Locally nilpotent derivations, a new ring invariant and applications

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SUMMARY

The main objects we are going to deal with in these lectures are locally nilpotent derivations on various algebras and a new invariant which was recently introduced in a successful attempt to distinguish from $\mathbb{C}^3$ a threefold given as a hypersurface of $\mathbb{C}^4$ by $x + x^2y + z^2 + t^3 = 0$ or, in algebraic terms, to show that the factor ring $\mathbb{C}[x, y, z, t]/(x + x^2y + z^2 + t^3)$ is not isomorphic to the polynomial ring $\mathbb{C}[x, y, z]$. This hypersurface is one of the so-called Russell-Koras threefolds which appeared naturally in the works of Koras and Russell on the linearizing conjecture in the case of an algebraic $\mathbb{C}^*$ action on $\mathbb{C}^3$.

The main motivation here is the development of algebraic tools to distinguish between rings (in commutative and non-commutative settings).

The main objective of this research is to find applications of the invariant to algebraic and geometric questions and to understand the properties of the invariant.

We will see later how the invariant and its suitable modifications help distinguish between rings and between fields, facilitate a description of groups of automorphisms of a certain class of surfaces, and give additional information in cancellation type problems.

Though most of the definitions below make sense in a greater generality, we are going to consider only algebras without zero divisors over the field $\mathbb{C}$ of complex numbers. Actually nearly all the proofs below are correct over any field of characteristic zero.

Let us start with definitions.

Definitions.

Derivations and related notions

Let A be a (not necessarily commutative) algebra over a field $\mathbb{C}$. Then a $\mathbb{C}$-homomorphism ∂ of A is called a *derivation* of A if it satisfies the Leibniz rule: $\partial(ab) = \partial(a)b + a\partial(b)$.

Let us denote the set of all derivations of A by $\mathrm{Der}(A)$. It is well known (and easy to check) that $\mathrm{Der}(A)$ is a Lie algebra relative to the addition and "commutator" operations in the algebra of homomorphisms of A. (If $\partial_1, \partial_2 \in \mathrm{Der}(A)$ then $\partial_1\partial_2 - \partial_2\partial_1 \in \mathrm{Der}(A)$.)

Indeed $\partial_1\partial_2(ab) = \partial_1(\partial_2(a)b + a\partial_2(b)) = \partial_1\partial_2(a)b + \partial_2(a)\partial_1(b) + \partial_1(a)\partial_2(b) + a\partial_1\partial_2(b)$. So $(\partial_1\partial_2 - \partial_2\partial_1)(ab) = \partial_1\partial_2(a)b + \partial_2(a)\partial_1(b) + \partial_1(a)\partial_2(b) + a\partial_1\partial_2(b) - \partial_2\partial_1(a)b - \partial_1(a)\partial_2(b) - \partial_2(a)\partial_1(b) - a\partial_2\partial_1(b) = (\partial_1\partial_2 - \partial_2\partial_1)(a)b + a(\partial_1\partial_2 - \partial_2\partial_1)(b)$.)

If A is a commutative algebra then $\mathrm{Der}(A)$ is a left A-module.

With any element $r \in A$, it is possible to connect a derivation ad_r on A which is given by $ad_r(a) = ra - ar$, the so called *inner derivation*. Let us denote the set of all inner derivations of A by $\mathrm{IDer}(A)$.

(It is a derivation since $[r, ab] = [r, a]b + a[r, b]$, check it.)

Any derivation ∂ determines two subalgebras of A. One is the kernel of ∂ which is usually denoted by A^∂ and is called the *ring of ∂-constants*, by analogy with the ordinary derivative.

The other is $\mathrm{Nil}(\partial)$, the *ring of nilpotency of* ∂. It is determined by $\mathrm{Nil}(\partial) = \{a \in A | \partial^n(a) = 0, n >> 1\}$. In other words $a \in \mathrm{Nil}(\partial)$ if for a sufficiently large natural number n we have $\partial^n(a) = 0$.

Example 1 . If our ring is the ring of all infinitely differentiable functions in one variable (and ∂ is the ordinary derivative) then $\mathrm{Nil}(\partial)$ is just the subring of polynomials.

Both A^∂ and $\mathrm{Nil}(\partial)$ are subalgebras of A because of the Leibniz rule.

Check that $\mathrm{Nil}(\partial)$ is a subring.

Let us call a derivation *locally nilpotent* if $\mathrm{Nil}(\partial) = A$. Let us denote by $\mathrm{LND}(A)$ the set of all locally nilpotent derivations and by $\mathrm{ILnd}(A)$ the set of all locally nilpotent inner derivations.

