X agree up to dimension n .

For every simplicial scheme X , the unit map $X \rightarrow \text{cosk}_n(\text{sk}_n X)$ induces a natural map

$$X_m \rightarrow (\text{cosk}_k X)_m.$$

These maps will appear again and again.

Note that $(\text{cosk}_n X)_{n+1}$ is the n th matching object $M_n X$ of X [11, Definition 15.2.5].

Remark 18. For any finite simplicial set K and any scheme X , define $X \otimes K$ to be the simplicial scheme isomorphic to $\coprod_{K_n} X$ in dimension n . For any simplicial scheme Y , define the cotensor $\text{hom}(K, Y)$ such that the functors $(\cdot) \otimes K$ and $\text{hom}(K, \cdot)$ are adjoints. In these terms, the scheme $(\text{cosk}_n X)_m$ is isomorphic to $\text{hom}(\text{sk}_n \Delta[m], X)$. This is the notation used in [5].

5. Rigid covers

In this section, we review the notion of a rigid cover and introduce some constructions and results concerning them. Some of the material in this section can be found in [9].

For any point x_0 of a scheme X , a *geometric point* of X over x_0 is a map $x: \text{Spec } \bar{k} \rightarrow X$ with image x_0 , where \bar{k} is the separable closure of the residue field $k(x_0)$. If $f: X \rightarrow Y$ is a map of schemes and $y: \text{Spec } \bar{k} \rightarrow Y$ is a geometric point of Y , then a *lift* of y is a geometric point $x: \text{Spec } \bar{k} \rightarrow X$ such that $y = f \circ x$. Equivalently, x goes to y under the set map $f(\bar{k}): X(\bar{k}) \rightarrow Y(\bar{k})$. In this situation, we abuse notation and write $f(x) = y$.

Definition 19. A *rigid cover* U of a scheme X is

- (1) a map $f: U \rightarrow X$,
- (2) a decomposition $U = \coprod U_x$, where the coproduct is indexed by the geometric points of X , each U_x is connected, and each map $U_x \rightarrow X$ is étale and separated;
- (3) and a geometric point u_x of each component U_x such that $f(u_x) = x$.

Note that rigid covers are *not* étale covers. The problem is that rigid covers have infinitely many pieces in general. In fact, rigid covers are *infinite* étale covers. Also, we require that the maps in a rigid cover are separated. For technical precision, it is important to keep this difference in mind.

If U and U' are rigid covers of X and X' , then a rigid cover map over a scheme map $h: X \rightarrow X'$ consists of a commuting square

$$\begin{array}{ccc} U_x & \xrightarrow{g_x} & U'_{h(x)} \\ \downarrow & & \downarrow \\ X & \xrightarrow{h} & X' \end{array}$$

for each geometric point x of X such that $g_x(u_x) = u'_{h(x)}$. The idea is that the map of rigid covers preserves basepoints.

The importance of rigid covers is that there exists at most one rigid cover map between any two rigid covers of a scheme [9, Proposition 4.1].

5.1. Rigid pullbacks

Suppose that $f : X \rightarrow Y$ is a map of schemes and $U \rightarrow Y$ is étale surjective. Then the base change $f^*U \rightarrow X$ is the projection $X \times_Y U \rightarrow X$, which is again étale surjective. This idea generalizes to rigid covers.

Definition 20. Let $f : X \rightarrow Y$ be any map of schemes and let U be a rigid cover of Y . Then the *rigid pullback* f^*U is the rigid cover of X defined by the following construction. For each geometric point x of X , let $(f^*U)_x$ be the component of $X \times_Y U$ containing $x \times u_{f(x)}$, and let $x \times u_{f(x)}$ be the basepoint of $(f^*U)_x$.

Remark 21. Note that $(f^*U)_x$ is a component of $X \times_Y U_x$, but f^*U is not a subobject of $X \times_Y U$ since some components of $X \times_Y U$ may occur more than once as components of f^*U . Also note that there is a canonical rigid cover map from f^*U to U over the map $X \rightarrow Y$.

Proposition 22. Let $f : X \rightarrow Y$ be any map of schemes and let U be a rigid cover of Y . Then the rigid cover f^*U of X has the following universal property. Let V be an arbitrary rigid cover of Z . Rigid cover maps $V \rightarrow f^*U$ over a map $Z \rightarrow X$ correspond bijectively to rigid cover maps from V to U over the composition $Z \rightarrow X \rightarrow Y$.

