ON K_*-ULTRAHOMOGENEOUS GRAPHS

DANIEL C. ISAKSEN, CHRIS JANKOWSKI, AND STEPHANIE PROCTOR

Abstract. Let $\mathcal{C}$ be any class of graphs. A graph G is $\mathcal{C}$-ultrahomogeneous if every isomorphism between induced subgraphs belonging to $\mathcal{C}$ extends to an automorphism of G. We study graphs that are K_*-ultrahomogeneous, where K_* is the class of complete graphs. We also explicitly classify the graphs that are $\sqcup K_*$-ultrahomogeneous, where $\sqcup K_*$ is the class of disjoint unions of complete graphs.

1. Introduction

A mathematical structure is called ultrahomogeneous if every isomorphism between substructures can be extended to an automorphism. In the context of graph theory, a graph G is ultrahomogeneous if every isomorphism between induced subgraphs extends to an automorphism of G. Gardiner [4] gave an explicit classification of the ultrahomogeneous graphs using the previous work of Sheehan [12].

One can vary this basic definition in many interesting ways. For example, G is homogeneous if for every pair of isomorphic subgraphs H_1 and H_2, there exists an isomorphism $H_1 \to H_2$ that extends to an automorphism of G. The difference between homogeneity and ultrahomogeneity is whether or not the isomorphism $H_1 \to H_2$ is specified. Ronse [11] showed that a graph is homogeneous if and only if it is ultrahomogeneous. Two different variations are considered in [3] and [9].

Yet other variations arise by considering only certain types of subgraphs. A graph G is connected-ultrahomogeneous if every isomorphism between connected induced subgraphs extends to an automorphism of G. Gardiner [5] explicitly classified the connected-ultrahomogeneous graphs.

Definition 1.1. Let $\mathcal{C}$ be any class of graphs closed under isomorphism. A graph G is **$\mathcal{C}$-ultrahomogeneous** if every isomorphism between induced subgraphs belonging to $\mathcal{C}$ extends to an automorphism of G.

This definition recovers the original notion of ultrahomogeneity by taking $\mathcal{C}$ to be the class of all graphs. Similarly, we recover $\mathcal{C}$-ultrahomogeneity by taking $\mathcal{C}$ to be the class of all connected graphs.

If H is a given fixed graph, we say that another graph G is H-ultrahomogeneous if it is $\{H\}$-ultrahomogeneous. This is equivalent to requiring that the automorphism group $\mathrm{Aut}(G)$ of the graph G acts transitively on the set of induced subgraphs that are isomorphic to H.

In this paper, we choose a few specific examples of reasonable classes $\mathcal{C}$ and study the $\mathcal{C}$-ultrahomogeneous graphs. Because complete subgraphs are a common subject of study for graph theorists (see, for example, [8] or [10]), we are interested in the

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class K_* of all complete graphs. Thus, we are considering graphs G such that any isomorphism between complete subgraphs of G extends to an automorphism. Note that the K_*-ultrahomogeneous graphs include all ultrahomogeneous and connected-ultrahomogeneous graphs, since these are stronger conditions than K_*-ultrahomogeneity.

Unfortunately, we are able to obtain only partial results about the K_*-ultrahomogeneous graphs. In order to prove something more substantial, we also consider the $\sqcup K_*$-ultrahomogeneous graphs, where $\sqcup K_*$ is the class of graphs that are disjoint unions of complete graphs. The class $\sqcup K_*$ consists of all graphs of the form

$$K_{r_1} \sqcup K_{r_2} \sqcup \cdots \sqcup K_{r_n}.$$

In Section 4, we explicitly classify the $\sqcup K_*$-ultrahomogeneous graphs. We find that the class of $\sqcup K_*$-ultrahomogeneous graphs is only slightly larger than the class of ultrahomogeneous graphs. This enlightens us about the nature of ultrahomogeneity for graphs. Considering all subgraphs turns out to be a highly redundant condition. The seemingly much weaker condition of $\sqcup K_*$-ultrahomogeneity turns out be nearly equivalent.

In this paper, we consider ultrahomogeneity only for finite undirected graphs. However, ultrahomogeneity makes sense in many other combinatorial contexts, such as finite geometries [1] and infinite graph theory [7]. We strongly suspect that it is possible to show that ultrahomogeneity in these situations is also highly redundant.

1.1. **Notation.**

For any graph G and any positive integer t, let tG be the graph that consists of t disjoint copies of G.

If t and n are positive integers, let $K_{t;n}$ be the complete regular multipartite graph containing t partite sets, each of which has n elements. Thus, $K_{t;n}$ is the complete regular multipartite graph with order nt and degree $n(t-1)$. For example; $K_{1;n}$ is the graph nK_1, and $K_{t;1}$ is the graph K_t.

2. $\mathcal{C}$-ULTRAHOMOGENEOUS GRAPHS

We present in this section some general results about $\mathcal{C}$-ultrahomogeneity that will be useful later.