The best examples of locally nilpotent derivations are the partial derivatives on the rings of polynomials $\mathbb{C}[x_1, ..., x_n]$. But even for rings of polynomials the kernels of locally nilpotent derivations may fail to be finitely generated (see [De], [DF1], [EJ], [Es1], and [No]).

Example 2 . Let $M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices over $\mathbb{C}$ and let $a \in M_n$. We can define a derivation ad_a of M by $ad_a(m) = [a, m]$. Check that this derivation is nilpotent if a is a nilpotent matrix. Can you describe all matrices for which ad_a is nilpotent?

Example 3 . For those of you who know about Lie algebras here is another non-commutative example. If L is a nilpotent Lie algebra and UL is its universal enveloping algebra then the adjoint action of any element $x \in L$ gives a locally nilpotent derivation of UL.

With the help of a derivation acting on a ring A, one can define a *function* deg_∂ on $\text{Nil}(\partial)$ by $\deg_\partial(f) = \max(n|\partial^n(f) \neq 0)$ if $f \in \text{Nil}(\partial) = \text{Nil}(\partial) \setminus 0$ and $\deg_\partial(0) = -\infty$.

If A is a domain and the characteristic of A is zero, then $\deg_\partial$ is a degree function, i.e.,
$\deg_\partial(a+b) \leq \max(\deg_\partial(a), \deg_\partial(b))$ and
$\deg_\partial(ab) = \deg_\partial(a) + \deg_\partial(b)$
whenever $\deg_\partial(a)$ and $\deg_\partial(b)$ are defined.

Check these properties.

Example 4. If our ring is the ring of all infinitely differentiable functions in one variable then $\deg_\partial$ is finite only on polynomials and gives on them the ordinary degree of polynomials.

If ∂ is a locally nilpotent derivation then this function is defined on all A and $\deg_\partial$ has only non-negative values on A^*. By definition $a \in A^\partial$ if $\deg_\partial(a) = 0$. So it is clear that the ring A^∂ is "factorially closed"; i. e., if $a, b \in A^*$ and $ab \in A^\partial$, then $a, b \in A^\partial$, and that all units of A (invertible elements of A) must belong to A^∂. So, if A is a skew field then $\text{LND}(A) = 0$.

Nevertheless it is possible to modify the definition and to extend the notion of locally nilpotent derivation to the case when A is a skew field. Let us call a derivation ∂ *locally nilpotent for skew field* if $\text{Nil}(\partial)$ is sufficiently large; i.e., if the minimal skew subfield which contains $\text{Nil}(\partial)$ is A. Let us denote the corresponding set by $\text{LND}_s(A)$. This definition of course is applicable when A is a field.

Example 5. If $\mathbb{C}(x)$ is the field of rational functions in one variable then the ordinary derivative is a locally nilpotent derivation.

Give an example of a derivation of $\mathbb{C}(x)$ which is not locally nilpotent.

Non-commutative examples.

Example 6. Let $M = M_2(\mathbb{C})$ be the algebra of 2×2 matrices over $\mathbb{C}$. Then we can define a derivation ∂ of M by $\partial(m) = [e_{1,2}, m]$. Since $\partial(e_{1,1}) = -e_{1,2}$, $\partial(e_{1,2}) = 0$, $\partial(e_{2,1}) = e_{1,1} - e_{2,2}$, $\partial(e_{2,2}) = e_{1,2}$, this derivation is locally nilpotent.

What can you say about the degree function which corresponds to this derivation?

What can you say about the kernel of this derivation?

Prove that A has only inner derivations.

Example 7. Let $A = \mathbb{C} < a, b >$ be a free associative algebra with generators a and b. (Algebra A consists of finite linear combinations of non-commutative monomials $a^{i_1}b^{j_1}...a^{i_k}b^{j_k}$ with the vector space addition and the natural multiplication of monomials (it is called concatenation) which extends on all algebra by linearity.)

Let us define a derivation ∂ by $\partial(a) = 1$ and $\partial(b) = 0$. By the Leibniz rule ∂ can be uniquely extended on A. For example $\partial(aba) = ba + ab$.

Prove that this derivation is locally nilpotent.

What can you say about the degree function which corresponds to this derivation? Is it true that it is the same as the degree of an element relative to a? Is it true for monomials? If you cannot prove it in general can you give a counterexample?

What can you say about the kernel of this derivation?

Prove that A does not have inner locally nilpotent derivations.

Example 8. Next example is the skew field $\mathbb{H}$ of real quaternions. It is a skew field which is a four dimensional space over the field of real numbers $\mathbb{R}$ with a basis 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and relations $\mathbf{i}^2 = \mathbf{j}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$. (It is a skew field because there is a notion of conjugate number in $\mathbb{H}$ just like in $\mathbb{C}$ and for any $x \in \mathbb{H}$ the product $x\overline{x}$ is a non-zero positive real number if $x \neq 0$.)