Proof. The category of connected pointed schemes has finite limits. To construct such limits, just take the basepoint component of the usual limit of schemes. The proposition now follows from this observation and the universal property of pullbacks of schemes. \square

5.2. Rigid limits

The goal of this section is to generalize Proposition 13 from étale covers to rigid covers. The following lemma shows that the usual notion of limit does not quite work.

Lemma 23. Let $f : U \rightarrow X$ be a finite diagram of maps of schemes such that each $U^a \rightarrow X^a$ is a rigid cover and such that each map $U^a \rightarrow U^b$ is a rigid cover map over $X^a \rightarrow X^b$. Then the map

$$\lim_a f^a : \lim_a U^a \rightarrow \lim_a X^a$$

is surjective.

Proof. We need to show that every geometric point x of $\lim X$ lifts to $\lim U$. Let x^a be the composition of x with the projection map $\lim X \rightarrow X^a$. Since each U^a is a rigid cover of X^a , there exist canonical lifts u^a of each x^a to U^a . They assemble to give a geometric point u of $\lim U$ because f is a diagram of rigid cover maps.

The above proposition is not true if each f^a is only surjective. A limit of surjective maps is not necessarily surjective. \square

Note that $\lim U$ is not in general a rigid cover of $\lim X$. As the proof above indicates, there are canonical lifts for each geometric point of $\lim X$, but the components of $\lim U$ may not correspond bijectively to the geometric points of $\lim X$. Since ordinary finite limits do not preserve rigid covers, the notion of limit must be refined in order to get a rigid cover-preserving construction.

Definition 24. Let $f : U \rightarrow X$ be a finite diagram of rigid cover maps. Then the *rigid limit*

$$\mathbf{R}\lim_a f^a : \mathbf{R}\lim_a U^a \rightarrow \lim_a X^a$$

is the rigid cover defined as follows. For each geometric point $x = \lim_a x^a$ of $\lim_a X^a$, let $(\mathbf{R}\lim_a U^a)_x$ be the connected component of $\lim_a U^a$ containing $u_x = \lim_a u_{x^a}^a$, and let u_x be the basepoint of $(\mathbf{R}\lim_a U^a)_x$.

Note that there is a natural map $\mathbf{R}\lim U \rightarrow \lim U$ over $\lim X$. The geometric points $u_{x^a}^a$ are compatible and induce a geometric point u_x of $\lim_a U^a$ because f is a diagram of rigid cover maps.

First, we must show that rigid limits are in fact rigid covers.

Lemma 25. *The rigid limit of a finite diagram of rigid cover maps is a rigid cover.*

Proof. The map $\mathbf{R}\lim_a U^a \rightarrow \lim_a X^a$ factors as a local isomorphism $\mathbf{R}\lim_a U^a \rightarrow \lim_a U^a$ followed by the map $\lim_a U^a \rightarrow \lim_a X^a$. The latter is étale and separated by Proposition 13, so the composition is also étale and separated. The other parts of the definition of a rigid cover are satisfied by construction. \square

The symbols \prod^R and \times^R denote rigid limits in the case of products or fiber products. Similarly, if U and X are n -truncated schemes and $f : U \rightarrow X$ is a diagram of rigid cover maps, then

$$(\mathbf{R}\text{cosk}_n f)_k : (\mathbf{R}\text{cosk}_n U)_k \rightarrow (\text{cosk}_n X)_k$$

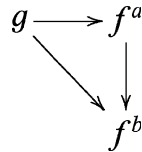
is the rigid limit of the finite diagram whose ordinary limit is $(\text{cosk}_n f)_k$. Because of the functoriality expressed below in Remark 27, these constructions assemble

into a map

$$\mathbf{Rcosk}_n f : \mathbf{Rcosk}_n U \rightarrow \mathbf{cosk}_n X$$

of simplicial schemes that is a simplicial object in the category of rigid covers.

Proposition 26. *Let $f : U \rightarrow X$ be a finite diagram of rigid cover maps. Then $\mathbf{Rlim}_a f^a$ is universal in the following sense. Let $g : V \rightarrow Y$ be any rigid cover of a scheme Y . Rigid cover maps $g \rightarrow \mathbf{Rlim} f$ are in one-to-one correspondence with collections of rigid cover maps $g \rightarrow f^a$ such that for every map $f^a \rightarrow f^b$, the diagram*



of rigid cover maps commutes.

