Hamiltonian circuits in Cayley digraphs

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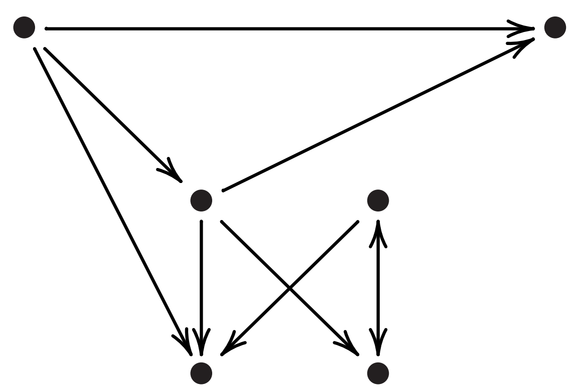
Digraphs

Definition. A digraph is a set V and a subset E of $V \times V$.

The elements of V are called vertices. We think of vertices as points.

The elements of E are called edges, or directed edges. We think of an edge (v, w) as an arrow from the vertex v to the vertex w.



Cayley digraphs

Definition. Let G be a finite group, and let S be a subset of G. The Cayley digraph $\text{Cay}(G; S)$ is the digraph whose vertices consist of the elements of G, and there is an edge from g to h if and only if $g^{-1}h$ belongs to S.

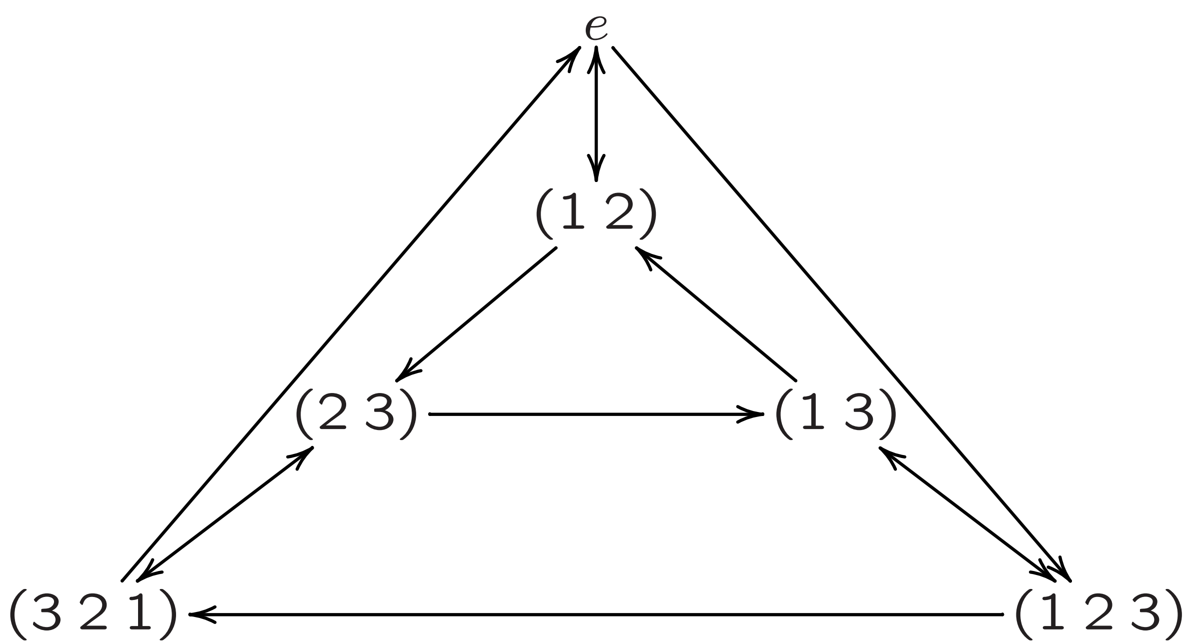
There is an edge from g to h if and only if there exists an element s of S such that $h = gs$.

In other words, you can get from one vertex to another by right multiplication by elements of S.

Arthur Cayley, 1821–1895

An example of a Cayley digraph

Let $G = S_3$ be the symmetric group of order 6, and let S be the set $\{(1\,2), (1\,2\,3)\}$.