Let us prove that $\mathrm{LND}(\mathbb{H}) = 0$. Suppose $\partial \in \mathrm{LND}(\mathbb{H})$ and there exists an $x \in \mathbb{H}$ for which $\partial(x) \neq 0$. Let $\partial^{k+1}(x) = 0$, $\partial^k(x) = y \neq 0$, and $\partial^{k-1}(x) = z$. Let $t = zy^{-1}$. Then $\partial(t) = 1$.

Since $\mathbb{H}$ is four-dimensional t must satisfy a polynomial p of degree at most four with real coefficients. Then $0 = p(t) = \partial(p(t)) = p'(t)$ where $p'(t)$ is the ordinary derivative of p. (If $\partial(t) = 1$ then $\partial(t^n) = nt^{n-1}$ by the Leibniz rule.) If we repeat this several times we will get that the leading coefficient of p is zero which is absurd. So $\partial(x) = 0$ for any $x \in \mathbb{H}$ and so $\partial = 0$.

Think why this proof does not work for M_2 above.

Prove that all derivations of $\mathbb{H}$ are inner. (Some of you may know Nether-Scholem theorem from which it follows that for any finite-dimensional skew field all derivations are inner.)

Example 9. Let L be a two-dimensional solvable algebra $L = \text{span}(x, y)$ where $[xy] = x$ and let UL be its universal enveloping algebra. In other words UL is an algebra with two generators which satisfy a relation $[x, y] = xy - yx = x$. We can rewrite it as $yx = x(y - 1)$. Any element of UL may be written as $\sum c_{i,j} x^i y^j$ since $(x^i y^j)x = x^{i+1}(y-1)^j$.

Let us show that $\partial(x) = 0$ if $\partial \in \text{LND}(UL)$ using the degree notion. Let $\partial \in \text{LND}(UL)$. Then $\partial(x) = [\partial(x), y] + [x, \partial(y)] = xf$ where $f \in UL$ because the commutator of any two elements from UL is "divisible" by x.

If $\partial(x) \neq 0$ then $\deg_\partial(\partial(x)) = \deg_\partial(x) - 1 = \deg_\partial(x) + \deg_\partial(f)$ which is impossible since degrees of elements from UL^* are non-negative numbers.

So $\partial(x) = 0$. Then $\partial(y) = p(x)$ since $[x, \partial(y)] = 0$. Since $[x^n, y] = nx^n$ we can see that not all locally nilpotent derivations are inner. We cannot present "partial derivative relative to y", that is, ϵ for which $\epsilon(y) = 1$ as an inner derivation. So $\text{LND}(UL) = \text{ILnd}(UL) \bigoplus \mathbb{C}\epsilon$ and all inner locally nilpotent derivations are given by commutators with polynomials of x.

Algebra UL is an *Ore ring* and can be embedded in the skew field of fractions B. Then any locally nilpotent derivation of UL is locally nilpotent on B.

Can you give examples of other locally nilpotent derivations of B?

Is it true that $\partial(x) = 0$ for any $\partial \in \text{LND}(B)$?

Now for the main new notion.

AK and EC invariants.

The intersection of the rings of constants of all locally nilpotent derivations will be called the *ring of absolute constants* and denoted by $\text{AK}(A)$.

The intersection of the rings of constants of all inner locally nilpotent derivations will be called the *extended center* and denoted by $\mathrm{EC}(A)$.

Of course $\mathrm{EC}(A) = A$ for a commutative A.

For example $\mathrm{AK}(\mathbb{C}[x]) = \mathrm{AK}(\mathbb{C}(x)) = \mathbb{C}$ since only complex numbers are the constants for ordinary derivative.

For quaternions $\mathrm{AK}(\mathbb{H}) = \mathrm{EC}(\mathbb{H}) = \mathbb{H}$ since only zero derivation is locally nilpotent on $\mathbb{H}$.

For M_n invariants $\mathrm{AK}(M_n(\mathbb{C})) = \mathrm{EC}(M_n(\mathbb{C}))$ coincide with the center of M_n and are the subring of scalar matrices.

It is easy to show that $\mathrm{EC}(A < a, b >) = A < a, b >$ and it is possible to show that $\mathrm{AK}(A < a, b >)$ is a polynomial ring in one variable $ab - ba$, but it is rather hard.

We know that $\mathrm{AK}(UL) = \mathrm{EC}(UL) = \mathbb{C}[x]$ for UL from example 9 since $\partial(h) = [x, h]$ is a locally nilpotent derivation and we checked that any locally nilpotent derivation send x to zero.