Definition 2.1. If $\mathcal{C}$ is any class of graphs, then $\overline{\mathcal{C}}$ is the class of graphs whose complements belong to $\mathcal{C}$.

For example, the class $\overline{K_*}$ contains all graphs that consist entirely of disjoint vertices. Also, the class $\overline{\sqcup K_*}$ consists of all multipartite complete graphs.

The following theorem will allow us to check whether a graph is $\mathcal{C}$-ultrahomogeneous simply by examining its complement, which in some cases is an easier task.

Theorem 2.2. *A graph G is $\mathcal{C}$-ultrahomogeneous if and only if its complement $\overline{G}$ is $\overline{\mathcal{C}}$-ultrahomogeneous.*

Proof. We prove the backward direction. The proof in the other direction is identical because the complement of the complement of a graph is the original graph.

Suppose that $\overline{G}$ is $\overline{\mathcal{C}}$-ultrahomogeneous. We need to show that G is $\mathcal{C}$-ultrahomogeneous. Let $\phi : H_1 \to H_2$ be an isomorphism of subgraphs in G, where H_1 and H_2 belong to $\mathcal{C}$. Then ϕ canonically determines an isomorphism $\overline{\phi} : \overline{H}_1 \to \overline{H}_2$ in $\overline{G}$, where $\overline{H}_1$ and $\overline{H}_2$ belong to $\overline{\mathcal{C}}$. Since $\overline{G}$ is $\overline{\mathcal{C}}$-ultrahomogeneous, there exists an automorphism $\overline{\psi}$ of $\overline{G}$ taking $\overline{H}_1$ to $\overline{H}_2$. Now $\overline{\psi}$ canonically determines an automorphism ψ of G taking H_1 to H_2, and ψ extends ϕ. □

3. K_*-ULTRAHOMOGENEOUS GRAPHS

In this section, we give some examples of K_*-ultrahomogeneous graphs. Of course, any ultrahomogeneous graph or connected-ultrahomogeneous graph is K_*-ultrahomogeneous. These are not the only examples. For example, any vertex-transitive, strongly edge-transitive, triangle-free graph is K_*-ultrahomogeneous.

Proposition 3.1. *Let G be a K_*-ultrahomogeneous graph. Then the t-fold Cartesian product G^t is K_*-ultrahomogeneous.*

Proof. Suppose that H is an induced subgraph of G^t such that H is isomorphic to K_n for some n. Then H is of the form $\{v_1\} \times \{v_2\} \times \cdots \times J \times \cdots \times \{v_{t-1}\}$, where J is isomorphic to K_n. By permuting factors, there exists an automorphism of G taking H to H', where H' is of the form $J \times \{v_1\} \times \{v_2\} \times \cdots \times \{v_{t-1}\}$. Since G is K_1-ultrahomogeneous (i.e., vertex-transitive), there exists an automorphism of G taking H' to H'', where H'' is of the form $J \times \{v\} \times \{v\} \times \cdots \times \{v\}$, for some fixed vertex v. Finally, since G is K_*-ultrahomogeneous, there exists an automorphism taking H'' to any other subgraph of the form $L \times \{v\} \times \{v\} \times \cdots \times \{v\}$, where L is isomorphic to K_n. $\square$

Note that for an ultrahomogeneous graph G, the graph G^t is not necessarily ultrahomogeneous. In fact, $K_n \times K_n$ is K_*-ultrahomogeneous but not ultrahomogeneous for $n > 3$. This shows that there are K_*-ultrahomogeneous graphs which are not already ultrahomogeneous.

It may seem natural to guess that G is ultrahomogeneous if it is both K_*-ultrahomogeneous and $\overline{K}_*$-ultrahomogeneous, but this is not true. For example, $K_n \times K_n$ for $n > 3$ is K_*-ultrahomogeneous and $\overline{K}_*$-ultrahomogeneous but not ultrahomogeneous, as will be shown below in Lemma 3.3.

Lemma 3.2. *The graph $\overline{K_m \times K_n}$ is K_*-ultrahomogeneous.*

Proof. In the graph $\overline{K_m \times K_n}$, two vertices (v_1, w_1) and (v_2, w_2) are adjacent if and only if $v_1 \neq v_2$ and $w_1 \neq w_2$. Let H and H' be induced subgraphs isomorphic to K_t, and let $\{(v_1, w_1), (v_2, w_2,), \ldots, (v_t, w_t)\}$ and $\{(v'_1, w'_1), (v'_2, w'_2,), \ldots, (v'_t, w'_t)\}$ be the sets of vertices of H and H' respectively. Let ϕ be the isomorphism that takes (v_i, w_i) to (v'_i, w'_i). Note that the vertices $v_1, v_2, \ldots, v_t$ are distinct. Similarly, the vertices $w_1, w_2, \ldots, w_t$ are distinct; the vertices $v'_1, v'_2, \ldots, v'_t$ are distinct; and the vertices $w'_1, w'_2, \ldots, w'_t$ are distinct.

