COMPUTING DIMENSIONS OF ANNIHILATORS IN CAYLEY-DICKSON ALGEBRAS

Abstract. The purpose of this paper is to provide an algorithm for computing the dimensions of annihilators of certain elements in Cayley-Dickson algebras. Specifically, these elements are formed by adding two standard basis vectors.

1. Introduction

This article serves as an appendix to [1]. Accordingly, much of the notation and many results are borrowed. In particular, we will often make reference to the various subspaces used in [1]. Also, we let the associator of x, y, and z be $[x, y, z] = (xy)z - x(yz)$. Other results and notation may be referenced later on, but for a full understanding of the following material the reader should see [1] for background.

2. Associators

Lemma 2.1. *For any a and b, $\mathrm{Ass}[a, b]$ is closed under conjugation.*

Proof. Let x be an element of $\mathrm{Ass}[a, b]$. Since $x^* = 2\,\mathrm{Re}(x) - x$ we calculate $[a, x^*, b] = [a, 2\,\mathrm{Re}(x) - x, b] = [a, 2\,\mathrm{Re}(x), b] - [a, x, b] = 0$. The last equality follows because $2\,\mathrm{Re}(x)$ is central and by the assumption on x. Thus, x^* belongs to $\mathrm{Ass}[a, b]$. □

Corollary 2.2. *For any a and b, $\mathrm{Ass}[a, b]$ equals $\mathrm{Ass}[b, a]$.*

Proof. By linearity we may assume a and b are imaginary. Therefore, $a^* = -a$ and $b^* = -b$. By conjugation we have $(ax)b - a(xb) = 0$ if and only if $(bx^*)a - b(x^*a) = 0$. This shows $x \in \mathrm{Ass}[a, b]$ if and only if $x^* \in \mathrm{Ass}[b, a]$.

Since $\mathrm{Ass}[b, a]$ is closed under conjugation by Lemma 2.1, we have $x \in \mathrm{Ass}[a, b]$ if and only if $x \in \mathrm{Ass}[b, a]$. □

The following is a corollary to [1, Lemma 8.4].

Corollary 2.3. *Let a and b be a quaternionic pair. Then*

1. $\mathrm{Ass}[a, b] = (\mathrm{Ass}[a, b] \cap \mathbb{H}\langle a, b\rangle) \oplus (\mathrm{Ass}[a, b] \cap \mathbb{H}\langle a, b\rangle^{\perp})$.
2. $\mathrm{Ass}'[a, b] = (\mathrm{Ass}'[a, b] \cap \mathbb{H}\langle a, b\rangle) \oplus (\mathrm{Ass}'[a, b] \cap \mathbb{H}\langle a, b\rangle^{\perp})$.

Proof. Using the notation from [1, Lemma 8.4], let $L = A_{a,b} = R_b L_a - L_a R_b$. Then, $Lx = (ax)b - a(xb)$ and $\ker L = \mathrm{Ass}[a, b]$. The first claim now follows directly from [1, Lemma 9.2].

Similarly, if we let $L = A'_{a,b} = R_b L_a + L_a R_b$, the second claim also follows from [1, Lemma 9.2]. □

Corollary 2.4. *Let a and b be a quaternionic pair. Then*

$$\dim(\mathrm{Ass}[a, b] \cap \mathbb{H}\langle a, b\rangle^{\perp}) = \dim \mathrm{Ass}[a, b] - 4.$$

Proof. Notice that $\mathbb{H}\langle a,b\rangle$ is contained in $\mathrm{Ass}[a,b]$, so the dimension of their intersection equals 4. With this observation, the result follows from the first part of Corollary 2.3. $\square$

Lemma 2.5. *Let a and b be a quaternionic pair. Then $\mathrm{Ass}'[a,b]$ is contained in $\mathbb{H}\langle a,b\rangle^\perp$.*

Proof. Suppose x belongs to $\mathrm{Ass}'[a,b]\cap\mathbb{H}\langle a,b\rangle$. Since $\mathbb{H}\langle a,b\rangle$ is associative, we have $2axb=0$. We can cancel to obtain $x=0$. With the second part of Corollary 2.3, the result follows. $\square$

Corollary 2.6. *Let a and b be a quaternionic pair. Then $\mathrm{Ass}'[a,b]$ is closed under conjugation.*

Proof. Let x be an element of $\mathrm{Ass}'[a,b]$. By Lemma 2.5 we know x is imaginary. This implies $x^*=-x\in\mathrm{Ass}'[a,b]$. $\square$

Corollary 2.7. *Let a and b be a quaternionic pair. Then $\mathrm{Ass}'[a,b]$ equals $\mathrm{Ass}'[b,a]$.*

Proof. Since a and b are a quaternionic pair, $a^*=-a$ and $b^*=-b$. By conjugation we then have $(ax)b+a(xb)=0$ if and only if $(bx^*)a+b(x^*a)=0$. Thus, $x\in\mathrm{Ass}'[a,b]$ if and only if $x^*\in\mathrm{Ass}'[b,a]$.