Product of two cyclic groups

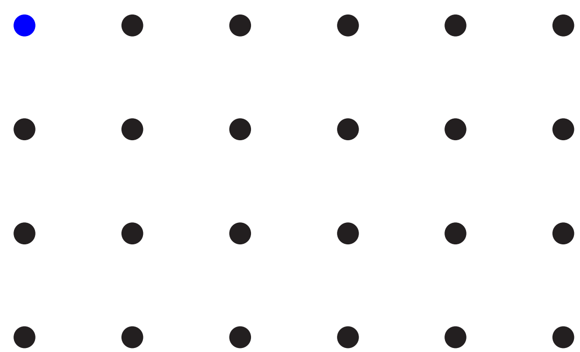
Let $G = \mathbb{Z}_a \times \mathbb{Z}_b$.

Elements of G are ordered pairs (x, y), where x belongs to $\mathbb{Z}_a$ and y belongs to $\mathbb{Z}_b$.

x is an integer modulo a, and y is an integer modulo b.

(Strictly speaking, we should write $\overline{x}$ and $\overline{y}$.)

Product of two cyclic groups

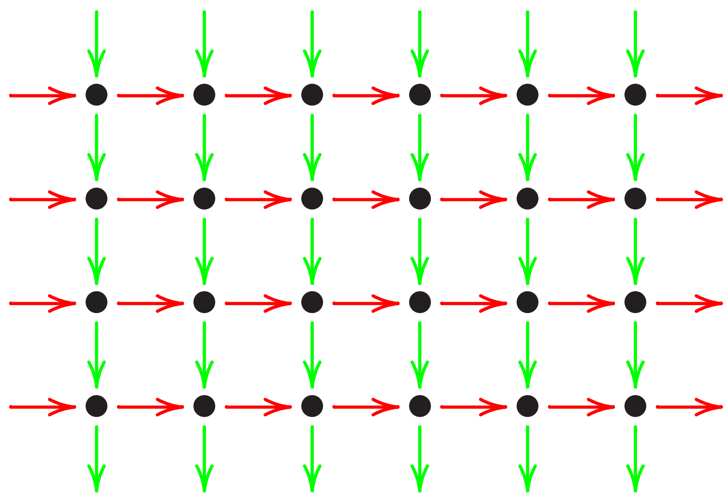


Place the identity element $(0, 0)$ in the upper left corner.

x-coordinates increase to the right, and y-coordinates increase downward.

Torus grids

Now add "horizontal edges" $(x, y) \to (x+1, y)$ and "vertical edges" $(x, y) \to (x, y+1)$.



We have constructed the Cayley digraph of $G = \mathbb{Z}_a \times \mathbb{Z}_b$ with respect to $S = \{(1,0),(0,1)\}$.

Hamiltonian circuits

Definition. A Hamiltonian circuit in a digraph is an ordered sequence of edges

$$v_0 \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_n \longrightarrow v_0$$

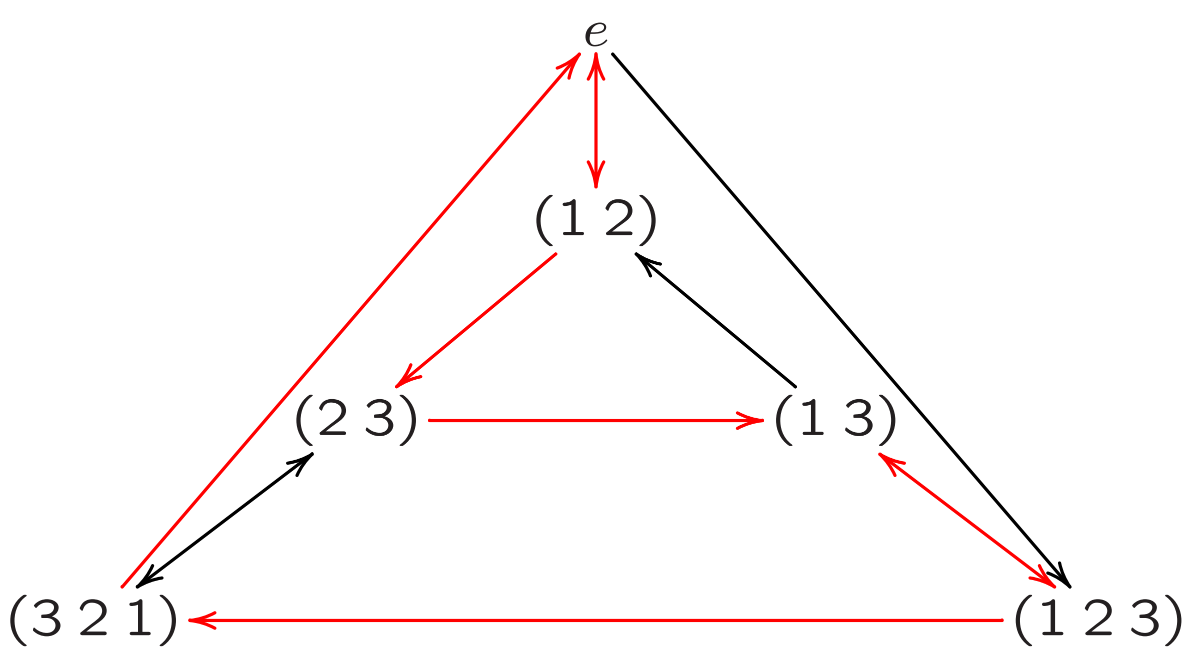
such that each vertex appears exactly once.