Choose any permutation of the vertices of K_m that takes each v_i to v'_i, and choose any permutation of the vertices of K_n that takes each w_i to w'_i. These permutations induce an automorphism of $\overline{K_m \times K_n}$, and this automorphism extends ϕ. $\square$

Lemma 3.3. *For $n > 3$, the graph $K_n \times K_n$ is K_*-ultrahomogeneous and $\overline{K}_*$-ultrahomogeneous but not ultrahomogeneous.*

Proof. The graph $K_n \times K_n$ is K_*-ultrahomogeneous because of Proposition 3.1. Also, by Lemma 3.2, $\overline{K_n \times K_n}$ is K_*-ultrahomogeneous. Therefore, by Theorem 2.2, $K_n \times K_n$ is $\overline{K}_*$-ultrahomogeneous. But the classification of [4] shows that $K_n \times K_n$ is not ultrahomogeneous. $\square$

It is reasonable to ask whether all the graphs in the class K_* are relevant. Is there any subclass $\mathcal{C}$ of K_* such that the K_*-ultrahomogeneous graphs are precisely equal to the $\mathcal{C}$-ultrahomogeneous graphs? We do not know the answer, but the following example is a preliminary step.

Example 3.4. We construct a graph G that is K_1-ultrahomogeneous and K_2-ultrahomogeneous but *not* K_3-ultrahomogeneous. The idea to use Cayley graphs to find such an example was brought to our attention through the work of Doyle [2] who employed similar techniques to show that there are vertex-transitive, edge-transitive graphs that are not strongly edge-transitive.

Let A be the group $\mathbb{Z}_5 \times \mathbb{Z}_5$. Let $S = \{(x,0),(0,x),(x,x) \mid x \in \mathbb{Z}_5, x \neq 0\}$ be the generating set. It can be shown that the resulting Cayley graph $G = \mathrm{Cay}(A;S)$ is K_1-ultrahomogeneous and K_2-ultrahomogeneous. To see that G is not K_3-ultrahomogeneous, just consider the induced subgraphs H_1 whose vertices are $(0,0)$, $(1,0)$, and $(1,1)$, and H_2 whose vertices are $(0,0)$, $(1,0)$, and $(2,0)$. Then any isomorphism between these two subgraphs does not extend to an automorphism.

The following proposition tells us that we only need to consider connected K_*-ultrahomogeneous graphs if we wish to understand all K_*-ultrahomogeneous graphs.

Proposition 3.5. *If G is K_*-ultrahomogeneous, then G is isomorphic to tH, where H is a connected, K_*-ultrahomogeneous graph.*

Proof. Suppose G is K_*-ultrahomogeneous. The case when G is connected is trivial, so suppose G is not connected. Then G has at least two components. Suppose for sake of contradiction that G has two components H_1 and H_2 that are not isomorphic. Let v_1 be a vertex of H_1 and v_2 a vertex of H_2. Since G is K_*-ultrahomogeneous, there exists an automorphism taking v_1 to v_2. This is a contradiction since H_1 is not isomorphic to H_2. □

4. $\sqcup K_*$-ULTRAHOMOGENEOUS GRAPHS

Let $\mathcal{M}$ be the class of all complete (not necessarily regular) multipartite graphs $K_{r_1,r_2,\ldots,r_n}$, with $n \geq 1$ and each $r_i \geq 1$. Note that $\mathcal{M}$ is the complement of the class $\sqcup K_*$. In order to understand the $\sqcup K_*$-ultrahomogeneous graphs, we will study the $\mathcal{M}$-ultrahomogeneous graphs and then apply Theorem 2.2.

Note in particular that K_n belongs to $\mathcal{M}$ (by taking $r_i = 1$) and that the disjoint union tK_1 of t vertices also belongs to $\mathcal{M}$ (by taking $n = 1$ and $r_1 = t$). Thus, every $\mathcal{M}$-ultrahomogeneous graph is K_*-ultrahomogeneous and also $\overline{K}_*$-ultrahomogeneous.

Definition 4.1. For any induced subgraph H of a graph G, let $\boldsymbol{N_H(G)}$ be the induced subgraph of G consisting of the vertices β of G such that β is adjacent to every vertex of H but not in H. Also, let $\boldsymbol{\overline{N}_H(G)}$ be the induced subgraph of G consisting of the vertices β of G such that β does not belong to either H or $N_H(G)$.

Note that when H consists just of a single vertex α, then $N_H(G)$ (which we also write as $N_\alpha(G)$) is the induced subgraph on the neighbors of α. Also $\overline{N}_\alpha(G)$ is the induced subgraph on the vertices that are not adjacent to α.

We will classify the $\mathcal{M}$-ultrahomogeneous graphs by induction on the number of vertices. The key inductive step is given by the following proposition.