Proof. As in the proof of Proposition 22, it is important that the category of connected pointed schemes has finite limits. The lemma now follows from this observation and the universal property of limits. \square

Remark 27. Rigid limits have the same kind of functoriality as ordinary limits. We make this more precise. Let $f : U \rightarrow X$ and $g : V \rightarrow Y$ be diagrams of rigid cover maps indexed by finite categories A and B , respectively. Suppose given a functor $F : B \rightarrow A$, and let F^*f be the diagram of rigid cover maps indexed by B given by the formula $(F^*f)^b = f^{F(b)}$. Suppose given a natural transformation $\eta : F^*f \rightarrow g$. Then η induces a natural map $\mathbf{Rlim}_A f \rightarrow \mathbf{Rlim}_B g$. This is precisely what happens for ordinary limits.

6. Hypercovers

Much of the material in this section can be found in [9]. We review the basic notions of hypercovers and rigid hypercovers and formalize some useful constructions concerning them. Our investment in language and machinery clarifies some of the technical complexities in the proofs of [9, Chapter 4].

Definition 28. A *hypercove* (resp., Nisnevich hypercover) of a scheme X is an augmented simplicial scheme U such that $U_{-1} = X$ and the map

$$U_n \rightarrow (\mathbf{cosk}_{n-1}^X U)_n$$

is étale surjective (*resp.*, Nisnevich surjective) for all $n \geq 0$. A hypercover of a simplicial scheme X is an augmented bisimplicial scheme U such that $U_{\cdot, -1} = X$ and $U_{n, \cdot}$ is a hypercover of X_n for each n .

By convention, the map

$$U_n \rightarrow (\text{cosk}_{n-1}^X U)_n$$

is equal to the map $U_0 \rightarrow X$ when $n = 0$. It is important to remember that $U_0 \rightarrow X$ must be étale surjective.

Maps of hypercovers are just maps of augmented simplicial schemes or augmented bisimplicial schemes.

Definition 29. A rigid hypercover of a scheme X is an augmented simplicial scheme U such that $U_{-1} = X$ and the map

$$U_n \rightarrow (\text{cosk}_{n-1}^X U)_n$$

is a rigid cover for all $n \geq 0$.

Note that rigid hypercovers are not hypercovers; the maps $U_n \rightarrow (\text{cosk}_{n-1}^X U)_n$ are rigid covers, not étale covers. This causes some confusion in the notation, and it is an important technical point.

If U and U' are rigid hypercovers of schemes X and X' , then a rigid hypercover map $U \rightarrow U'$ is a map of augmented simplicial schemes such that for every $n \geq 0$, the map $U_n \rightarrow U'_n$ is a rigid cover map over $(\text{cosk}_{n-1}^X U)_n \rightarrow (\text{cosk}_{n-1}^{X'} U')_n$.

Definition 30. A rigid hypercover of a simplicial scheme X is an augmented bisimplicial scheme such that $U_{\cdot, -1} = X$, $U_{n, \cdot}$ is a rigid hypercover of X_n for each n , and $U_{n, \cdot} \rightarrow U_{m, \cdot}$ is a rigid hypercover map over $X_n \rightarrow X_m$ for every $[m] \rightarrow [n]$.

If U and U' are rigid hypercovers of simplicial schemes X and X' , then a rigid hypercover map $U \rightarrow U'$ is a map of augmented bisimplicial schemes such that $U_{n, \cdot} \rightarrow U'_{n, \cdot}$ is a rigid hypercover map for each n .

Similarly to rigid covers, there exists at most one map between two rigid hypercovers of a scheme (or simplicial scheme) [9, Proposition 4.3]. On the other hand, maps between hypercovers are unique only in a certain homotopical sense [1, Corollary 8.13].

Definition 31. For a scheme (or simplicial scheme) X , let $\text{HRR}(X)$ be the category of rigid hypercovers of X .

The notation comes from [9]. The critical property of this category is that it is cofiltered [9, Proposition 4.3]. Since there is at most one map between any two objects, $\text{HRR}(X)$ is actually a directed set.