We also know that $\mathrm{LND}(UL) = \mathrm{ILnd}(UL) \bigoplus \mathbb{C}\epsilon$ where $\epsilon(y) = 1$ and all inner locally nilpotent derivations are given by commutators with polynomials of x. Let us use this information to describe the group $\mathrm{Aut}(UL)$ of automorphisms of UL.

First some general observation. Let A be an algebra. Let α be an automorphism of A and let ∂ be a derivation of A. Then $\alpha^{-1}\partial\alpha$ is also a derivation of A. (It is clear that the Leibniz rule is satisfied.) If ∂ is a locally nilpotent derivation then $\alpha^{-1}\partial\alpha$ is also a locally nilpotent derivation. So $\alpha^{-1}\mathrm{LND}(A)\alpha = \mathrm{LND}(A)$ and $\alpha^{-1}\mathrm{AK}(A) = \mathrm{AK}(A)$.

Now back to our example.

Let $\alpha \in \mathrm{Aut}(UL)$. Since $\alpha(\mathrm{AK}(UL)) = \mathrm{AK}(UL)$, α gives an automorphism of $\mathbb{C}[x]$. But an image of x under any automorphism of $\mathbb{C}[x]$ must be a polynomial of degree one. Therefore $\alpha(x) = cx + d$ where $c \in \mathbb{C}^*$ and $d \in \mathbb{C}$.

Next we know that $\partial^2(y) = 0$ and $\partial(y) \neq 0$ for any non-zero $\partial \in \mathrm{LND}(UL)$. So $\partial^2(\alpha(y)) = 0$ and $\partial(\alpha(y)) \neq 0$ also should be true. Then $\alpha(y) = ey + f$ where $e \in \mathbb{C}[x]^*$ and $f \in \mathbb{C}[x]$.

Finally, since $[\alpha(x), \alpha(y)] = \alpha(x)$ we can easily check that $d = 0$ and $e = 1$.

So $\alpha(x) = cx$ and $\alpha(y) = y + f(x)$ for any $\alpha \in \mathrm{Aut}(UL)$ which is a pretty good description of the automorphism group of UL.

This is a bit artificial example as far as the description of automorphisms through derivations is concerned, since the same "divisibility" considerations can be used directly to show that $\alpha(x) = cx$ for any $\alpha \in \mathrm{Aut}(UL)$. But the approach itself is rather general.

The sets $\mathrm{LND}(A)$ and $\mathrm{ILnd}(A)$ and the rings $\mathrm{AK}(A)$ and $\mathrm{EC}(A)$ are invariants of A. Thus, they may be useful if we need to show that two rings are not isomorphic. On the other hand these objects may help to describe isomorphisms between two algebras and, in particular, automorphism groups of algebras.

Example 10. It is not hard to see that $\mathrm{AK}\mathbb{C}[x_1, ..., x_n] = \mathbb{C}$. Though description of all locally nilpotent derivations is not known if $n > 2$ we can compute the invariant easily since all partial derivatives are locally nilpotent.

Example 11. On the other hand let us look at the ring $R = \mathbb{C}[t^2, t^3]$. This is the ring with a basis consisting of monomials t^i where $i = 0, 2, 3, 4, ...$ (You can think about this ring as the ring of all "nice" functions on the curve

$x^3 = y^2$. From this point of view $\mathbb{C}[t]$ is the ring of all "nice" functions on the line.)

Then $\mathrm{AK}(R) = R$, that is, only zero derivation is locally nilpotent on R. Let us check it. Let $\partial \in Der(R)$ be not zero. If $\partial(t^2) = 1$ then $\partial(t^6) = \partial((t^2)^3) = 3t^4 = 2t^3\partial^(t^3)$. So $\partial(t^3) = \frac{3}{2}t$. But this is impossible since $t \notin R$. (If $\partial(t^2) = 0$ then $\partial(t^3) = 0$ and $\partial = 0$.)

So $\partial(t^2) = p(t)$ where degree of $p(t)$ is at least 2. Let us assume now that ∂ is locally nilpotent. Let us look at degree induced by ∂. Then $\deg_\partial(t^2) > 0$ and $\deg_\partial(t^2) - 1 = \deg_\partial(t)\deg_t(p)$ which does not have a positive solution for $\deg_\partial(t^2)$ if $\deg_t(p) > 1$. (Here $\deg_\partial(t) = \deg_\partial(t^3) - \deg_\partial(t^2) = \frac{1}{2}\deg_\partial(t^2)$.) So ∂ cannot be locally nilpotent.

Describe all derivations on the ring $R = \mathbb{C}[t^2, t^3]$.

We will use locally nilpotent derivations and the invariant to obtain the following results.