Proposition 4.2. *If G is $\mathcal{M}$-ultrahomogeneous and H is an induced subgraph of G that belongs to $\mathcal{M}$, then $N_H(G)$ is also $\mathcal{M}$-ultrahomogeneous.*

Proof. Let $\phi : K \to K'$ be any isomorphism between induced subgraphs of $N_H(G)$ such that K and K' belong to $\mathcal{M}$. We need to show that ϕ extends to an automorphism of $N_H(G)$.

Let L be the induced subgraph of G consisting of the vertices of H together with the vertices of K. Define L' similarly, using the vertices of H and of K'. Then there is an isomorphism $\tilde{\phi} : L \to L'$; it is the identity on the vertices of H, and it is ϕ on the vertices of K.

Note that L and L' also belong to $\mathcal{M}$. Therefore, $\tilde{\phi}$ extends to an automorphism ψ of G since G is $\mathcal{M}$-ultrahomogeneous. Since ψ fixes H, it restricts to an automorphism of $N_H(G)$. ☐

We now give the explicit classification of $\mathcal{M}$-ultrahomogeneous graphs.

Theorem 4.3. *A graph is $\mathcal{M}$-ultrahomogeneous if and only if it is isomorphic to:*

1. tK_n *for* $t \geq 1$ *and* $n \geq 1$*;*
2. $K_{t;n}$ *for* $t \geq 2$ *and* $n \geq 2$*;*
3. $K_n \times K_n$ *for* $n \geq 3$*; or*
4. C_5*.*

We give the main steps in the proof here, but the technical details are recorded in lemmas later in this section.

Proof. By inspection, each listed graph is $\mathcal{M}$-ultrahomogeneous. For the other implication, suppose that G is $\mathcal{M}$-ultrahomogeneous. If G is disconnected, then G belongs to the above list by Lemma 4.5 below.

Now we may assume that G is connected. The proof is by induction on the number of vertices in G. Choose any vertex α of G. By Proposition 4.2, $N_\alpha(G)$ is again $\mathcal{M}$-ultrahomogeneous, and it has strictly fewer vertices than G. By the induction assumption, $N_\alpha(G)$ belongs to the list in the statement of the theorem. Lemmas 4.7, 4.8, and 4.9 below indicate that $N_\alpha(G)$ must be isomorphic to

1. $2K_n$ with $n \geq 2$; or
2. tK_1 with $t \geq 1$.
3. $K_{t;n}$ with $t \geq 2$ and $n \geq 1$;

The situation when $N_\alpha(G) = K_n$ is included in Case (3). Lemmas 4.10, 4.11, and 4.12 provide an explicit description of G in cases (1), (2), and (3) respectively. ☐

The main point of Theorem 4.3 is to provide a classification of the $\sqcup K_*$-ultrahomogeneous graphs. This is stated in the following corollary.

Corollary 4.4. *A graph is $\sqcup K_*$-ultrahomogeneous if and only if it is isomorphic to:*

1. tK_n *for* $t \geq 1$ *and* $n \geq 1$*;*
2. $K_{t;n}$ *for* $t \geq 2$ *and* $n \geq 2$*;*
3. $\overline{K_n \times K_n}$ *for* $n \geq 3$*; or*
4. C_5*.*

Proof. By Theorem 2.2, we just need to find the complements of the graphs listed in Theorem 4.3. ☐

The only difference between our classification and the classification of ultrahomogeneous graphs [4] is that the graph $\overline{K_n \times K_n}$ is not ultrahomogeneous when $n > 3$.

The rest of this section is dedicated to proving the technical lemmas necessary for the proof of Theorem 4.3.

Lemma 4.5. *If G is disconnected and $\mathcal{M}$-ultrahomogeneous, then G is isomorphic to tK_n for some $t \geq 2$ and $n \geq 1$.*

Proof. Since G is $2K_1$-ultrahomogeneous, every pair of non-adjacent vertices belong to distinct components of G. This implies that G is a disjoint union of complete graphs. But G is also vertex-transitive (i.e., K_1-ultrahomogeneous), so each component has the same order. □

Remark 4.6. Many of the proofs that follow use similar techniques, which we introduce here. Some of these ideas were inspired by the methods of [11].

Let G be a connected $\mathcal{M}$-ultrahomogeneous graph. Then G is vertex-transitive (i.e., K_1-ultrahomogeneous), so the graph $N_\alpha(G)$ is independent (up to isomorphism) of a choice of vertex α in G.

Let β be any vertex of $\overline{N}_\alpha(G)$ (i.e., β is not adjacent to α). Since G is connected and since G is $2K_1$-ultrahomogeneous, the distance between any two non-adjacent vertices must equal 2. In other words, α and β have at least one common neighbor. In fact, every pair of non-adjacent vertices has the same number of common neighbors. Let $\boldsymbol{m}$ be this number; note that it is at least one.