Lemma 32. *Let X be a scheme. The category of rigid hypercovers over X is equivalent to the category of rigid hypercovers over the constant simplicial scheme cX .*

Proof. Consider the functor $\text{HRR}(X) \rightarrow \text{HRR}(cX)$ that takes a rigid hypercover U of X to the hypercover V of cX given by the formula $V_{m,n} = U_n$. This functor is full and faithful, so it suffices to show that every rigid hypercover of cX belongs to the image of this functor.

Let V be an arbitrary rigid hypercover of cX . Then V is a simplicial diagram in the category $\text{HRR}(X)$. There is at most one rigid hypercover map between any two rigid hypercovers of X , so the map $V_{n,\cdot} \rightarrow V_{n,\cdot}$ is the identity map for all $[n] \rightarrow [n]$. It follows that all of the maps $V_{n,\cdot} \rightarrow V_{m,\cdot}$ are isomorphisms; in fact, they are all the same isomorphism for all maps from $[m]$ to $[n]$. \square

The following lemma is a key property of hypercovers. It provides a technical ingredient in the construction of rigid pullbacks and rigid limits of rigid hypercovers later in this section.

Proposition 33. *Every hypercover of a scheme is split. Also, every rigid hypercover of a scheme is split.*

Proof. Let U be a hypercover of X . By induction and Proposition 13, each U_n and each $(\text{cosk}_n^X U)_{n-1}$ are étale schemes over X . Thus, U is a simplicial object in the category of étale schemes over X . The remark after [1, Definition 8.1] finishes the argument.

The proof of the second claim is similar. Instead of considering étale schemes over X , we must consider disjoint unions of étale schemes over X . \square

6.1. Rigid pullbacks

Using rigid pullbacks of rigid covers, we can also construct rigid pullbacks of rigid hypercovers.

Definition 34. Suppose $f : X \rightarrow Y$ is a map of schemes and U is a rigid hypercover of Y . Then the *rigid pullback* f^*U is the rigid hypercover of X constructed as follows. Let $(f^*U)_0$ be the rigid pullback along f of the rigid cover $U_0 \rightarrow Y$. Inductively define $(f^*U)_n$ to be the rigid pullback along $(\text{cosk}_{n-1}^X f^*U)_n \rightarrow (\text{cosk}_{n-1}^Y U)_n$ of the rigid cover $U_n \rightarrow (\text{cosk}_{n-1}^Y U)_n$.

Remark 35. The face maps of f^*U are easy to describe; they are induced by the map $(f^*U)_n \rightarrow (\text{cosk}_{n-1}^X f^*U)_n$. The degeneracy maps are somewhat more complicated. We need to describe a map d from the latching object $L_n(f^*U)$ to $(f^*U)_n$. There is a natural map from $L_n(f^*U)$ to the pullback of the diagram

$$U_n \rightarrow (\text{cosk}_{n-1}^Y U)_n \leftarrow (\text{cosk}_{n-1}^X f^*U)_n,$$

but this pullback is *not* exactly equal to $(f^*U)_n$. See Remark 21 for the difference between the pullback and $(f^*U)_n$. In order to produce the desired map $d : L_n f^*U \rightarrow (f^*U)_n$, we must specify which component of $(f^*U)_n$ is the target of each component of $L_n f^*U$. Since $L_n f^*U$ is a disjoint union of copies of $(f^*U)_m$ for $m < n$, each component has a basepoint. Let C be a component of $L_n f^*U$ with basepoint c . Then d is defined to take C into the component $((f^*U)_n)_{c'}$ of $(f^*U)_n$, where c' is the image of c under the map $L_n(f^*U) \rightarrow (\text{cosk}_{n-1}^X f^*U)_n$.

This complication with defining the degeneracies is not really important; all that matters is that it is possible to define them in a natural way.

A careful inspection of the definitions indicates that rigid pullbacks of rigid hypercovers are functorial. This means that the definition of rigid pullbacks extends to rigid hypercovers of simplicial schemes.

Also note that there is a canonical rigid hypercover map $f^*U \rightarrow U$ over the map $f : X \rightarrow Y$.

Proposition 36. *Let U be a rigid hypercover of a scheme Y , and let $f : X \rightarrow Y$ be any map of schemes. The rigid hypercover f^*U of X has the following universal property. Let V be an arbitrary rigid hypercover of a scheme Z . Rigid hypercover maps $V \rightarrow f^*U$ over a map $Z \rightarrow X$ correspond bijectively to rigid hypercover maps $V \rightarrow U$ over the composition $Z \rightarrow X \rightarrow Y$.*

Proof. This follows from Proposition 22 and induction. Because V , U , and f^*U are all split by Proposition 33, the degeneracy maps take care of themselves. \square

6.2. Rigid limits

We will now use rigid limits of rigid covers to make a similar construction for rigid hypercovers. The next lemma demonstrates the problem with ordinary limits.