(1) A description of the group of automorphisms of a plane,

(2) A description of the group of automorphisms of a surface given by $x^n y = P(z)$, where $n > 1$ and $\deg(P) > 1$,

(3) A new proof of the Abhyankar-Eakin-Heinzer theorem that a curve can be extracted from a cylinder of any dimension over this curve,

(4) A proof that only variety which admit a $\mathbb{C}$ action cannot be extracted from a "one-dimensional" cylinder over this variety,

(5) A proof that a threefold $x + x^2y + z^2 + t^3 = 0$ is not isomorphic to $\mathbb{C}^3$.

We shall also discuss some non-commutative results which are obtained with the help of inner locally nilpotent derivations and modifications of the invariant.

Most probably we will be able to fulfill only a part of this program but then if you are interested I shall suggest you some reading.

Let us prove now some

1. Lemmas on derivations

First a definition. Let A be a domain over $\mathbb{C}$ and let F be its field of fractions. Then the *transcendence degree* of F and A is the biggest number n for which it is possible to embed the field $\mathbb{C}(x_1, ..., x_n)$ into F.

So the transcendence degree of $\mathbb{C}$ is zero and the transcendence degree of algebra $\mathbb{C}[t^2, t^3]$ from Example 11 is one.

This number can also be defined as the biggest number n for which it is possible to embed the ring $\mathbb{C}[x_1, ..., x_n]$ into A or as the biggest number n for which it is possible to find n algebraically independent elements in A.

In the non-commutative setting this notion is replaced by so-called *Gelfand-Kirillov dimension* which for domains coincides with the transcendence degree. So we will use notation $\operatorname{GK}\dim(A)$ for the transcendence degree as well as $\operatorname{trdim}(A)$.

Try to prove that $\operatorname{GK}\dim(\mathbb{C}[x, y]) = 2$, that is, that any three elements from $\mathbb{C}[x, y]$ are algebraically dependent.

If B is a subring of A we can talk about the transcendence degree of A over B. Each of the equivalent definitions above can be appropriately reformulated. For example the first one. Let E be the field of fractions of B.

Then the transcendence degree of F and A over B is the biggest number n for which it is possible to embed the field $E(x_1, ..., x_n)$ into F.

Reformulate other definitions of transcendence degree.

Prove that $\mathrm{trdim}_{C[x]}(\mathbb{C}[x, y]) = 1$.

If A has transcendence degree n let us call any n algebraically independent elements of A a *transcendence basis* of A. It is clear from this definition that for any element $a \in A$ and any transcendence basis $a_1, ..., a_n$ of A there exists a non-trivial polynomial dependence $P(a, a_1, ..., a_n) = 0$ over $\mathbb{C}$.

Another two things to recall.

A subring B of the ring A is *algebraically closed* in A if any element $a \in A \setminus B$ is algebraically independent over B, that is, if the powers of a are linearly independent over B (and so they generate an infinite dimensional B-module). In other words, a subring B of the ring A is algebraically closed in A if any element which is *algebraic* over B belongs to B. (Algebraic over B means that it satisfies a polynomial relation $p(a) = 0$ where the coefficients of p belong to B.)

A subring B of the ring A is *integrally closed* in A if an element $a \in A$ belongs to B whenever it satisfies a monic polynomial relation $p(a) = 0$ where the coefficients of p belong to B.

Let A be a ring and let F be its field of fractions. The ring A is integrally closed if it is integrally closed in F.

Give a module reformulation of this notion.

Any ring is algebraically closed in itself but not necessarily integrally closed. For example the ring $\mathbb{C}[t^2, t^3]$ is not integrally closed. Its field of fractions contains t which satisfies a monic polynomial $(t)^2 - t^2 = 0$.

Give an example when subring B is integrally closed in A but is not algebraically closed.

Lemma 1. Let A be a domain and let ∂ be a derivation of A. Then A^∂ is algebraically closed in A, that is if $g \in A$ then $\partial(g) \neq 0$ if and only if g is algebraically independent over A^∂. If the transcendence degree of A^∂ is equal to the transcendence degree of A then $\partial = 0$.

Proof. Assume that $p(g) = 0$ where $p(x) \in A^\partial[x]$ and has minimal possible degree. Then $0 = \partial(p(g)) = p'(g)\partial(g)$ where p' is the ordinary derivative. But $p'(g) \neq 0$. Therefore $\partial(g) = 0$. Of course, if $\partial(g) = 0$ then $g \in A^\partial$ and so g is algebraically dependent over A^∂. So if $\partial(g) \neq 0$ then g is algebraically independent over A^∂. Now if the transcendence degree of A^∂ is equal to the transcendence degree of A then any $g \in A$ is algebraically dependent over A^∂ and $\partial(g) = 0$.