Corollary 2.6 then tells us that $x\in\mathrm{Ass}'[a,b]$ if and only if $x\in\mathrm{Ass}'[b,a]$. $\square$

Lemma 2.8. *If b is imaginary, then*

1. $A_{(0,a),(0,b)}(x,y)=\big(b(ax^*)-(x^*b)a,\,a(y^*b)-b(y^*a)\big)$.
2. $A'_{(0,a),(0,b)}(x,y)=\big(b(ax^*)+(x^*b)a,\,-a(y^*b)-b(y^*a)\big)$.

Proof. Compute using the definition of multiplication and $b^*=-b$. $\square$

Lemma 2.9. *Let a and b be a quaternionic pair in A_n. Then*

$$\mathrm{Ass}[(0,a),(b,0)]=(\mathrm{Ass}'[a,b]\oplus\mathbb{R}\oplus\mathbb{R}b)\times(\mathrm{Ass}'[a,b]\oplus\mathbb{R}a\oplus\mathbb{R}ab).$$

In particular, the dimension of $\mathrm{Ass}[(0,a),(b,0)]$ equals $2\dim\mathrm{Ass}'[a,b]+4$.

Proof. Since a and b are a quaternionic pair, [1, Lemma 8.8] implies

$$\mathrm{Ass}[(b,0),(0,a)]=(\mathrm{Ass}'[b,a]\oplus\mathbb{R}\oplus\mathbb{R}b)\times(\mathrm{Ass}'[b,a]\oplus\mathbb{R}a\oplus\mathbb{R}ba).$$

By Corollary 2.2 we know $\mathrm{Ass}[(b,0),(0,a)]=\mathrm{Ass}[(0,a),(b,0)]$. Corollary 2.7 tells us $\mathrm{Ass}'[b,a]=\mathrm{Ass}'[a,b]$. Also, since $\mathbb{R}ba=\mathbb{R}ab$, the first claim is verified.

It follows that $\dim\mathrm{Ass}[(0,a),(b,0)]=2\dim\mathrm{Ass}'[a,b]+4$. $\square$

Often we will use [1, Remark 4.7] so we summarize it here. Two vectors a and b are a quaternionic pair if and only if they are orthogonal imaginary unit vectors such that $[a,b,b]$ and $[a,a,b]$ both vanish. Notice that $\|a\|^2=\|b\|^2=1$.

Lemma 2.10. *Let a and b be a quaternionic pair. Then $\mathrm{Ass}[(0,a),(0,b)]$ equals*

$$\big((\mathrm{Ass}[a,b]\cap\mathbb{H}\langle a,b\rangle^\perp)\oplus\mathbb{R}\oplus\mathbb{R}ab\big)\times\big((\mathrm{Ass}[a,b]\cap\mathbb{H}\langle a,b\rangle^\perp)\oplus\mathbb{R}a\oplus\mathbb{R}b\big).$$

In particular, the dimension of $\mathrm{Ass}[(0,a),(0,b)]$ equals $2\dim\mathrm{Ass}[a,b]-4$.

Proof. According to part (1) of Lemma 2.8, we need to find all x and y such that $b(ax^*) - (x^*b)a = 0$ and $a(y^*b) - b(y^*a) = 0$. Respectively, let K_1 and K_2 be the solution spaces to the two equations. By applying [1, Lemma 8.4] to $L_bL_a - R_aR_b$ and $L_aR_b - L_bR_a$, we see that K_1 and K_2 split as the direct sums of their intersections with $\mathbb{H}\langle a, b\rangle$ and $\mathbb{H}\langle a, b\rangle^\perp$.