Lemma 37. *Suppose that U is a finite diagram of rigid hypercovers, and let X equal U_{-1} . Then*

$$(\lim U)_n \rightarrow \text{cosk}_{n-1}^{\lim X} (\lim U)_n$$

is an infinite étale cover.

Proof. First note that

$$\text{cosk}_{n-1}^{\lim X} (\lim U)_n \cong \lim_a (\text{cosk}_{n-1}^{X^a} U^a)_n.$$

Thus Lemma 23 gives us the surjectivity. Proposition 13 finishes the proof. \square

As in Lemma 23, the above proposition is not true if each U^a is only a hypercover. Also, $\lim U$ is not a rigid hypercover because the components of $(\lim U)_n$ do not necessarily correspond to geometric points of the target.

Let U be a finite diagram of rigid hypercover maps, and let X equal U_{-1} . Let V be the simplicial scheme $\lim_a U^a$ over $Y = \lim_a X^a$. Lemma 37 implies that V is almost a hypercover of Y ; the only problem is that the étale covers have infinitely many pieces. As observed above, it is also not quite a rigid hypercover. As for rigid covers, we need a more refined construction in order to obtain a rigid hypercover $W = \text{Rlim}_a U^a$ of Y and a natural map $W \rightarrow V$ over Y .

Begin by defining W_0 to be the rigid limit $\text{Rlim}_a U_0^a$ of the rigid covers $U_0^a \rightarrow X^a$. There is a canonical map from W_0 to $V_0 = \lim_a U_0^a$.

Suppose for sake of induction that W_m and the map $W_m \rightarrow V_m$ have been defined for $m < n$. Thus there is a map $(\text{cosk}_{n-1}^Y W)_n \rightarrow (\text{cosk}_{n-1}^Y V)_n$. Let x be a geometric point of $(\text{cosk}_{n-1}^Y W)_n$, and let y be its image in $(\text{cosk}_{n-1}^Y V)_n$. Since $(\text{cosk}_{n-1}^Y V)_n$ is isomorphic to $\lim_a (\text{cosk}_{n-1}^{X^a} U^a)_n$, y gives compatible geometric points y^a in each of the schemes $(\text{cosk}_{n-1}^{X^a} U^a)_n$. Each y^a has a canonical lift z^a in U_n^a since each U^a is a rigid hypercover. Moreover, these lifts are compatible since U is a diagram of rigid hypercover maps. This means that they assemble to give a geometric point z of $V_n = \lim_a U_n^a$, and z is a lift of y .

Now define $(W_n)_x$ to be the connected component of

$$V_n \times_{(\text{cosk}_{n-1}^Y V)_n} (\text{cosk}_{n-1}^Y W)_n$$

containing $z \times x$, and let $z \times x$ be the basepoint of $(W_n)_x$. This extends the definition of W to dimension n .

Remark 38. To describe the degeneracy maps of W , one must use a technical argument similar to that given in Remark 35.

Proposition 39. *Rigid limits of rigid hypercovers have the following universal property. Suppose that U is a diagram of rigid hypercover maps, and let V be an arbitrary rigid hypercover. Rigid hypercover maps from V to $\text{Rlim } U$ are in one-to-one correspondence with collections of rigid hypercover maps $V \rightarrow U^a$ such that for every map $U^a \rightarrow U^b$, the diagram*

$$\begin{array}{ccc} V & \longrightarrow & U^a \\ & \searrow & \downarrow \\ & & U^b \end{array}$$

of rigid hypercover maps commutes.

Proof. This follows from Proposition 26 and induction. The degeneracy maps take care of themselves because V , each U^a , and $\lim U$ are all split by Proposition 33 (for $\lim U$, one also needs Lemma 37). \square

Remark 40. As for rigid limits of rigid covers, rigid limits of rigid hypercovers have the same kind of functoriality as ordinary limits. See Remark 27 for more details.

We use the notations \prod^R , \otimes^R , and Rcosk_n for rigid limits of rigid hypercovers analogously to our use of these notations for rigid covers as in Section 5.2.