A Hamiltonian circuit is just a closed loop that visits each vertex exactly once.

It's a famously difficult question to determine whether an arbitrary digraph has a Hamiltonian circuit.

Sir William Rowan Hamilton, 1805–1865

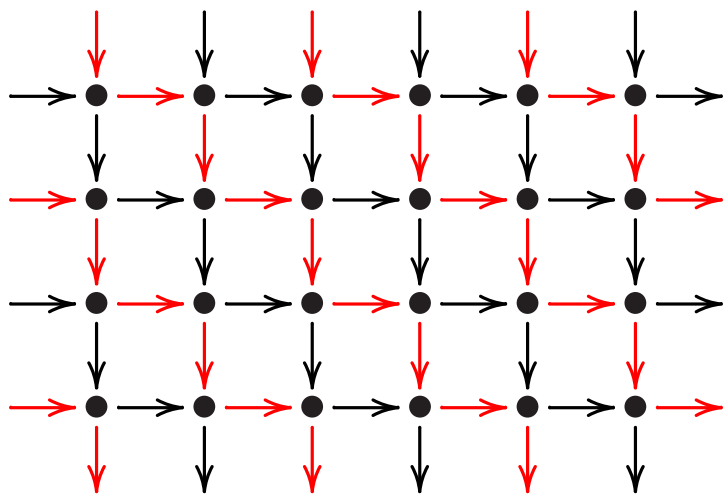
Hamiltonian circuits of Cayley digraphs



This digraph has a Hamiltonian circuit.

Hamiltonian circuits on torus grids

How about $\mathbb{Z}_4 \times \mathbb{Z}_6$?

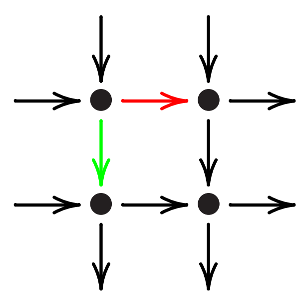


This digraph happens to have a Hamiltonian circuit.

Edge forcing

Definition. A Hamiltonian circuit on a torus grid travels horizontally at a vertex (x, y) if the edge $(x, y) \to (x+1, y)$ belongs to the circuit. It travels vertically at (x, y) if the edge $(x, y) \to (x, y+1)$ belongs to the circuit.

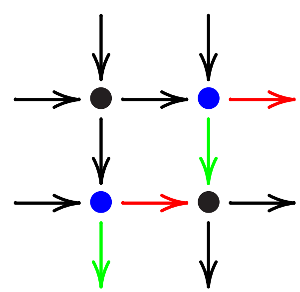
At any vertex (x, y), a Hamiltonian circuit must travel horizontally or vertically.



Edge forcing

Lemma. A Hamiltonian circuit travels horizontally at the vertex (x, y) if and only if it travels horizontally at the vertex $(x+1, y-1)$.

Proof. What happens at the vertex $(x+1, y)$?



Note. A Hamiltonian circuit travels vertically at (x, y) if and only if it travels vertically at $(x+1, y-1)$.

