POINCARÉ SUBMERSIONS

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Abstract. We prove two kinds of fibering theorems for maps $X \to P$, where $X$ and $P$ are Poincaré spaces. The special case of $P = S^1$ yields a Poincaré duality analogue of the fibering theorem of Browder and Levine.

1. Introduction

One of the early successes of surgery theory was the fibering theorem of Browder and Levine [B-L], which gives criteria for when a smooth map $f: M \to S^1$ is homotopic to a submersion. Here $M$ is assumed to be a connected closed, smooth manifold of dimension $\geq 6$, and we also require $f$ to induce an isomorphism of fundamental groups. The Browder-Levine fibering theorem then says that $f$ is homotopic to a submersion if and only if the homotopy groups of $M$ are finitely generated in each degree.

The purpose of the current note is to prove fibering results in the Poincaré duality category. Note that a submersion of closed manifolds is a smooth fiber bundle with closed manifold fibers. Replacing the closed manifolds with finitely dominated Poincaré spaces and the fiber bundle with a fibration yields the notion of Poincaré submersion: this is a map between Poincaré spaces whose homotopy fibers are Poincaré spaces.

Our first result concerns the case when the target is acyclic (this includes the Browder-Levine situation). Let $X$ be a connected, finitely dominated Poincaré duality space of (formal) dimension $d$ and fundamental group $\pi$. Let $f: X \to B\pi$ be the classifying map for the universal cover of $X$. We will be assuming that the classifying space $B\pi$ is a finitely dominated Poincaré space of dimension $p$.

**Theorem A.** Let $F$ denote the homotopy fiber of $f$. Then $F$ is a homotopy finite Poincaré duality space of dimension $d - p$ if and only if the homotopy groups of $X$ are finitely generated in each degree.

Our second result considers the case when the target is simply connected. Let $f: X \to P$ be a 2-connected map in which $X$ and $P$ are 1-connected Poincaré duality spaces. Assume $X$ has dimension $d$ and $P$ has dimension $p$. We will give criteria for deciding when the homotopy fiber of $f$ satisfies Poincaré duality.

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The idea will be that a candidate fundamental class for the fiber is always available. Furthermore, duality with respect to this candidate automatically regulates homotopy finiteness.

Here is how the candidate class is constructed: associated with the map \( i : F \to X \) there is an umkehr homomorphism

\[
i_!^*: H_*(X) \to H_{*-p}(F)
\]

(cf. §4). The pushforward of a fundamental class \([X] \in H_d(X)\) for \( X \) with respect to \( i ! \) then gives a class

\[
x_f := i_!^*([X]) \in H_{d-p}(F).
\]

This will be our candidate for a fundamental class of \( F \).

**Theorem B.** With respect to these assumptions, the following are equivalent:

1. \( H_*(F) = 0 \) in sufficiently large degrees.
2. \( F \) is homotopy finite.
3. The homomorphism

\[
\cap x_f : H^*(F; \mathbb{Z}) \to H_{d-p-\ast}(F)
\]

is an isomorphism in all degrees.
4. \( F \) is a Poincaré duality space.

**Remark.** When \( P = S^p \) is a sphere, (1) \( \Rightarrow \) (4) overlaps with [C, lem. 1.1]. The implication (2) \( \Rightarrow \) (4) is a consequence of [Kl1, th. B].

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**Conventions.** A space is homotopy finite if has the homotopy type of a finite cell complex. A space is finitely dominated if it is the retract of a homotopy finite space.

A Poincaré space of formal dimension \( d \) is a space \( X \) for which there exists a pair \((L, [X])\) consisting of a rank one abelian system of local coefficients \( L \) on \( X \) and a (fundamental) class \([X] \in H_d(X ; L)\) such that the cap product homomorphism

\[
\cap [X] : H^*(X ; \mathcal{A}) \to H_{d-\ast}(X ; L \otimes \mathcal{A})
\]

is an isomorphism, for all local coefficient modules \( \mathcal{A} \) on \( X \) (cf. [W1], [Kl2]). If \( X \) is connected, then it is enough to establish the isomorphism when \( \mathcal{A} \) is the integral group ring of the fundamental group of \( X \). We do not at assume any finiteness conditions in the definition of Poincaré space appearing here. However, in the 1-connected case, homotopy finiteness is actually a consequence of Poincaré duality (see 3.2 below).

**2. Proof of Theorem A**

We first prove the ‘only if’ part. Assume that \( F \) is a homotopy finite Poincaré space. Since \( F \) is 1-connected and homotopy finite, we infer that its homology is finitely generated. Apply the mod \( \mathcal{C} \) Hurewicz theorem (with \( \mathcal{C} = \) the Serre class of finitely generated abelian groups) to see that the homotopy groups of \( F \) are finitely generated [S, cor. 9.6.16].

We now prove the ‘if’ part. Note that \( F \) has the homotopy type of the universal cover of \( X \), so \( F \) is homotopy finite dimensional because \( X \) is. By the long exact homotopy sequence and the fact that \( \pi_*(X) \) is degreewise finitely generated, we
infer that $\pi_*(F)$ is degreewise finitely generated. Since $F$ is simply connected, the mod $C$ Hurewicz theorem shows that the homology groups of $F$ are finitely generated. By a result of Wall [W2], we see that $F$ is homotopy finite.

We now know that each space in the homotopy fiber sequence

$$F \to X \to B\pi$$

is finitely dominated. It follows directly from [Kl1, th. B] (see also [G]) that $F$ satisfies Poincaré duality and has formal dimension $d - p$. This completes the proof of Theorem A. □

3. Duality and finiteness

A chain complex $C$ of abelian groups is said to be dualizable if there is chain complex $D$ and a map

$$d: Z \to C \otimes D$$

($\otimes$ = derived tensor product, and $d$ is allowed to be degree shifting) such that, for all $P$, we get that the induced map of complexes

$$\text{hom}(C, P) \to \text{hom}(Z, P \otimes D)$$

(derived hom) given by $f \mapsto (f \otimes 1_D) \circ d$ is a quasi-isomorphism, where $1_D$ denote the identity map of $D$.