6.3. Cofinal functors of rigid hypercovers

For every simplicial scheme X and every $n \geq 0$, there is a forgetful functor $\text{HRR}(X) \rightarrow \text{HRR}(X_n)$ taking a rigid hypercover U of X to the rigid hypercover $U_{n,\cdot}$ of X_n . These functors assemble to give a functor

$$\text{HRR}(X) \rightarrow \text{HRR}(X_0) \times \text{HRR}(X_1) \times \cdots \times \text{HRR}(X_n).$$

The idea is that this functor forgets the face and degeneracy maps and only remembers the objects $U_{m,\cdot}$ for $m \leq n$.

Proposition 41. *Let X be a simplicial scheme. The functor*

$$\text{HRR}(X) \rightarrow \text{HRR}(X_0) \times \text{HRR}(X_1) \times \cdots \times \text{HRR}(X_n).$$

is cofinal.

This proposition is closely related to [9, Corollary 4.6], which shows that the functor $\text{HRR}(X) \rightarrow \text{HRR}(X_n)$ is cofinal for every simplicial scheme X and every $n \geq 0$.

Proof. For convenience, let I be the category

$$\text{HRR}(X_0) \times \text{HRR}(X_1) \times \cdots \times \text{HRR}(X_n).$$

Since each $\text{HRR}(X_m)$ is actually a directed set, so is I . The category $\text{HRR}(X)$ is also a directed set, so it suffices to show that if $(U_{0,\cdot}, U_{1,\cdot}, \dots, U_{n,\cdot})$ is any object of I , then there is an object V of $\text{HRR}(X)$ and a rigid hypercover map $V_{m,\cdot} \rightarrow U_{m,\cdot}$ over X_m for every $m \leq n$. \square

For each m , define $V_{m,\cdot}$ to be

$$\text{Rlim}_{\substack{\phi : [k] \rightarrow [m] \\ k \leq n}} U_{k,\cdot}.$$

The idea is that $V_{m,\cdot}$ is a “rigid right Kan extension”. The rigid limit is finite because k is at most n .

The functoriality of rigid limits as expressed in Remark 40 assures us that V is in fact a rigid hypercover of X . The projections

$$V_{m,\cdot} \rightarrow U_{m,\cdot}$$

are the desired maps.

7. Realizations of pro-spaces

Let \mathcal{C} be a simplicial category; this means that objects of \mathcal{C} can be tensored and cotensored with simplicial sets, and these operations satisfy appropriate adjointness conditions. We assume that \mathcal{C} is complete and cocomplete. Our application involves pro-spaces, which is a complete and cocomplete category [13, Proposition 11.1].

Recall the definition of the realization of a simplicial object in \mathcal{C} .

Definition 42. Given a simplicial object X in a simplicial category \mathcal{C} , its *realization* $\text{Re } X$ is the coequalizer of the diagram

$$\coprod_{\phi : [m] \rightarrow [n]} X_n \otimes \Delta[m] \rightrightarrows \coprod_n X_n \otimes \Delta[n],$$

where the upper arrow is induced by maps $\text{id} \otimes \phi_* : X_n \otimes \Delta[m] \rightarrow X_n \otimes \Delta[n]$ and the lower arrow is induced by maps $\phi^* \otimes \text{id} : X_n \otimes \Delta[m] \rightarrow X_m \otimes \Delta[m]$.

The realization of X is a coend over Δ of the simplicial object X with the cosimplicial object $\Delta[\cdot]$. The most important property of realization is that it is left adjoint to the functor sending an object Y of \mathcal{C} to the simplicial object $Y^{\Delta[\cdot]}$.

Remark 43. Rather than think of $\text{Re } X$ as a coequalizer, we prefer to think of it as the colimit of the following diagram. The diagram has one object $X_n \otimes \Delta[n]$ for each $n \geq 0$ and one object $X_n \otimes \Delta[m]$ for each $\phi : [m] \rightarrow [n]$. The maps of the diagram are of two types. The first type is of the form $\text{id} \otimes \phi_* : X_n \otimes \Delta[m] \rightarrow X_n \otimes \Delta[n]$, and the second type is of the form $\phi^* \otimes \text{id} : X_n \otimes \Delta[m] \rightarrow X_m \otimes \Delta[m]$. The colimit of this diagram is the realization $\text{Re } X$ of X . Note that the diagram has no non-identity endomorphisms. This fact makes the analysis of realizations of pro-spaces simpler.