Cosets

Definition. A coset is a non-empty subset S of $\mathbb{Z}_a \times \mathbb{Z}_b$ such that (x, y) belongs to S if and only if $(x + 1, y - 1)$ belongs to S.

Strictly speaking, these are cosets of the subgroup H generated by $(1, -1)$.

A little elementary number theory shows that the size of any coset is $\text{lcm}(a, b)$, and the number of cosets is $\frac{ab}{\text{lcm}(a,b)}$.

Corollary. Constructing a Hamiltonian circuit amounts to choosing a direction of travel for each coset.

Torus knots

Definition. A torus knot is a closed loop embedded in a torus.

Definition. A torus knot has winding number (m, n) if it winds m times “horizontally” and n times “vertically” around the torus.

Theorem. Positive integers m and n are relatively prime if and only if (m, n) is the winding number of some torus knot.

Hamiltonicity of torus grids

Theorem. [Curran] $\mathbb{Z}_a \times \mathbb{Z}_b$ has a Hamiltonian circuit if and only if there exist relatively prime positive integers m and n such that $ma + nb = ab$.

Proof. Suppose that a Hamiltonian circuit C exists.

It has ab edges.

Let (m, n) be the winding number of C, so m and n are relatively prime.

There must be ma horizontal edges in C, and there must be nb vertical edges in C.

Thus $ab = ma + nb$.

Hamiltonicity of torus grids

Now suppose that there exist relatively prime m and n such that $ma + nb = ab$.

Since $ma = b(n - a)$, both a and b divide ma. Similarly, both a and b divide nb.

Construct C by choosing $\frac{ma}{\text{lcm}(a,b)}$ cosets to travel horizontally, and by choosing $\frac{nb}{\text{lcm}(a,b)}$ cosets to travel vertically.

Hamiltonicity of torus grids

This works because

$$\frac{ma}{\operatorname{lcm}(a,b)} + \frac{nb}{\operatorname{lcm}(a,b)} = \frac{ab}{\operatorname{lcm}(a,b)}$$

is the number of cosets in $\mathbb{Z}_a \times \mathbb{Z}_b$.

You have to worry about whether C is connected. But it is connected because it has “total” winding number (m, n), and m and n are relatively prime.

Exercise. Count the distinct Hamiltonian circuits on $\mathbb{Z}_a \times \mathbb{Z}_b$.

Example

Consider $\mathbb{Z}_6 \times \mathbb{Z}_4$, so $a = 6$ and $b = 4$.

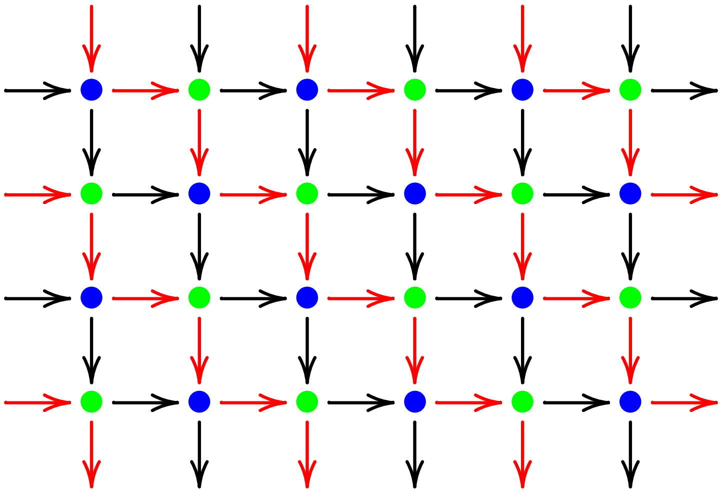
If $m = 2$ and $n = 3$, then $ma + nb = ab$.

The size of each coset is $12 = \text{lcm}(a, b)$, and the number of cosets is $2 = \frac{24}{12} = \frac{ab}{\text{lcm}(a,b)}$.

$\frac{ma}{\text{lcm}(a,b)} = \frac{12}{12} = 1$, and $\frac{nb}{\text{lcm}(a,b)} = \frac{12}{12} = 1$.

Example

One coset should travel horizontally, and one coset should travel vertically.



