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## On maps into a co-H-space

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**ABSTRACT.** We prove that the domain of a map  $X \rightarrow Y$  to a co-H-space inherits a co-H-structure provided some dimensionality and connectivity properties hold. Then we deduce that a space  $X$  admits a co-H-structure if and only if on all its skeletons there is such a structure as well. Moreover, if  $X$  is 1-connected then a co-H-structure on  $X$  is equivalent to such a structure on all its homology decomposition stages.

### Introduction.

Recall [5, Chapter IX], where it was shown that a map  $X \rightarrow Y$  of based spaces extends an H-space structure on  $X$  to one on  $Y$  provided some connectivity properties hold. In particular, the result implies that an H-structure on  $X$  is inherited on all its Postnikov stages.

This paper examines the dual problem. However, the arguments used for H-spaces do not seem to dualize. Therefore, we make use of the one-to-one correspondence (see [1] or [2, p. 209–212]) between homotopy classes of co-H-structures on a space  $X$  and those of coretractions  $X \rightarrow \Sigma\Omega X$ . A result due to Hilton (see [2, p. 185]) says that if a space  $X$  is dominated by a co-H-space  $Y$  then  $X$  also admits a co-H-structure. We prove (Theorem 1) that the domain of a map  $X \rightarrow Y$  to a co-H-space inherits a co-H-structure provided some dimensionality and connectivity properties hold. Then we deduce (Corollary 1) that a space  $X$  admits a co-H-structure if and only if on all its skeletons there is such a structure as well. Moreover, if  $X$  is 1-connected then Corollary 3 states that a co-H-structure on  $X$  is equivalent to such a structure on all its homology decomposition stages and a dualization of Corollary 5.6 in [5, page 443] is obtained provided  $X$  is 2-connected.

### 1. Preliminaries.

We consider based spaces of the based homotopy type of a based CW-complex; the basepoints are assumed to be non-degenerate. All maps and

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homotopies are supposed to preserve basepoints and all spaces denoted by the symbols  $X, Y$  and their derivatives ( $X_n, Y_n$  etc.) are implicitly assumed to be based CW-complexes. It follows from [4] the constructions we will perform do not lead outside the class of spaces considered. For a space  $X$  let  $\Delta : X \rightarrow X \times X$  be the diagonal map and  $j : X \vee X \rightarrow X \times X$  the inclusion map of the wedge into the product. A *co-H-structure* on  $X$  is a map  $\sigma : X \rightarrow X \vee X$  such that the diagram

$$\begin{array}{ccc} & & X \vee X \\ & \nearrow \sigma & \downarrow j \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

commutes up to homotopy. A *co-H-space* is a pair  $(X, \sigma)$ , where  $X$  is a space and  $\sigma$  a co-H-structure on  $X$ . An *inversion* for  $\sigma$  is a map  $\eta : X \rightarrow X$  such that the composites

$$X \xrightarrow{\sigma} X \vee X \xrightarrow{\text{id}_X \vee \eta} X \vee X \xrightarrow{\nabla} X \quad \text{and} \quad X \xrightarrow{\sigma} X \vee X \xrightarrow{\eta \vee \text{id}_X} X \vee X \xrightarrow{\nabla} X$$

are both nullhomotopic maps;  $\nabla$  is the folding map. We point out that by [3] any co-H-structure on a 1-connected space admits an inversion.

With any space  $X$  we may associate  $\Sigma\Omega X$ , the (reduced) suspension of the loop space of  $X$ ; the natural projection  $p_X : \Sigma\Omega X \rightarrow X$  is given by  $p_X([t, \omega]) = \omega(t)$  for  $[t, \omega] \in \Sigma\Omega X$ . A *coretraction* is a map  $\phi : X \rightarrow \Sigma\Omega X$  such that  $p_X \phi \simeq \text{id}_X$ , the identity map of  $X$ . For a space  $X$  the suspension co-H-structure  $s_X : \Sigma X \rightarrow \Sigma X \vee \Sigma X$  is the pinch map and the coretraction  $\Sigma e : \Sigma X \rightarrow \Sigma\Omega\Sigma X$  is given by  $e(x)(t) = [t, x]$  for  $x \in X$  and  $t \in [0, 1]$ . As it was pointed out in [1] and [2, p. 209–212] the map  $(p_X \vee p_X)s_{\Omega X} : \Sigma\Omega X \rightarrow X \vee X$  induces a bijection between homotopy classes of coretractions and those of comultiplications on  $X$ .