Realizations present some problems because they are colimits of infinite diagrams. Sometimes only techniques involving finite colimits are applicable. Hence the following definition is useful.

Definition 44. If X is a simplicial object in a simplicial category \mathcal{C} , then the *n-truncated realization* $\text{Re}_n X$ of X is the coequalizer of the diagram

$$\coprod_{\substack{\phi : [m] \rightarrow [k] \\ m, k \leq n}} X_k \otimes \Delta[m] \rightrightarrows \coprod_{m \leq n} X_m \otimes \Delta[m].$$

This is essentially the same construction as ordinary realization except that only the objects X_m for $m \leq n$ are considered. It can be described as a coend over $\Delta_{\leq n}$ of $\text{sk}_n X$ with the n -truncated standard cosimplicial complex $\Delta_{\leq n}[\cdot]$.

Remark 45. As for realizations, we prefer to think of n -truncated realizations not as coequalizers but as colimits of diagrams with no non-identity endomorphisms. See Remark 43 for more details.

Like ordinary realization, n -truncated realization is also a left adjoint. Namely, it is left adjoint to the functor sending an object Y of \mathcal{C} to the simplicial object that is the n th coskeleton of the simplicial object $Y^{\Delta[\cdot]}$.

There is a canonical map $\text{Re}_n X \rightarrow \text{Re } X$ for every simplicial object X . Of course, this map is not an isomorphism in general. However, for simplicial sets, it is an isomorphism on low-dimensional simplices as stated in the next proposition.

Proposition 46. *Let X be a simplicial space. Then the natural map $\text{sk}_n \text{Re}_n X \rightarrow \text{sk}_n \text{Re } X$ is an isomorphism.*

Proof. We show that both functors $\text{sk}_n \text{Re}_n$ and $\text{sk}_n \text{Re}$ have the same right adjoint. The right adjoint of $\text{sk}_n \text{Re}$ is the functor taking a space Y to the simplicial space $(\text{cosk}_n Y)^{\Delta[\cdot]}$. On the other hand, the right adjoint of $\text{sk}_n \text{Re}_n$ is the functor taking a space Y to the n th coskeleton of the simplicial space $(\text{cosk}_n Y)^{\Delta[\cdot]}$. For formal reasons, this last simplicial space is isomorphic to the simplicial space $(\text{cosk}_n Y)^{\text{sk}_n \Delta[\cdot]}$. To show that $(\text{cosk}_n Y)^{\text{sk}_n \Delta[m]}$ and $(\text{cosk}_n Y)^{\Delta[m]}$ are isomorphic, use adjunction and the fact that $\text{sk}_n(X \times Z)$ is isomorphic to $\text{sk}_n(X \times \text{sk}_n Z)$ for every X and Z . \square

Corollary 47. *Let X be a simplicial space. Then for every $i < n$, the map $\pi_i \text{Re}_n X \rightarrow \pi_i \text{Re } X$ is an isomorphism.*

Proof. When $i < n$, the i th homotopy group of X only depends on $\text{sk}_n X$. Hence Proposition 46 gives the result. \square

Now we specialize the above ideas about realizations to the category of pro-spaces.

Given any pro-space X , apply sk_n to each X_s to obtain another pro-space $\text{sk}_n X$. Define $\text{cosk}_n X$ similarly. A straightforward computation shows that sk_n and cosk_n are adjoint functors from pro-spaces to pro-spaces.

The following proposition is a direct analogue for pro-spaces of Proposition 46.

Proposition 48. *Let X be a simplicial object in the category of pro-spaces. Then the natural map $\text{sk}_n \text{Re}_n X \rightarrow \text{sk}_n \text{Re } X$ is an isomorphism of pro-spaces.*

Proof. The proof is basically the same as the proof of Proposition 46. One just needs to check that the ingredients used there also apply to pro-spaces. \square

Corollary 49. *Let X be a simplicial object in the category of pointed pro-spaces. Then for every $i < n$, the map $\pi_i \text{Re}_n X \rightarrow \pi_i \text{Re } X$ is an isomorphism of pro-groups.*

Proof. When $i < n$, the i th homotopy pro-group of X only depends on $\mathrm{sk}_n X$. Hence Proposition 48 gives the result. \square