Counterexample

Consider $\mathbb{Z}_6 \times \mathbb{Z}_3$, so $a = 6$ and $b = 3$.

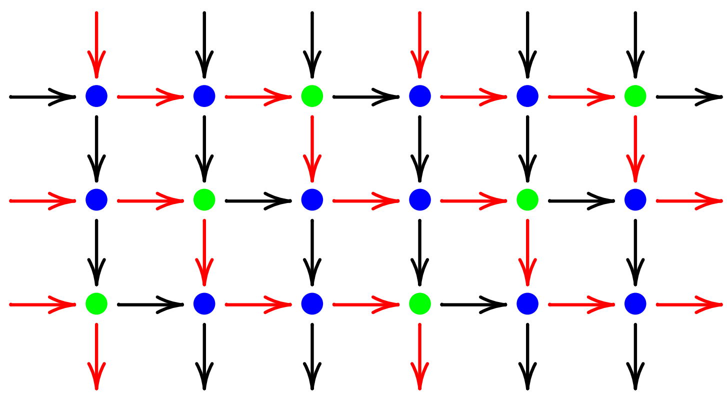
If $m = 2$ and $n = 2$, then $ma + nb = ab$.

The size of each coset is $6 = \text{lcm}(a, b)$, and the number of cosets is $3 = \frac{18}{6} = \frac{ab}{\text{lcm}(a,b)}$.

$\frac{ma}{\text{lcm}(a,b)} = \frac{12}{6} = 2$, and $\frac{nb}{\text{lcm}(a,b)} = \frac{6}{6} = 1$.

Counterexample

Two cosets should travel horizontally, and one coset should travel vertically.



But m and n are not relatively prime, and the circuit is not connected.

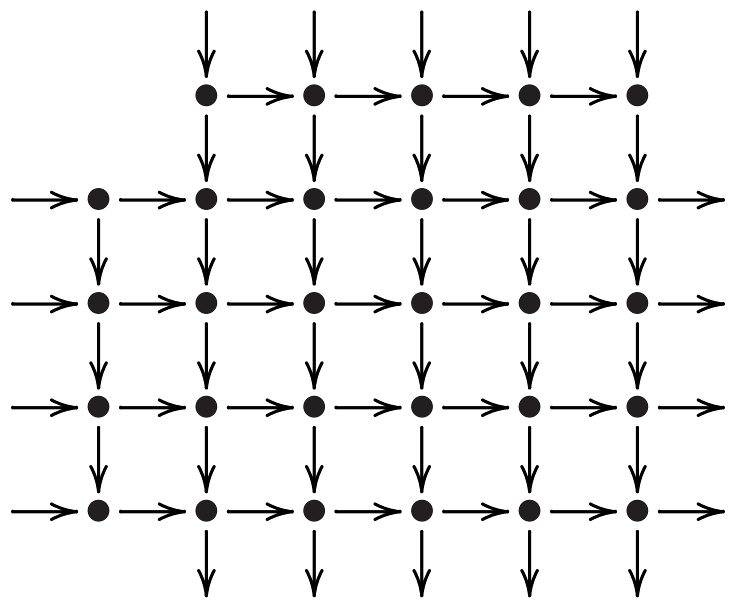
Competing theorems

Theorem. [Curran] $\mathbb{Z}_a \times \mathbb{Z}_b$ has a Hamiltonian circuit if and only if there exist relatively prime positive integers m and n such that $ma + nb = ab$.

Theorem. [Trotter-Erdös] $\mathbb{Z}_a \times \mathbb{Z}_b$ has a Hamiltonian circuit if and only if there exist positive integers m and n such that $m + n = \gcd(a, b)$ and $\gcd(a, m) = 1 = \gcd(b, n)$.

Hypohamiltonicity

Now let's delete the vertex $(0, 0)$ from $\mathbb{Z}_a \times \mathbb{Z}_b$.