Let  $f : X \rightarrow Y$  be any map and consider the diagrams

$$\begin{array}{ccc} \Sigma\Omega X & \xrightarrow{\Sigma\Omega f} & \Sigma\Omega Y \\ \phi \uparrow & & \uparrow \psi \\ X & \xrightarrow{f} & Y \end{array} \quad \begin{array}{ccc} X \vee X & \xrightarrow{f \vee f} & Y \vee Y \\ \sigma \uparrow & & \uparrow \tau \\ X & \xrightarrow{f} & Y \end{array}$$

in which  $\phi, \psi$  are coretractions and  $\sigma, \tau$  the corresponding comultiplications. Taking into account [1] the second square homotopy commutes if and only if the first does; we way then that  $f$  is a *map of co-H-spaces*  $(X, \sigma)$  and  $(Y, \tau)$ .

We now show how maps from some spaces into a co-H-space reflect co-H-structures on their domains. For a map  $f : X \rightarrow Y$  we write  $\text{conn } f = n$ , if the induced map of homotopy groups  $\pi_k(f) : \pi_k(X) \rightarrow \pi_k(Y)$  is an isomorphism for  $k < n$  and an epimorphism for  $k = n$ . In particular, for the map  $f : X \rightarrow *$  to the single point space we put  $\text{conn } X = \text{conn } f$ . Write  $X \flat Y$  and  $X \wedge Y$  for the flat and smash product of spaces  $X$  and  $Y$ , respectively; now there is a homotopy equivalence  $X \flat Y \simeq \Sigma \Omega X \wedge \Omega Y$  (see e.g., [2, page 216]).

LEMMA 1. For a map  $f : X \rightarrow Y$ ,  $\text{conn}(f \wedge f) = \text{conn } f + \min\{\text{conn } X, \text{conn } Y\} + 1$  and  $\text{conn}(f \flat f) = \text{conn } f + \min\{\text{conn } X, \text{conn } Y\}$ .

PROOF. Observe that  $f \wedge f = (f \wedge \text{id}_X) \circ (\text{id}_Y \wedge f)$  and  $f \flat f = (f \text{id}_X) \circ (\text{id}_Y \flat f)$ . From the above  $f \text{id}_X \simeq \Omega f \wedge \text{id}_{\Sigma \Omega X}$  and  $\text{id}_Y \flat f \simeq \text{id}_{\Sigma \Omega Y} \wedge \Omega f$ . Hence  $\text{conn}(f \wedge \text{id}_X) = \text{conn } f + \text{conn } X + 1$ ,  $\text{conn}(\text{id}_Y \wedge f) = \text{conn } f + \text{conn } Y + 1$  and  $\text{conn}(f \text{id}_X) = \text{conn } X + \text{conn } f$ ,  $\text{conn} \text{id}_Y \flat f = \text{conn } Y + \text{conn } f$ . The result follows.  $\square$

This lemma will be also useful in the next section.

LEMMA 2. Let  $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \dots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} \dots$  be a countable direct system of co-H-spaces and co-H-maps. Then its homotopy colimit  $\text{holim } X_n$  admits a co-H-structure such that the canonical imbeddings  $X_n \rightarrow \text{holim } X_n$  are co-H-maps for all  $n \geq 0$ .

PROOF. By the ‘‘small object’’ argument we get that the canonical map

$$\text{holim } \Sigma \Omega X_n \rightarrow \Sigma \text{holim } \Omega X_n = \Sigma \Omega \text{holim } X_n$$

is a homotopy equivalence. Then the coretractions  $\phi_n : X_n \rightarrow \Sigma \Omega X_n$  corresponding to co-H-structures on  $X_n$  for  $n \geq 0$  yield the required coretraction  $\phi = \text{holim } \phi_n : \text{holim } X_n \rightarrow \Sigma \Omega \text{holim } X_n$ .  $\square$

## 2. Main results.

The behaviour of co-H-structures with respect to maps is taken into account in this section.

THEOREM 1. Let  $f : X \rightarrow Y$  be a map with  $Y$  a co-H-space. If  $\dim X \leq \text{conn } f + \min\{\text{conn } X, \text{conn } Y\}$  then there exists a (which is unique up to homotopy if strict inequality holds) co-H-structure on  $X$  such that  $f : X \rightarrow Y$  is a co-H-map.