Let $\boldsymbol{r}$ be the number of vertices in $\overline{N}_\alpha(G)$ (i.e., the number of vertices not adjacent to α). The total number of paths of length 2 from α to $\overline{N}_\alpha(G)$ is equal to rm.

Now let $\boldsymbol{d}$ be the number of vertices in $N_\alpha(G)$ (i.e., the degree of α) and let $\boldsymbol{k}$ be the degree of each vertex in $N_\alpha(G)$ (i.e., the number of copies of K_3 containing any given edge). Then each vertex in $N_\alpha(G)$ has exactly $d-k-1$ neighbors in $\overline{N}_\alpha(G)$, so the total number of paths of length 2 from α to $\overline{N}_\alpha(G)$ is equal to $d(d-k-1)$. Thus,

$$rm = d(d-k-1). \tag{4.1}$$

We will use this equation frequently in the following proofs.

The graph $H = N_{\alpha,\beta}(G)$ does not depend (up to isomorphism) on the choice of β in $\overline{N}_\alpha(G)$ because G is $2K_1$-ultrahomogeneous. Let $\boldsymbol{s}$ be the number of vertices in $\overline{N}_\alpha(G)$ that are adjacent to every vertex of H. Since β is such a vertex, $s \geq 1$. The graph spanned by α together with H might belong to $\mathcal{M}$; in practice, this condition will be satisfied in all the cases that we study. Then s is also independent of the choice of β. If $\boldsymbol{p}$ is the number of subgraphs of $N_\alpha(G)$ that are isomorphic to H, then

$$r = sp. \tag{4.2}$$

Let $\boldsymbol{v}$ be the number of vertices in G, and let $\boldsymbol{q}$ be the number of copies of K_n in the subgraph $N_\alpha(G)$. Then α is contained in exactly q copies of K_{n+1}. Since G is K_1-ultrahomogeneous, every vertex is contained in exactly q copies of K_{n+1}. Therefore, qv equals $n+1$ times the number of copies of K_{n+1} in G, so

$$n+1 \text{ divides } qv. \tag{4.3}$$

Lemma 4.7. *If G is connected and $\mathcal{M}$-ultrahomogeneous and α is any vertex of G, then $N_\alpha(G)$ is not isomorphic to C_5.*

Proof. For sake of contradiction, suppose that $N_\alpha(G)$ is isomorphic to C_5. Now G is regular of degree 5, and G has $6+r$ vertices. Equation 4.1 tells us that $rm = 10$, so r is a divisor of 10. Letting $n = 2$, Equation 4.3 tells us that 3 divides $5(6+r)$, so r is a multiple of 3. But there are no divisors of 10 that are also multiples of 3. □

Lemma 4.8. *If G is connected and $\mathcal{M}$-ultrahomogeneous and α is any vertex of G, then $N_\alpha(G)$ is not isomorphic to $K_n \times K_n$ with $n \geq 3$.*

Proof. For sake of contradiction, suppose that $N_\alpha(G)$ is isomorphic to $K_n \times K_n$ with $n \geq 3$. To fix notation, let $\{(\delta_i, \delta_j) | i, j = 1, 2, \ldots, n\}$ be the vertices of $N_\alpha(G)$, where (δ_i, δ_j) and (δ_k, δ_l) are adjacent if and only if $i = k$ or $j = l$.

Let β be any vertex of $\overline{N}_\alpha(G)$, and let γ be a vertex in $N_\alpha(G)$ that is also adjacent to β. We may assume that γ is the vertex (δ_1, δ_1). Consider $N_\gamma(G)$, which is also isomorphic to $K_n \times K_n$; we know that it contains α, β, (δ_i, δ_1), and (δ_1, δ_i) for every i. It follows that β is adjacent to at least two more vertices of $N_\alpha(G)$, one of the form (δ_1, δ_j) and one of the form (δ_k, δ_1). Moreover, β is not adjacent to any other vertices of the form (δ_1, δ_l) or (δ_l, δ_1).

This implies that $N_{\alpha,\beta}(G)$ is isomorphic to a non-empty disjoint union of cycles of even length. But $N_{\alpha,\beta}(G)$ must be $\mathcal{M}$-ultrahomogeneous by Proposition 4.2, so $N_{\alpha,\beta}(G)$ is actually isomorphic to C_4 and $m = 4$.

Note that the graph spanned by α together with $N_{\alpha,\beta}(G)$ is isomorphic to $K_{1,2,2}$; in particular, it belongs to $\mathcal{M}$. Therefore, the number s is independent of the choice of β. There are $\binom{n}{2}^2$ copies of C_4 in $K_n \times K_n$, so Equation 4.2 tell us that

$$r = s\binom{n}{2}^2.$$

On the other hand, Equation 4.1 tells us that

$$4r = n^2(n-1)^2.$$

It follows that $s = 1$, which means that for every copy C of C_4 in $N_\alpha(G)$, there exists exactly one vertex β in $\overline{N}_\alpha(G)$ such that β is adjacent to each vertex of C.

