Homework 12
Due April 17, 2007

1. Let R be a commutative ring, and let $M_n(R)$ be the non-commutative ring of $n \times n$ matrices with coefficients in R.

(a) For A and B in $M_n(R)$, write $A \sim B$ if there exists an invertible matrix C such that $A = CAC^{-1}$. Prove that this is an equivalence relation.

(b) For A and B in $M_n(R)$, write $A \sim B$ is there exist invertible matrices C and D such that $A = CBD$. Prove that this is an equivalence relation.

2. Let $T : V \to W$ be a linear transformation of F-vector spaces. Let $\{v_1, \ldots, v_n\}$ be a basis for $\ker(T)$, and let $\{w_1, \ldots, w_m\}$ be a basis for $\text{Image}(T)$. For each j, choose u_j in V such that $T(u_j) = w_j$. Prove that $\{v_1, \ldots, v_n, u_1, \ldots, u_m\}$ is a basis for V.

3. Let T be the linear transformation on $\mathbb{R}^3$ whose matrix with respect to the standard basis is

$$\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

(a) Find the characteristic polynomial of T.

(b) Find the eigenvalues of T.

(c) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors.

(d) Find the minimal polynomial of T.

4. Let F be a field. Show that the polynomial ring $F[x]$ is an F-vector space, where addition is the usual addition and scalar multiplication is the usual multiplication by constants.

Define the function $D : F[x] \to F[x]$ by $D(f(x)) = f'(x)$. Prove that D is a linear transformation.

What can you say about the eigenvalues and eigenvectors of D?

5. Let V be an F-vector space, and let $T : V \to V$ be a linear transformation with a T-invariant subspace W of V. Prove that if

$T|_W : W \to W$ and $\overline{T} : V/W \to V/W$ are both isomorphisms, then T is an isomorphism.

6. Let V be an F-vector space, and let $T : V \to V$ be a linear transformation. A vector v of V is T-cyclic if the only T-invariant subspace of V containing v is V itself. Assume that V is 2-dimensional. Prove that every non-zero vector of V is either T-cyclic or is an eigenvector for T.