Lemma 2. Let F be a field and let ∂ be a derivation of F. Then the field $\mathrm{Frac}(\mathrm{Nil}(\partial))$ is either F^∂ or a purely transcendental extension of F^∂. The ring $\mathrm{Nil}(\partial)$ is integrally closed.

Proof. If $\mathrm{Nil}(\partial) = F^\partial$ then it is a field and we are done. Otherwise let us take $p \in (\mathrm{Nil}(\partial) \setminus F^\partial)$. Let $\partial^{k+1}(p) = 0$, $\partial^k(p) = s \neq 0$, and $\partial^{k-1}(p) = r$. Let $t = rs^{-1}$. Then $\partial(t) = 1$. Let us use induction on $\deg_\partial(f) = n$ to show that $f \in F^\partial[t]$ if $f \in \mathrm{Nil}(\partial)$.

If $\deg(f) = 0$ then $f \in F^\partial$. Let us make the step from $\deg(f) = n-1$ to $\deg(f) = n$. If $\deg(f) = n$ then $\deg(\partial(f)) = n-1$ and by induction $\partial(f) = \sum_{i=0}^{n-1} f_i r^{n-1-i}$ for some $f_i \in F^\partial$. Let $g = \sum_{i=0}^{n-1} (n-i)^{-1} f_i r^{n-i}$. Then $\partial(g) = \partial(f)$. So $\partial(f-g) = 0$ which means that $f = g + f_n$ where $f_n \in F^\partial$. So $\mathrm{Nil}(\partial) = F^\partial[t]$ and $\mathrm{Frac}(\mathrm{Nil}(\partial)) = F^\partial(t)$ which implies the lemma.

Check that $E[t]$ is integrally closed for any field E.

Remark. We proved that $\mathrm{Nil}(\partial)$ is either F^∂ or $F^\partial[t]$ where $\partial(t) = 1$. Unlike F^∂ the ring $\mathrm{Nil}(\partial)$ is not necessarily algebraically closed in F.

Let A be an algebra. We will tell that A has a $\mathbb{Z}$ filtration if there is the family of linear subspaces $\{A_i\}$ where $i \in \mathbb{Z}$ such that A_i is a subspace of A_{i+1}, $A_i \times A_j \subset A_{i+j}$ and the union of all A_i is equal to A ($\cup_i A_i = A$). We will also consider $\mathbb{Z}^+$ filtrations where i are only non-negative (it is the same as to say that $A_i = 0$ if $i < 0$). (It is possible to consider filtration with $\mathbb{Z}$ replaced by any ordered semigroup.)

Algebra A is graded if $A = \bigoplus_i B_i$ where $B_i \times B_j \subset B_{i+j}$. Then A also has a filtration with $A_i = \bigoplus_{i=\infty}^{i-1} B_j$.

If algebra A has a filtration then it is possible to construct a graded algebra $\mathrm{Gr}(A)$ which often is easier to deal with then with A itself. Algebra $\mathrm{Gr}(A) = \bigoplus A_i/A_{i-1}$ with linear algebra addition and multiplication defined as follows. Let $a, b \in \mathrm{Gr}(A)$. Let us assume that $a \in A_i/A_{i-1}$ and

$b \in A_j/A_{j-1}$. So $a = \alpha + A_{i-1}$ and $b = \beta + A_{j-1}$ where we can think about elements α and β as elements from A_i and A_j. Then $ab = \alpha\beta + A_{i+j-1}$. This multiplication can be extended on all $\mathrm{Gr}(A)$ with the help of distributive law.

Let A be an algebra with filtration. For any $a \in A$ we can find such an i that $a \in (A_i \setminus A_{i-1})$. We can define a natural map from A into $\mathrm{Gr}(A)$ sending a into its factor class in A_i/A_{i-1}.

Examples.

Example 12. Any algebra A has a filtration where $A_0 = A_i = A$ for all positive i and $A_i = 0$ for all negative i.

Example 13. Let $A = \mathbb{C}[x]$ and let $a \in A_i$ if x-degree of a is less then or equal to i. Then we can present $\mathbb{C}[x]$ as $\bigoplus \mathbb{C}x^i$. So in this case A itself is graded and $Gr(A) \simeq A$.

Example 14. Let $A = \mathbb{C}(a)$ and let $a \in A_i$ if degree of $a \leq i$. Degree of a rational function is equal to the difference of the degrees of numerator and denominator. Then $Gr(A) \simeq \mathbb{C}[x^{-1}, x]$.

Prove that $\mathbb{C}(x)$ cannot be presented as a graded algebra with $B_0 \neq A$.