PROOF. Consider the commutative (up to homotopy) diagram

$$\begin{array}{ccccc}
 P & \xrightarrow{\pi'} & \Sigma\Omega Y & \xrightarrow{(p_Y \vee p_Y)_{s_Y}} & Y \vee Y \\
 \pi \downarrow & & \downarrow p_Y & & \downarrow j \\
 X & \xrightarrow{f} & Y & \xrightarrow{d} & Y \times Y
 \end{array}$$

where the first square is the homotopy pullback and the second one is also a homotopy pullback by [1]. Therefore, the homotopy fibres of the maps  $p_Y : \Sigma\Omega Y \rightarrow Y$  and  $\pi : P \rightarrow X$  have the homotopy type of the space  $Y \wr Y$ . Let now  $\phi : Y \rightarrow \Sigma\Omega Y$  be the coretraction corresponding to the co-H-structure on  $Y$ . Then the pairs of maps  $\Sigma\Omega f : \Sigma\Omega X \rightarrow \Sigma\Omega Y$ ,  $p_X : \Sigma\Omega X \rightarrow X$  and  $\phi f : X \rightarrow \Sigma\Omega Y$ ,  $\text{id}_X$  yield maps  $\alpha : \Sigma\Omega X \rightarrow P$  and  $\gamma : X \rightarrow P$ , respectively with the appropriate composite property.

Analyze now the map of fibrations

$$\begin{array}{ccccc}
 X \wr X & \longrightarrow & \Sigma\Omega X & \xrightarrow{p_X} & X \\
 f \wr \downarrow & & \downarrow \alpha & & \parallel \\
 Y \wr Y & \longrightarrow & P & \xrightarrow{\pi} & X.
 \end{array}$$

From the 5-lemma we get that  $\text{conn } \alpha = \text{conn } f \wr \downarrow$ . But by Lemma 1  $\text{conn } f \wr \downarrow = \text{conn } f + \min\{\text{conn } X, \text{conn } Y\}$  and the obstruction theory yields a (unique up to homotopy if there is strict inequality) map  $\psi : X \rightarrow \Sigma\Omega X$  such that  $\alpha\psi \simeq \gamma$ . Then  $\pi\alpha\psi \simeq p_X\psi \simeq \pi\gamma \simeq \text{id}_X$  and the map  $\psi$  determines a co-H-structure on  $X$ . Moreover,  $(\Sigma\Omega f)\psi \simeq \pi'\alpha\psi \simeq \pi'\gamma \simeq \phi f$  and  $f : X \rightarrow Y$  is a co-H-map.  $\square$

LEMMA 3. Let  $f_1 : X_1 \rightarrow Y$ ,  $f_2 : X_2 \rightarrow Y$  and  $g : X_1 \rightarrow X_2$  be maps with  $f_2 g \simeq f_1$  and  $Y$  a co-H-space. If  $\dim X_1 \leq \min\{\text{conn } f_2 - 1, \text{conn } f_1 + \min\{\text{conn } X_1, \text{conn } Y\}\}$  and  $\dim X_2 \leq \text{conn } f_2 + \min\{\text{conn } X_2, \text{conn } Y\}$  then a co-H-structure on  $Y$  yields co-H-structures on  $X_1$  and  $X_2$  such that  $g : X_1 \rightarrow X_2$  is a co-H-map.

PROOF. Let  $\phi : Y \rightarrow \Sigma\Omega Y$  be the coretraction determined by the co-H-structure on  $Y$  and  $\phi_1 : X_1 \rightarrow \Sigma\Omega X_1$ ,  $\phi_2 : X_2 \rightarrow \Sigma\Omega X_2$  the coretractions determined by the co-H-structures on  $X_1$  and  $X_2$  described in Theorem 1. To show the commutativity (up to homotopy) of the diagram

$$\begin{array}{ccc}
 X_1 & \xrightarrow{\phi_1} & \Sigma\Omega X_1 \\
 g \downarrow & & \downarrow \Sigma\Omega g \\
 X_2 & \xrightarrow{\phi_2} & \Sigma\Omega X_2
 \end{array}$$

observe that  $(\Sigma\Omega f_2)\phi_2g \simeq \phi f_2g \simeq \phi f_1$  and  $(\Sigma\Omega f_2)(\Sigma\Omega g)\phi_1 \simeq (\Sigma\Omega f_2g)\phi_1 \simeq (\Sigma\Omega f_1)\phi_1 \simeq \phi f_1$ . But  $\text{conn } \Sigma\Omega f_2 = \text{conn } f_2$  and  $\dim X_1 \leq \text{conn } f_2 - 1$ , hence obstruction theory yields a homotopy  $(\Sigma\Omega g)\phi_1 \simeq \phi_2g$ .  $\square$

