1. Let $R$ be a commutative ring, and let $M_n(R)$ be the non-commutative ring of $n \times n$ matrices with coefficients in $R$.

(a) For $A$ and $B$ in $M_n(R)$, write $A \sim B$ if there exists an invertible matrix $C$ such that $A = CAC^{-1}$. Prove that this is an equivalence relation.

(b) For $A$ and $B$ in $M_n(R)$, write $A \sim B$ if there exist invertible matrices $C$ and $D$ such that $A = CBD$. Prove that this is an equivalence relation.

2. Let $T : V \rightarrow W$ be a linear transformation of $F$-vector spaces. Let $\{v_1, \ldots, v_n\}$ be a basis for $\text{ker}(T)$, and let $\{w_1, \ldots, w_m\}$ be a basis for $\text{Image}(T)$. For each $j$, choose $u_j$ in $V$ such that $T(u_j) = w_j$. Prove that $\{v_1, \ldots, v_n, u_1, \ldots, u_m\}$ is a basis for $V$.

3. Let $T$ be the linear transformation on $\mathbb{R}^3$ whose matrix with respect to the standard basis is

$$
\begin{pmatrix}
-9 & 4 & 4 \\
-8 & 3 & 4 \\
-16 & 8 & 7
\end{pmatrix}
$$

(a) Find the characteristic polynomial of $T$.

(b) Find the eigenvalues of $T$.

(c) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors.

(d) Find the minimal polynomial of $T$.

4. Let $F$ be a field. Show that the polynomial ring $F[x]$ is an $F$-vector space, where addition is the usual addition and scalar multiplication is the usual multiplication by constants.

Define the function $D : F[x] \rightarrow F[x]$ by $D(f(x)) = f'(x)$. Prove that $D$ is a linear transformation.

What can you say about the eigenvalues and eigenvectors of $D$?

5. Let $V$ be an $F$-vector space, and let $T : V \rightarrow V$ be a linear transformation with a $T$-invariant subspace $W$ of $V$. Prove that if
$T|_W : W \to W$ and $\overline{T} : V/W \to V/W$ are both isomorphisms, then $T$ is an isomorphism.

6. Let $V$ be an $F$-vector space, and let $T : V \to V$ be a linear transformation. A vector $v$ of $V$ is $T$-cyclic if the only $T$-invariant subspace of $V$ containing $v$ is $V$ itself. Assume that $V$ is 2-dimensional. Prove that every non-zero vector of $V$ is either $T$-cyclic or is an eigenvector for $T$. 