Example 15. Let $A = UL$ be the algebra from Example 9, that is, an algebra with two generators which satisfy a relation $[x, y] = xy - yx = x$. Let $a \in A_i$ if a can be presented as a polynomial of x and y of degree less then or equal to i. Then $\mathrm{Gr}(A) \simeq \mathbb{C}[u, v]$ a polynomial algebra with two generators. Indeed if $a \in A_i$ and $b \in A_j$ then $ab - ba \in A_{i+j-1}$ so $\mathrm{Gr}(A)$

is commutative. It is also clear that $u = \mathrm{gr}(x)$ and $v = \mathrm{gr}(y)$ generate $\mathrm{Gr}(A)$.

Let A be a ring with $\mathbb{Z}$ filtration $\{A_i\}$ and let ∂ be a derivation on A for which $\partial(A_i) \subset A_{i+k}$ for a fixed k and all i. Let $\mathrm{Gr}(A) = \bigoplus A_i/A_{i-1}$ be the corresponding graded ring and let $h \in A_i/A_{i-1}$. Let us write $h = a + A_{i-1}$ where $a \in A_i$. We can define a homomorphism ∂_1 on $\mathrm{Gr}(A)$ which acts on h by $\partial_1(h) = \partial(a) + A_{i+k-1} \in A_{i+k}/A_{i+k-1}$ and then extend ∂_1 on $\mathrm{Gr}(A)$ by linearity. It is clear that ∂_1 is a derivation of $\mathrm{Gr}(A)$.

Lemma 3. If ∂ is a locally nilpotent derivation on A then ∂_1 is a locally nilpotent derivation on $\mathrm{Gr}(A)$.

Proof. Let us denote by gr the natural mapping of A into $\mathrm{Gr}(A)$ which we described above. Let $a \in A$. It is clear that either $\partial_1(\mathrm{gr}(a)) = 0$ which means that $a \in A_i$ and $\partial(a) \in A_{i+k-1}$ for some i or $\partial_1(\mathrm{gr}(a)) = \mathrm{gr}(\partial(a))$ which means that $a \in A_i$ and $\partial(a) \in A_{i+k}$ for some i. Iterating this computation we see that either $\partial_1^n(\mathrm{gr}(a)) = 0$ or $\partial_1^n(\mathrm{gr}(a)) = \mathrm{gr}(\partial^n(a))$. Since ∂ is locally nilpotent it implies that ∂_1 is locally nilpotent on all elements of the form $\mathrm{gr}(A)$ and therefore on $\mathrm{Gr}(A)$.

Lemmas 1 and 2 can be made more precise for a locally nilpotent derivation

Lemma 4. If ∂ is a locally nilpotent nonzero derivation of a domain A then A has transcendence degree one over the subring A^∂ of constants of ∂, and $\mathrm{Frac}(A)^\partial = \mathrm{Frac}(A^\partial)$.

Proof. Since $A \neq A^\partial$ there exists an $r \in A \setminus A^\partial$ such that $\partial(r) \in A^\partial$.

Indeed let $a \in A \setminus A^\partial$. Then we can take $r = \partial^n(a)$ where $n = \deg_\partial(a) - 1$. Let $s = \partial(r)$. Let us again use induction on $\deg_\partial(a) = n$ to show that there exist elements a_i, $b \in A^\partial$ where $i = 0, 1, ..., n$ and $ba_0 \neq 0$ such that $ba = \sum_{i=0}^n a_i r^{n-i}$. If $\deg(a) = 0$ then $a \in A^\partial$. Let us make the step from $\deg(a) = n - 1$ to $\deg(a) = n$. If $\deg(a) = n$ then $\deg(\partial(a)) = n - 1$ and by induction $b\partial(a) = \sum_{i=0}^{n-1} a_i r^{n-1-i}$ for some a_i, $b \in A^\partial$ where $b \neq 0$. Let $f = \sum_{i=0}^{n-1} (n-i)^{-1} a_i r^{n-i}$. Then $\partial(f) = sb\partial(a)$. So $\partial(sba - f) = 0$ which means that $sba = f + a_n$ where $a_n \in A^\partial$. So $A \subset \mathrm{Frac}(A^\partial)[r]$ and any two elements of A are algebraically dependent over A^∂ while any element of $A \setminus A^\partial$ is transcendental over A^∂. The remaining claim follows from $\mathrm{Frac}(A) = \mathrm{Frac}(A^\partial)(r)$.

We shall use the presentation $ba = \sum_{i=0}^n a_i r^{n-i}$ of elements from a on several occasions later.

The function $\deg_\partial$ will be one of our major tools.

Lemma 5. If A is a domain and $\partial \in \mathrm{LND}(A)$ then $\deg_\partial$ is a degree function, i.e.
$\deg_\partial(a + b) \leq \max(\deg_\partial(a), \deg_\partial(b))$ and
$\deg_\partial(ab) = \deg_\partial(a) + \deg_\partial(b)$.