Our final results can be stated now. For a space  $X$  let  $X^{(n)}$  be its  $n$ -skeleton,  $n \geq 0$ . Then Theorem 1, Lemmas 2 and 3 yield

**COROLLARY 1.** *A connected space  $X$  admits a co-H-structure if and only if on each skeleton  $X^{(n)}$  there exists such a co-H-structure that the canonical imbedding  $X^{(n)} \rightarrow X^{(n+1)}$  is a co-H-map. Furthermore, in this case, the imbedding  $X^{(n)} \rightarrow X$  is a co-H-map for suitable co-H-structures on  $X$  and  $X^{(n)}$ .*

For a co-H-map  $f : X \rightarrow Y$  let  $C_f$  be its mapping cone and  $q : Y \rightarrow C_f$  the canonical imbedding. By [3] we may give  $C_f$  a co-H-structure in such a way that  $q$  is a co-H-map. We now show that the converse of this fact also holds, provided some conditions are satisfied. Recall first that for a co-H-space  $X$  with a co-H-structure  $\sigma$  and an inversion map, and any space  $Y$  there is a (naturally) split short exact sequence

$$1 \rightarrow [X, Y \triangleright Y] \rightarrow [X, Y \vee Y] \xrightleftharpoons[\gamma_Y]{j_*} [X, Y \times Y] \rightarrow 1,$$

where  $j_*\gamma_Y = \text{id}$  with  $\gamma_Y([\alpha_1, \alpha_2]) = [(\alpha_1 \vee \alpha_2)\sigma]$  for  $(\alpha_1, \alpha_2) : X \rightarrow Y \times Y$ . Denote the induced operation on  $[X, Y \vee Y]$  additively and let  $\beta_Y = \text{id} - \gamma_Y j_*$ . Then we get  $j_*\beta_Y = 0$ .

**THEOREM 2.** *Let  $f : X \rightarrow Y$  be a map with  $X$  and  $Y$  1-connected co-H-spaces. If the mapping cone  $C_f$  is a co-H-space,  $q : Y \rightarrow C_f$  a co-H-map and  $\dim X < \text{conn } X + \min\{\text{conn } Y, \text{conn } C_f\}$  then  $f : X \rightarrow Y$  is a co-H-map.*

**PROOF.** Observe that by [3] a co-H-structure on a 1-connected space  $X$  admits an inversion. Let  $\sigma : Y \rightarrow Y \vee Y$  and  $\sigma' : C_f \rightarrow C_f \vee C_f$  be co-H-structures on  $Y$  and  $C_f$ , respectively and consider the commutative diagram

$$\begin{array}{ccccc} [X, Y] & \xrightarrow{\sigma_*} & [X, Y \vee Y] & \xrightarrow{\beta_Y} & [X, Y \triangleright Y] \\ q_* \downarrow & & \downarrow (q \vee q)_* & & \downarrow (qbq)_* \\ [X, C_f] & \xrightarrow{\sigma'_*} & [X, C_f \vee C_f] & \xrightarrow{\beta_{C_f}} & [X, C_f \triangleright C_f] \end{array}$$

Then  $(qbq)_*\beta_Y\sigma_*([f]) = \beta_{C_f}\sigma'_*q_*([f]) = \beta_{C_f}\sigma'_*([qf]) = 0$ . But  $\text{conn } q = \text{conn } X$  and by Lemma 1  $\text{conn}(qbq) = \text{conn } q + \min\{\text{conn } Y, \text{conn } C_f\}$ , hence  $\dim X < \text{conn}(qbq)$ . By obstruction theory the map  $(qbq)_*$  is an isomorphism, so we get that  $\beta_Y\sigma_*([f]) = 0$ . From the definition of  $\beta_Y$  we derive that



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