Now we know that G contains $1 + n^2 + \binom{n}{2}^2$ vertices, and $N_\alpha(G)$ contains $2n$ copies of K_n. By Equation 4.3, $n+1$ divides $2n(1 + n^2 + \binom{n}{2}^2)$. Performing arithmetic modulo $n+1$,

$$0 \equiv 2n\left(1 + n^2 + \binom{n}{2}^2\right) \equiv 2(-1)\left(1 + 1(-1)^2 + \binom{n}{2}^2\right) \equiv -4 - 2\binom{n}{2}^2.$$

Analyzing the cases when n is even and odd separately, it turns out that $2\binom{n}{2}^2$ is always congruent to 2 modulo $n+1$, so $-4 - 2\binom{n}{2}^2$ is congruent to -6 modulo $n+1$. Thus 0 is congruent to -6 modulo $n+1$, so $n+1$ must divide 6. Since $n \geq 3$, the only possibility is $n = 5$.

We are left only with the case $n = 5$. Since $n \geq 3$, the graph $N_\alpha(G)$ contains three vertices that are pairwise non-adjacent. Since G is $3K_1$-ultrahomogeneous, every set of three pairwise non-adjacent vertices has a common neighbor. If β_1 and β_2 are vertices of $\overline{N}_\alpha(G)$ with no common neighbor in $N_\alpha(G)$, then β_1 and β_2 must be adjacent; otherwise, α, β_1, and β_2 are three pairwise non-adjacent vertices with no common neighbor.

Let β be the vertex of $\overline{N}_\alpha(G)$ that is adjacent to every vertex of a given copy of C_4 in $N_\alpha(G)$. There are 51 copies of C_4 in $N_\alpha(G)$ that do not intersect this given copy of C_4, and there are 51 vertices of $\overline{N}_\alpha(G)$ that correspond to these 51 copies of C_4. None of these 51 vertices has a common neighbor with β in $N_\alpha(G)$, so these 51 vertices must all be adjacent to β because of the remark in the previous paragraph. This is impossible since G is 25-regular. □

Lemma 4.9. *Let G be connected and $\mathcal{M}$-ultrahomogeneous, and let α be any vertex of G. If $N_\alpha(G)$ is isomorphic to tK_n, then $n = 1$ or $t \leq 2$.*

Proof. Let $A_1, A_2, \ldots, A_t$ be the components of $N_\alpha(G)$, so each A_j is a complete graph. Let γ be any vertex of $N_\alpha(G)$; then $N_{\alpha,\gamma}(G)$ is isomorphic to K_{n-1}. Because G is K_2-ultrahomogeneous, every pair of adjacent vertices has exactly $n-1$ common neighbors.

If γ_1 and γ_2 both belong to A_j, then they are adjacent. Their common neighbors consist entirely of α together with the other vertices of A_j. In other words, if β is not a neighbor of α, then $N_\beta(G) \cap A_j$ contains at most one vertex. This implies that $1 \leq m \leq t$. Equation 4.1 gives us the formula

$$rm = n^2 t(t-1). \tag{4.4}$$

The graph spanned by α together with $N_{\alpha,\beta}(G)$ is isomorphic to $K_{1,m}$, which belongs to $\mathcal{M}$. Therefore, the number s is independent of the choice of β. The number of copies of mK_1 in $N_\alpha(G)$ is $\binom{t}{m} n^m$, so Equation 4.2 gives the formula

$$r = \binom{t}{m} n^m s.$$

Combining the previous two equations, we get the formula

$$s n^{m-2} \binom{t-1}{m-1} = t-1. \tag{4.5}$$

This means that $t-1$ cannot be strictly smaller than $\binom{t-1}{m-1}$, so there are only four possible values for m: 1, 2, $t-1$, and t. We treat each case separately. In each case, we assume that $t \geq 3$ and $n \geq 2$ and reach a contradiction.

First, note that when $t \geq 3$, the graph $N_\alpha(G)$ has three pairwise non-adjacent vertices. Since G is $3K_1$-ultrahomogeneous, every set of three non-adjacent vertices in G must have a common neighbor.

Case 1: $m = 1$. Let β be any vertex in $\overline{N}_\alpha(G)$, and let γ be its unique neighbor in $N_\alpha(G)$. If δ is another vertex in $\overline{N}_\alpha(G)$ that is not adjacent to γ, then β and δ must be adjacent; otherwise α, β, and δ would form a set of three pairwise non-adjacent vertices with no common neighbor. In other words, every vertex of $\overline{N}_\alpha(G)$ must be adjacent to β or γ (or both). From Equation 4.5, we know that $s = n(t-1)$. Thus, γ has $s = n(t-1)$ neighbors in $\overline{N}_\alpha(G)$, so β has at least $r - n(t-1)$ neighbors in $\overline{N}_\alpha(G)$. Now $nt \geq n^2 t(t-1) - n(t-1)$ since G is nt-regular and since $r = n^2 t(t-1)$. This inequality can never hold when $n \geq 2$ and $t \geq 3$.