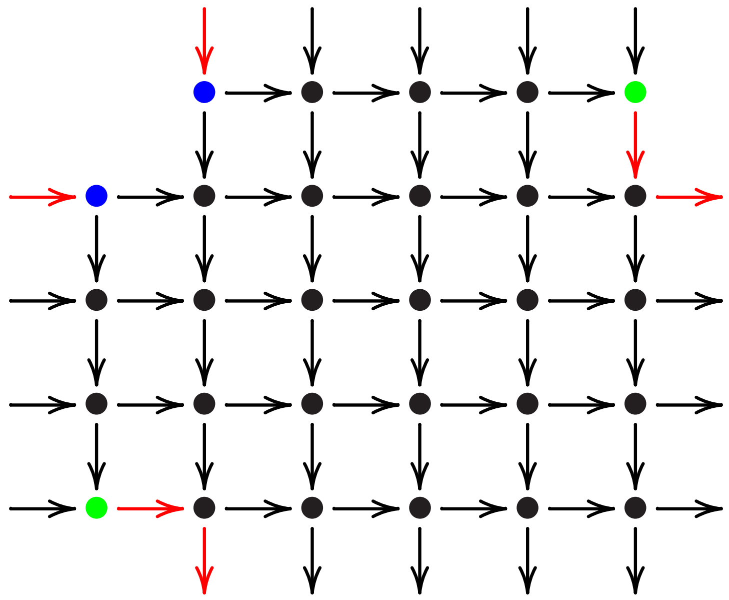


Also delete any hanging edges.

Hypohamiltonicity

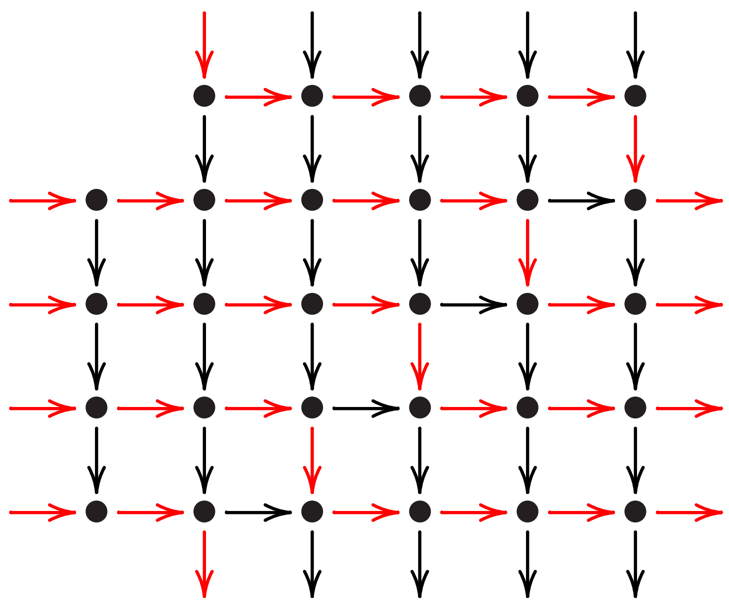
Two vertices have only one in-edge, and two vertices have only one out-edge.

This forces certain edges to be in the Hamiltonian circuit.



Hypohamiltonicity

By edge forcing, this determines the entire Hamiltonian circuit.



Hypohamiltonicity

Theorem. [Penn-Witte] $\mathbb{Z}_a \times \mathbb{Z}_b - \{(0,0)\}$ has a Hamiltonian circuit if and only if there exist relatively prime positive integers m and n such that $ma + nb = ab - 1$.

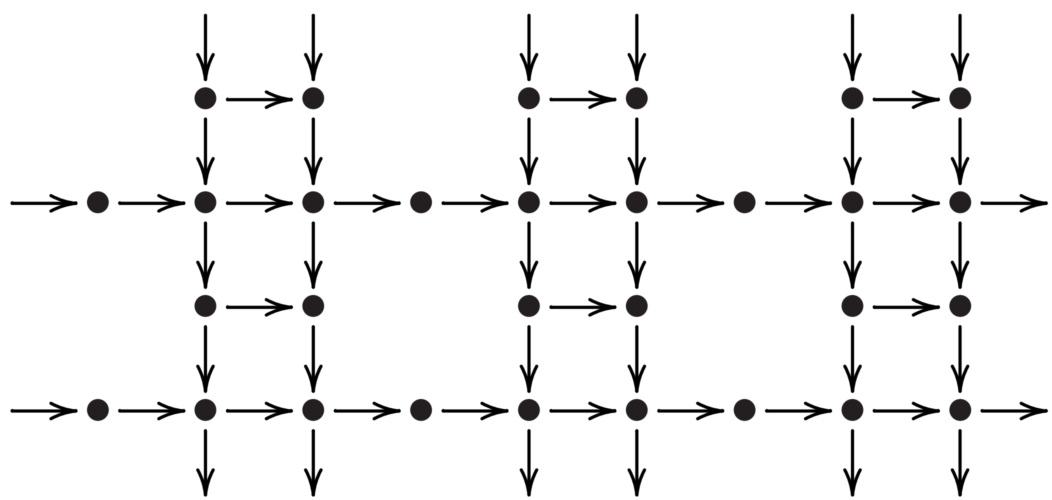
Example. Let $a = 6$ and $b = 5$. If $m = 4$ and $n = 1$, then $ma + nb = ab - 1$.