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References

- [1] M. Artin, B. Mazur, *Etale Homotopy*, in: *Lecture Notes in Mathematics*, Vol. 100, Springer, Berlin, 1969.
- [2] B.A. Blander, Local projective model structures on simplicial presheaves, *K-Theory* 24 (3) (2001) 283–301.
- [3] J.D. Christensen, D.C. Isaksen, *Duality and pro-spectra*, preprint.
- [4] D. Dugger, Universal homotopy theories, *Adv. in Math.* 164 (1) (2001) 144–176.
- [5] D. Dugger, S. Hollander, D.C. Isaksen, *Hypercovers and simplicial presheaves*, *Math. Proc. Cambridge Philos. Soc.*, to appear.
- [6] D. Dugger, D.C. Isaksen, *Hypercovers in topology*, to appear.
- [7] W. Dwyer, E. Friedlander, V. Snaith, R. Thomason, Algebraic K -theory eventually surjects onto topological K -theory, *Invent. Math.* 66 (3) (1982) 481–491.
- [8] A. Grothendieck, J. Dieudonné, *Étude locale des schémas et des morphismes des schémas (EGA IV)*, Tome 1–4, *Publ. Math. IHES* 20 (1964); 24 (1965); 28 (1966); 32 (1967).
- [9] E.M. Friedlander, *Etale Homotopy of Simplicial Schemes*, in: *Annals of Mathematics Stud.*, Vol. 104, Princeton University, Princeton, NJ, 1982.
- [10] R. Hartshorne, *Algebraic Geometry*, in: *Graduate Texts in Mathematics*, Vol. 52, Springer, Berlin, 1977.
- [11] P.S. Hirschhorn, *Model categories and their localizations*, in: *Mathematical Surveys and Monographs*, Vol. 99, American Mathematical Society, Providence, RI, 2003.
- [12] M. Hovey, *Model categories*, in: *Mathematical Surveys and Monographs*, Vol. 63, American Mathematical Society, Providence, RI, 1999.
- [13] D.C. Isaksen, A model structure for the category of pro-simplicial sets, *Trans. Amer. Math. Soc.* 353 (2001) 2805–2841.
- [14] D.C. Isaksen, Calculating limits and colimits in pro-categories, *Fund. Math.* 175 (2002) 175–194.
- [15] D.C. Isaksen, *The pro-Atiyah–Hirzebruch spectral sequence*, in preparation.
- [16] J.F. Jardine, *Simplicial presheaves*, *J. Pure Appl. Algebra* 47 (1987) 35–87.
- [17] J.P. May, *Simplicial Objects in Algebraic Topology*, in: *Van Nostrand Mathematical Studies*, Vol. 11, Van Nostrand, Princeton, 1967.
- [18] J.S. Milne, *Étale Cohomology*, Princeton University Press, Princeton, NJ, 1980.
- [19] F. Morel, Ensembles profinis simpliciaux et interprétation géométrique du foncteur T , *Bull. Soc. Math. France* 124 (1996) 347–373.
- [20] F. Morel, V. Voevodsky, \mathbb{A}^1 -homotopy theory of schemes, *Inst. Hautes Etudes Sci. Publ. Math.* 90 (1999) 45–143.
- [21] Y.A. Nisnevich, The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K -theory, in: J.F. Jardine, V.P. Snaith, (Eds.), *Algebraic K-Theory: Connections with Geometry and Topology*, Vol. 279, NATO Advanced Science Institute Series C *Mathematica and Physical Sciences*, Kluwer Academic Publishers, Dordrecht, 1989, pp. 241–342.

- [22] D. Quillen, Homotopical Algebra, in: *Lecture Notes in Mathematics*, Vol. 43, Springer, Berlin, 1967.
- [23] D. Quillen, Higher algebraic K -theory. I., in: *Algebraic K -theory, I: Higher K -theories*, *Lecture Notes in Mathematics*, Vol. 341, Springer, Berlin, 1973, pp. 85–147.
- [24] M. Artin, A. Grothendieck, J.L. Verdier, *Theorie des topos et cohomologie étale des schémas (SGA 4)*, in: *Lecture Notes in Mathematics*, Vol. 269, Springer, Berlin, 1972.
- [25] G. Tamme, *Introduction to Étale Cohomology*, Springer, Berlin, 1994.
- [26] V. Voevodsky, *The Milnor conjecture*, preprint.