Let $x^* \in K_1 \cap \mathbb{H}\langle a, b\rangle$. We can use the associativity of $\mathbb{H}\langle a, b\rangle$ to drop parentheses and we can use that a and b anti-commute (they are imaginary orthogonal vectors). We then have $bax^* - x^*ba = x^*ab - abx^* = 0$. This shows that x^* commutes with ab so that x^* must be in the 2-dimensional subspace generated by 1 and ab. Thus $K_1 \cap \mathbb{H}\langle a, b\rangle = \mathbb{R} \oplus \mathbb{R}ab$.

Let $x^* \in \mathbb{H}\langle a, b\rangle^\perp$. According to [1, Lemma 3.8], ax^* and x^*b belong to $\mathbb{H}\langle a, b\rangle^\perp$ so they are imaginary and orthogonal to a and b. Since imaginary orthogonal vectors anti-commute, we have

$$b(ax^*) - (x^*b)a = -(ax^*)b + a(x^*b) = -\big((ax^*)b - a(x^*b)\big).$$

So, $x^* \in K_1$ if and only if $x^* \in \operatorname{Ass}[a, b]$. This shows that $K_1 \cap \mathbb{H}\langle a, b\rangle^\perp = \operatorname{Ass}[a, b] \cap \mathbb{H}\langle a, b\rangle^\perp$.

Now we look at $K_2 \cap \mathbb{H}\langle a, b\rangle$. Suppose y^* belongs to this intersection. For the next set of computations we need to recall we can drop some parentheses, a and b anti-commute, $b^* = -b$, and $bb^* = b^*b = \|b\|^2 = 1$. Then we have, $ay^*b - by^*a = 0$. Multiplying both sides by b^* gives

$$0 = b^*(ay^*b - by^*a)b^* = b^*ay^*\|b\|^2 - \|b\|^2y^*ab^* = -bay^* + y^*ab.$$

This implies $y^*ab = bay^* = -aby^*$ so that y^* anti-commutes with ab. This indicates y^* is imaginary and orthogonal to ab. Since $y^* \in \mathbb{H}\langle a, b\rangle$, it must be in the 2-dimensional subspace generated by a and b. Thus, $K_2 \cap \mathbb{H}\langle a, b\rangle = \mathbb{R}a \oplus \mathbb{R}b$.

Finally, let $y^* \in \mathbb{H}\langle a, b\rangle^\perp$. By [1, Lemma 3.8], $y^*a \in \mathbb{H}\langle a, b\rangle^\perp$. Thus, y^*a and b are imaginary orthogonal vectors, so they anti-commute, as do y^* and a. Then we can compute

$$a(y^*b) - b(y^*a) = a(y^*b) + (y^*a)b = a(y^*b) - (ay^*)b.$$

Thus, y^* is in K_2 if and only if y^* is in $\operatorname{Ass}[a, b]$. Therefore, we have $K_2 \cap \mathbb{H}\langle a, b\rangle^\perp = \operatorname{Ass}[a, b] \cap \mathbb{H}\langle a, b\rangle^\perp$.

This verifies the main result. Using Corollary 2.4, it follows that $\dim \operatorname{Ass}[(0, a), (0, b)]$ equals

$$(\dim \operatorname{Ass}[a, b] - 4) + 2 + (\dim \operatorname{Ass}[a, b] - 4) + 2.$$

□

Lemma 2.11. *Let a and b be a quaternionic pair. Then* $\operatorname{Ass}'[(0, a), (b, 0)]$ *equals*

$$\big((\operatorname{Ass}[a, b] \cap \mathbb{H}\langle a, b\rangle^\perp) \oplus \mathbb{R}a \oplus \mathbb{R}ab\big) \times \big((\operatorname{Ass}[a, b] \cap \mathbb{H}\langle a, b\rangle^\perp) \oplus \mathbb{R} \oplus \mathbb{R}b\big).$$

In particular, the dimension of $\operatorname{Ass}'[(0, a), (b, 0)]$ *equals* $2 \dim \operatorname{Ass}[a, b] - 4$.

Proof. Since a and b form a quaternionic pair, we can apply [1, Lemma 8.10] to see that $\operatorname{Ass}'[(b, 0), (0, a)]$ equals

$$((\operatorname{Ass}[b, a] \cap \mathbb{H}\langle b, a\rangle^\perp) \oplus \mathbb{R}a \oplus \mathbb{R}ba) \times ((\operatorname{Ass}[b, a] \cap \mathbb{H}\langle b, a\rangle^\perp) \oplus \mathbb{R} \oplus \mathbb{R}b).$$

Note that $\mathbb{H}\langle b, a\rangle^\perp = \mathbb{H}\langle a, b\rangle^\perp$ and $\mathbb{R}ba = \mathbb{R}ab$. By Corollary 2.7, $\operatorname{Ass}'[(b, 0), (0, a)] = \operatorname{Ass}'[(0, a), (b, 0)]$. Also, Corollary 2.2 tells us $\operatorname{Ass}[b, a] = \operatorname{Ass}[a, b]$. From this, the main result follows.