Note. If $\mathbb{Z}_a \times \mathbb{Z}_b - \{(0,0)\}$ has a Hamiltonian circuit, then it is unique.

Deleting subgroups

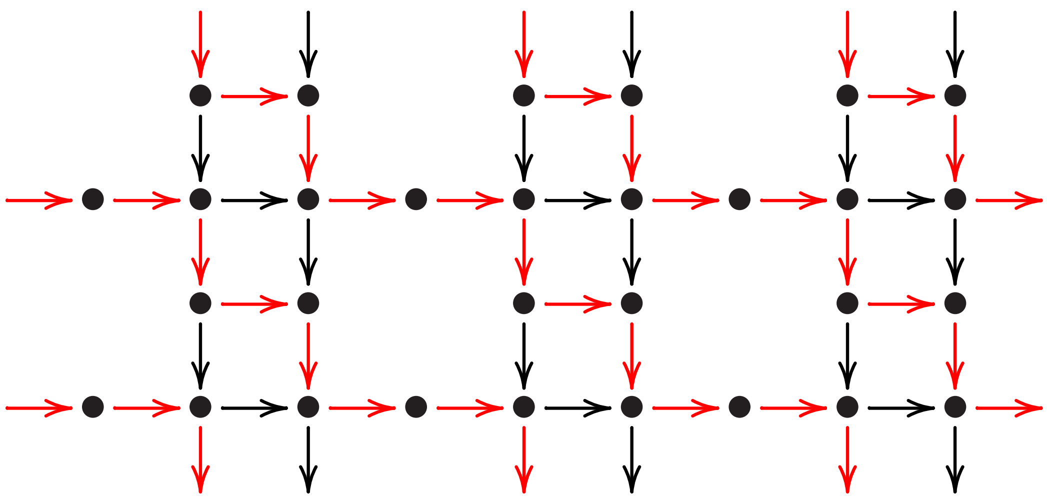
Now let's generalize further to $\mathbb{Z}_a \times \mathbb{Z}_b - \mathbb{Z}_c \times \mathbb{Z}_d$, where c divides a and d divides b.

Example. $\mathbb{Z}_9 \times \mathbb{Z}_4 - \mathbb{Z}_3 \times \mathbb{Z}_2$.



Deleting subgroups

Lots of forcing.



Deleting subgroups

Theorem. [Barone-Mauntel-Miller] $\mathbb{Z}_a \times \mathbb{Z}_b - \mathbb{Z}_c \times \mathbb{Z}_d$ has a Hamiltonian circuit if and only if there exist positive integers m and n such that:

1. $mA + nB = AB - 1$, where $A = \dfrac{a}{c}$ and $B = \dfrac{b}{d}$.

2. $\gcd(dm, cn) = 1$.

Example

Consider $\mathbb{Z}_9 \times \mathbb{Z}_4 - \mathbb{Z}_3 \times \mathbb{Z}_2$, so $a = 9$, $b = 4$, $c = 3$, and $d = 2$. Then $A = 3$ and $B = 2$.

If $m = 1$ and $n = 1$, then $mA+nB = AB-1$, and $\gcd(dm, cn) = 1$. This digraph has a Hamiltonian circuit.

Further variations and generalizations

1. Hyperhamiltonicity [Gallian-Witte] [Jungreis] [Miller].

2. Delete an arbitrary subgroup [Wu].

3. Closed circuits of various lengths [Wilkinson].