(a) Let $a_0 = 1$ and define $a_n = \frac{10 + a_{n-1}}{3}$ for all integers $n \geq 1$. Use the monotone convergence theorem to prove that $a_n$ is a convergent sequence. Then find $\lim_{n \to \infty} a_n$.

(b) Let $a_n$ be a bounded sequence. Prove that $\liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n$.

(c) Give an example of a bounded sequence where the inequality from party (b) is strict. That is, define a bounded sequence $a_n$ such that $\liminf_{n \to \infty} a_n < \limsup_{n \to \infty} a_n$. No need for a proof here; just define the sequence.
(a) Let $a_n = \frac{3n - 2}{2n + 1}$. Prove that $a_n$ is Cauchy using the definition of Cauchy.

(b) Prove that if a sequence $a_n$ is Cauchy, then $a_n$ is bounded.
Topic 7: Limits of functions

(a) Find the following limit, and prove that it is correct: \( \lim_{x \to 2} x^3 \).

(b) Define functions \( f \) and \( g \) such that \( \lim_{x \to \infty} f(x)g(x) \) exists (finite), but either \( \lim_{x \to \infty} f(x) \) does not exist or \( \lim_{x \to \infty} g(x) \) does not exist. You do not need to prove this.
Topic 8: Limit theorems

(a) Assume that \( \lim_{x \to \infty} f(x) = L \) for some \( L \in \mathbb{R} \). Prove that there exists \( M > 0 \) and \( R > 0 \) such that \( |f(x)| \leq M \) for all \( x \geq R \).

(b) Assume that \( \lim_{x \to 0} f(x) = \infty \). Prove that \( \lim_{x \to 0} \frac{1}{f(x)} = 0 \).