Case 2: $m = 2$. First of all, Equation 4.5 tells us that $s = 1$. Let β be any vertex in $\overline{N}_\alpha(G)$, and let γ_1 and γ_2 be its neighbors in $N_\alpha(G)$. Since γ_1 and γ_2 have exactly $n-1$ neighbors in $N_\alpha(G)$ and since G is tn regular, γ_1 and γ_2 have exactly $n(t-1)$ neighbors in $\overline{N}_\alpha(G)$. Thus, there are exactly $2n(t-1)-1$ vertices in $\overline{N}_\alpha(G)$ that are adjacent to γ_1 or γ_2 (or both). If δ is another vertex in $\overline{N}_\alpha(G)$ that is adjacent to neither γ_1 nor γ_2, then β and δ must be adjacent; otherwise α, β, and δ would form a set of three pairwise non-adjacent vertices with no common neighbor.

Therefore, β has at least $r - 2n(t-1) + 1$ neighbors in $\overline{N}_\alpha(G)$. Taking into account the two neighbors of β in $N_\alpha(G)$, $nt \geq \frac{1}{2} n^2 t(t-1) - 2n(t-1) + 3$ since G is nt-regular and since $r = n^2 t(t-1)/2$. This inequality can never hold when $n \geq 2$ and $t \geq 3$.

Case 3: $m = t-1$. Now Equation 4.5 becomes $sn^{t-3} = 1$, so $t \leq 3$. Thus $t = 3$ and $m = 2$. We have reduced this case to Case 2.

Case 4: $m = t$. Now Equation 4.5 becomes $sn^{t-2} = t - 1$, so $t - 1 \geq n^{t-2}$. Since $n \geq 2$ and $t \geq 3$, this can only happen when $n = 2$, $s = 1$, and $t = 3$. It follows that $m = 3$ and $r = 8$. Exhaustion of cases shows that G cannot exist. □

Lemma 4.10. *Let G be an $\mathcal{M}$-ultrahomogeneous graph, and let α be any vertex of G. If $N_\alpha(G)$ is isomorphic to $2K_n$ with $n \geq 2$, then G is isomorphic to $K_{n+1} \times K_{n+1}$.*

Proof. As in the proof of Lemma 4.9, $1 \leq m \leq 2$.

Suppose that $m = 1$. Using Equation 4.4, $r = 2n^2$. Thus G has $1 + 2n + 2n^2$ vertices. Also, $N_\alpha(G)$ contains exactly 2 copies of K_n. Equation 4.3 tells us that $n + 1$ divides $2 + 4n + 4n^2$. Performing arithmetic modulo $n + 1$,

$$0 \equiv 2 + 4n + 4n^2 \equiv 2 + 4(-1) + 4(-1)^2 \equiv 2.$$

This can never happen when $n \geq 2$.

We have contradicted the assumption that $m = 1$, so m must equal 2. Again using the Equation 4.4, $r = n^2$. Thus G has $1 + 2n + n^2$ vertices.

Let γ be any neighbor of α; then $N_{\alpha,\gamma}(G)$ is isomorphic to K_{n-1}. Since G is K_2-ultrahomogeneous, $N_{\gamma_1,\gamma_2}(G)$ is isomorphic to K_{n-1} for any pair of adjacent vertices γ_1 and γ_2.

Let A and B be the two copies of K_n in $N_\alpha(G)$, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the vertices of A. Now each vertex α_i in A is adjacent to the copy of K_n consisting of α together with the other vertices of A. The vertex α_i must be adjacent to another copy B_i of K_n whose vertices are not in A, B, or equal to α. The subgraph $N_{\alpha_i,\alpha_j}(G)$ consists of the vertex α together with the other vertices of A. Therefore, none of the B_i intersect.

The vertex α together with the vertices of $A, B, B_1, \ldots, B_n$ give us $1 + 2n + n^2$ vertices, so we have accounted for all the vertices. We still must account for the edges between vertices belonging to $B, B_1, \ldots, B_n$. In fact, we must account for exactly n edges incident to each such vertex.

Let β be any vertex of B. If β had two neighbors γ_1 and γ_2 in B_i, then β would belong to $N_{\gamma_1,\gamma_2}(G)$. But this is not possible because $N_{\gamma_1,\gamma_2}(G)$ is the copy of K_{n-1} consisting of α_i and the other $n - 2$ vertices of B_i. Therefore, β has at exactly one neighbor in each B_i. From this fact it follows that G must be isomorphic to $K_{n+1} \times K_{n+1}$. □

Lemma 4.11. *Let G be connected and $\mathcal{M}$-ultrahomogeneous, and let α be any vertex of G. Suppose that $N_\alpha(G)$ is isomorphic to tK_1.*

(1) *If $t = 1$, then G is isomorphic to K_2.*
(2) *If $t = 2$, then G is isomorphic to C_4 or C_5.*
(3) *If $t \geq 3$, then G is isomorphic to $K_{2;t}$.*

Proof. When $t = 1$, the graph G is 1-regular, so it is isomorphic to K_2.