From Corollary 2.4, it follows that $\dim \mathrm{Ass}'[(0,a),(b,0)]$ equals

$$(\dim \mathrm{Ass}[a,b]-4)+2+(\dim \mathrm{Ass}[a,b]-4)+2.$$

□

Lemma 2.12. *Let a and b be a quaternionic pair in A_n. Then*

$$\mathrm{Ass}'[(0,a),(0,b)] = (\mathrm{Ass}'[a,b]\oplus \mathbb{R}a\oplus\mathbb{R}b)\times(\mathrm{Ass}'[a,b]\oplus\mathbb{R}\oplus\mathbb{R}ab).$$

In particular, the dimension of $\mathrm{Ass}'[(0,a),(0,b)]$ *equals* $2\dim \mathrm{Ass}'[a,b]+4$.

Proof. According to part (2) of Lemma 2.8, we need to find all x and y such that $b(ax^*)+(x^*b)a=0$ and $-b(y^*a)-a(y^*b)=0$. Let K_1 and K_2 be the respective solution spaces. By applying [1, Lemma 9.2] to $L_bL_a+R_aR_b$ and $-L_bR_a-L_aR_b$, we see that K_1 and K_2 split as the direct sums of their intersections with $\mathbb{H}\langle a,b\rangle$ and $\mathbb{H}\langle a,b\rangle^\perp$.

Let $x^*\in K_1\cap\mathbb{H}\langle a,b\rangle$. Then, we compute

$$b(ax^*)+(x^*b)a = bax^*+x^*ba = -(abx^*+x^*ab).$$

Since $x^*\in K_1$, this expression equals zero. So x^* anti-commutes with ab. This shows x^* must be in the two dimensional subspace generated by a and b.

Suppose $x^*\in\mathbb{H}\langle a,b\rangle^\perp$. Then, ax^* and x^*b belong to $\mathbb{H}\langle a,b\rangle^\perp$, so that b and ax^* anti-commute, as do a and x^*b. This allows us to compute $b(ax^*)+(x^*b)a = -(ax^*)b-a(x^*b)$. This implies $x^*\in K_1$ if and only if $x^*\in\mathrm{Ass}'[a,b]$. Therefore, $K_1\cap\mathbb{H}\langle a,b\rangle^\perp$ equals $\mathrm{Ass}'[a,b]\cap\mathbb{H}\langle a,b\rangle^\perp$. However, since $\mathrm{Ass}'[a,b]\subseteq\mathbb{H}\langle a,b\rangle^\perp$ by Lemma 2.5, we have $\mathrm{Ass}'[a,b]\cap\mathbb{H}\langle a,b\rangle^\perp=\mathrm{Ass}'[a,b]$.

We can perform some quaternionic arithmetic, similar to that in the proof of Lemma 2.10, to show the intersection of K_2 with $\mathbb{H}\langle a,b\rangle$ is equal to the 2-dimensional subspace generated by 1 and ab.

Finally, let $y^*\in\mathbb{H}\langle a,b\rangle^\perp$. Then, y^*a is an element of $\mathbb{H}\langle a,b\rangle^\perp$. Using that imaginary orthogonal vectors anti-commute, we have

$$-b(y^*a)-a(y^*b) = (y^*a)b-a(y^*b) = -(ay^*)b-a(y^*b).$$

This shows that $y^*\in K_2$ if and only if $y^*\in\mathrm{Ass}'[a,b]$. Again, since $\mathbb{H}\langle a,b\rangle^\perp$ contains $\mathrm{Ass}'[a,b]$, we have $K_2\cap\mathbb{H}\langle a,b\rangle^\perp=\mathrm{Ass}'[a,b]$.

This verifies the first claim and it follows that $\dim\mathrm{Ass}'[(0,a),(0,b)]$ equals

$$\dim\mathrm{Ass}'[a,b]+2+\dim\mathrm{Ass}'[a,b]+2.$$

□

References

[1] D. Biss, D. Dugger, and D.C. Isaksen, *Large Annihilators in Cayley-Dickson Algebras*, ArXiv preprint math.RA/0511691.