**Lemma 3.1.** If $C$ is dualizable, then it is homotopy finite over $Z$, i.e., it is quasi-isomorphic to a chain complex of finitely generated free abelian groups which is non-trivial in all but finitely many degrees.

**Proof.** Since $Z$ is “compact,” there exists a finite chain complex $C_0$, a map $i: C_0 \to C$ and a map $d_0: Z \to C_0 \otimes D$ such that

$$Z \xrightarrow{d_0} C_0 \otimes D \xrightarrow{i \otimes 1} C \otimes D$$

is homotopic to $d$. Consider the homotopy commutative diagram

$$
\begin{array}{ccc}
\text{hom}(C, C) & \xrightarrow{(- \otimes 1_C)d} & \text{hom}(Z, C \otimes D) \\
\uparrow i_* & & \uparrow i_* \\
\text{hom}(C, C_0) & \xrightarrow{(- \otimes 1_{C_0})d} & \text{hom}(Z, C_0 \otimes D)
\end{array}
$$

The map $d_0$ lives in the lower right corner and maps to $d$ under the right vertical map. The map $1_C$ maps to $d$ under the top horizontal map. Since the lower horizontal map is an equivalence, we get a map $j: C \to C_0$ such that $i_*(j) = j \circ i$ is homotopic to $1_C$. We conclude that the identity map of $C$ factors up to homotopy through the finite object $C_0$. □

Note now if $X^d$ is a 1-connected space which is equipped with a chain level fundamental class $[X]$ for which Poincaré duality holds, then $C(X) = \text{the singular chains on } X$ is dualizable using the maps

$$Z \xrightarrow{[X]} C(X) \xrightarrow{\text{diagonal}} C(X \times X) \simeq C(X) \otimes C(X),$$

where the first map is the homomorphism (of degree $d$) induced by a choice of fundamental class. By the above lemma, we infer that $C(X)$ is chain homotopy finite.
A result of Wall says that a 1-connected space is homotopy finite if and only if its chain complex is chain homotopy finite (see [W3]). Hence,

**Corollary 3.2.** Let $X$ be a 1-connected space which satisfies Poincaré duality. Then $X$ is also homotopy finite.

4. **The umkehr homomorphism**

According to [W1, th. 2.4], if $\dim B \geq 3$ there is a homotopy equivalence

$$B \simeq B_0 \cup D^p,$$

in which $B_0$ is a CW complex of dimension $\leq p-1$. Using the fact that $B_0$ is 1-connected, we can further conclude that $B_0$ has the homotopy type of a CW complex of dimension $\leq p-2$. If $B$ has dimension $\leq 2$, then $B \simeq S^p$, and the above decomposition is also available.

Furthermore, once an orientation for $B$ has been chosen, the above cell decomposition is unique up to oriented homotopy equivalence. From now on, we fix an identification $B := B_0 \cup D^p$, where $\dim B_0 \leq p-2$.

Without loss in generality, let us assume that $f : X \to B$ has been converted into a Hurewicz fibration. Let $X_0 = f^{-1}(B_0)$. Then we obtain a pushout square

$$\begin{array}{ccc}
  f^{-1}(S^{p-1}) & \longrightarrow & X_0 \\
  \downarrow & & \downarrow \\
  f^{-1}(D^p) & \longrightarrow & X.
\end{array}$$

Using the homotopy lifting property, we see that the pair $(f^{-1}(D^p), f^{-1}(S^{p-1}))$ has the homotopy type of the pair $(F \times D^p, F \times S^{p-1})$. Taking vertical cofibers in the diagram, we get an umkehr map

$$i_1 : X \longrightarrow X/X_0 = f^{-1}(D^p)/f^{-1}(S^{p-1}) \simeq F_+ \wedge S^p.$$  

The umkehr homomorphism

$$i_1^* : H_*(X) \to H_{*-p}(F)$$

is the effect of applying singular homology to $i_1^*$, and using the suspension isomorphism to perform the degree shift.

5. **Proof of Theorem B**

(1) $\Rightarrow$ (2). By the long exact homotopy sequence of the fibration, we see that $\pi_*(F)$ is degreewise finitely generated. By the mod $C$ Hurewicz theorem, we infer that $H_*(F)$ is finitely generated. Then $F$ is homotopy finite by [W2].

(2) $\Rightarrow$ (1). Trivial.

(2) $\Rightarrow$ (4). Follows from [K1I, th. B].

(3) $\Rightarrow$ (2). This follows from 3.2.
It will be enough to show that the class $x_f$ is a generator of $H_{d-p}(F) \cong \mathbb{Z}$. By definition of $x_f$, this is equivalent to knowing that the homomorphism

$$i_+^! : H_d(X) \to H_{d-p}(F)$$

is of degree one.

This can be seen as follows: the space $X_0$ is the pullback of the fibration $f : X \to B$ along a CW complex $B_0$ of dimension $\leq p-2$. As $F$ has formal dimension $\leq d-p$, it is straightforward to check that $X_0$ has the homotopy type of a CW complex of dimension $\leq d-2$. Using the homotopy cofiber sequence

$$X_0 \longrightarrow X \longrightarrow F_+ \wedge S^p$$

and the fact that the homology of $X_0$ vanishes above degree $d-2$, we see that $i_+^!$ induces an isomorphism in homology in degree $d$.

\[ \square \]

References


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