When $t = 2$, the graph G is connected and 2-regular. This means that G is isomorphic to a cycle. By inspection, the only $\mathcal{M}$-ultrahomogeneous cycles are C_4 and C_5.

In the rest of the proof, we assume that $t \geq 3$. We follow the same outline as the proof of Lemma 4.9. In the case $m = 1$, we obtain the inequality $t \geq (t-1)^2$; this can never hold when $t \geq 3$.

In the case $m = 2$, we get $t \geq \frac{1}{2}t(t-1) - 2(t-1) + 3$, which implies that $t \leq 5$. Handling the possibilities $t = 3$, $t = 4$, and $t = 5$ separately, exhaustion of cases shows that G cannot exist.

In the case $m = t - 1$, Equation 4.1 tells us that $r = t$. Each vertex β in $\overline{N}_\alpha(G)$ has $t-1$ neighbors in $N_\alpha(G)$, so it must have exactly one neighbor γ in $\overline{N}_\alpha(G)$. Since β and γ both have $t-1$ neighbors amongst the t vertices of $N_\alpha(G)$, they must have a common neighbor δ. Then β, γ, and δ are the vertices of a copy of K_3. This is a contradiction since G contains no copies of K_3.

Finally, in the case $m = t$, the graph $\overline{N}_\alpha(G)$ consists entirely of vertices that are adjacent to every vertex of $N_\alpha(G)$. Equation 4.1 tells us that $r = t - 1$. Since G is t-regular, there are no more vertices and G is isomorphic to $K_{2;t}$. $\square$

Lemma 4.12. *Let G be connected and $\mathcal{M}$-ultrahomogeneous, and let α be any vertex of G. If $N_\alpha(G)$ is isomorphic to the complete regular multipartite graph $K_{t;n}$ with $t \geq 2$ and $n \geq 1$, then G is isomorphic to $K_{t+1;n}$.*

Proof. Let $A_1, A_2, \ldots, A_t$ be the partite sets of $N_\alpha(G)$. Choose any vertex β of A_1. Since $N_\beta(G)$ is also isomorphic to $K_{t;n}$, there exist $n-1$ vertices $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ of G such that each α_j is adjacent to β and to every vertex of $A_2, \ldots A_t$ but is not adjacent to α. Also, the vertices α_j and α_k are not adjacent for $j \neq k$.

It remains only to show that each α_j is adjacent to every vertex of A_1. To do this, choose a vertex γ in A_2 and consider $N_\gamma(G)$, which is again isomorphic to $K_{t;n}$. $\square$

References

[1] A. Devillers, *Ultrahomogeneous semilinear spaces*, Proc. London Math. Soc. (3) **84** (2002), no. 1, 35–58.

[2] P. Doyle, *A 27-vertex graph that is vertex-transitive and edge-transitive but not 1-transitive*, unpublished, 1976.

[3] H. Enomoto, *Combinatorially homogeneous graphs*, J. Combin. Theory Ser. B **30** (1981), no. 2, 215–223.

[4] A. Gardiner, *Homogeneous graphs*, J. Combin. Theory Ser. B **20** (1976), 94-102.

[5] A. Gardiner, *Conditions for homogeneity*, J. Combin. Theory Ser. B **24** (1981), 301-310.

[6] F. Harary, *Graph Theory*, Perseus Books, 1969.

[7] A. H. Lachlan, and R. E. Woodrow, *Countable ultrahomogeneous undirected graphs*, Trans. Amer. Math. Soc. **262** (1980), no. 1, 51–94.

[8] S. D. Monson, N. J. Pullman, and R. Rees, A survey of clique and biclique coverings and factorizations of $(0, 1)$-matrices, *Bull. Inst. Combin. Appl.* **14** (1995), 17–86.

[9] R. W. Myers, Jr., 1-*homogeneous graphs*, Discrete Math. **57** (1985), no. 1-2, 67–88.

[10] N. J. Pullman, Clique coverings of graphs—a survey, Combinatorial mathematics X (Adelaide, 1982), 72–85, Lecture Notes in Mathematics **1036**, Springer, 1983.

[11] C. Ronse, *On homogeneous graphs*, J. London Math. Soc (2) **17** (1978), 375–379.

[12] J. Sheehan, *Smoothly embeddable subgraphs*, J. London Math. Soc. (2) **9** (1974), 212–218.

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556 USA

20690 Via Ventana, Yorba Linda, CA 92886
E-mail address: `isaksen@math.wayne.edu`
E-mail address: `sp057175@student.fullerton.edu`
E-mail address: `cjankows@nd.edu`