Proof. Follows immediately from the above presentation of elements of A since it is clear that $\sum_{i=0}^n a_i r^{n-i}$ has degree n if $a_0 \neq 0$.

Remarks. Our main application will be that if a product is a ∂-constant then each factor is also a ∂-constant or in other words that A^∂ is factorially closed, that is, if $a, b \in A^*$ and $ab \in A^\partial$, then $a, b \in A^\partial$. It is clear since degree has only non-negative values on A^*.

Any degree function on A can be extended naturally to the field of fractions of A.

Any degree function on A induces a filtration on A by $a \in A_i$ if $\deg(a) \leq i$.

Locally nilpotent derivations, a new ring invariant and applications

2. Derivations of polynomial rings and fields of rational functions

Let us consider a polynomial ring $\mathbb{C}_n = \mathbb{C}[x_1, x_2, ..., x_n]$ in n variables. Then clearly any partial derivative is a locally nilpotent derivation. Therefore $\mathrm{AK}(\mathbb{C}_n) = \mathbb{C}$, since if all partial derivatives of a polynomial are zeros this polynomial is a constant.

Though it is easy to describe $\mathrm{Der}(\mathbb{C}_n)$ (any derivation is completely determined by its values on the generating set $\{x_i\}$ and these values can be taken arbitrarily), description of $\mathrm{LND}(\mathbb{C}_n)$ is known only if $n = 1$ (an easy exercise) or $n = 2$ ([Re]).

Namely, $\mathrm{LND}(\mathbb{C}_1) = \{c\frac{d}{dx}\}$ where $c \in \mathbb{C}$.

Description of $\mathrm{LND}(\mathbb{C}_2)$ is more involved and it is more natural if we use the Jacobians.

Let $f_1, f_2, ..., f_n \in \mathbb{C}_n$. Let us denote by $Jac(f_1, f_2, ..., f_n)$ the Jacobian of these elements of $\mathbb{C}_n$, that is the determinant of the corresponding Jacobi matrix.

$$Jac(f_1, f_2, ..., f_n) = \begin{vmatrix} \frac{df_1}{dx_1} & ... & \frac{df_n}{dx_1} \\ ... & ... & ... \\ \frac{df_1}{dx_n} & ... & \frac{df_n}{dx_n} \end{vmatrix}.$$

It is clear that then $\partial(h) = Jac(f_1, f_2, ..., f_{n-1}, h)$ is a derivation.

For example if we have just one variable x then $\partial(h) = \frac{dh}{dx}$ and if we have two variables then $\partial(h) = \frac{\partial f_1}{\partial x_1}\frac{\partial h}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial h}{\partial x_1}$.

If $f_1, ..., f_n$ are algebraically dependent then $Jac(f_1, f_2, ..., f_n) = 0$. Indeed, let $P(f_1, ..., f_n) = 0$. Then $\frac{\partial P}{\partial x_j} = \sum_i \frac{\partial P}{\partial f_i}\frac{\partial f_i}{\partial x_j} = 0$. Therefore the Jacobi matrix applied to a non-zero vector $\frac{\partial P}{\partial f_1}, ..., \frac{\partial P}{\partial f_n}$ gives zero vector. But it is possible only if the determinant of this matrix is zero.

On the other hand if $f_1, ..., f_n$ are algebraically independent then $Jac(f_1, f_2, ..., f_n) \neq 0$. Indeed, let $X_i(f_1, ..., f_n, x_i) = 0$ be an algebraic dependence between $f_1, ..., f_n$ and x_i. Such a dependence exists since the transcendence degree of $\mathbb{C}_{(n)}$ is n. Then $\frac{\partial X_i}{\partial x_j} = \sum_k \frac{\partial X_i}{\partial f_k}\frac{\partial f_k}{\partial x_j} + X_{i,n+1}\frac{\partial x_i}{\partial x_j} = 0$ where $X_{i,n+1}$ is not zero. (If X_i does not contain x_i then f_j's are algebraically dependent.) Therefore the Jacobi matrix applied to the vector $\frac{\partial X_i}{\partial f_1}, ..., \frac{\partial X_i}{\partial f_n}$ gives the vector $X_{i,n+1}e_i$ where e_i has 1 in position i and zeros in all other positions. Since vectors $X_{i,n+1}$ are linearly independent over $\mathbb{C}_{(n)}$ the Jacobian cannot be zero.

We shall prove now several lemmas about derivations of $\mathbb{C}_n$ and $\mathbb{C}_{(n)} = \text{Frac}(\mathbb{C}_n)$.

First of all it is clear that for any $\{f_i\} \subset \mathbb{C}_n$ the Jacobian $\partial(h) = Jac(f_1, f_2, ..., f_{n-1}, h)$ is a derivation of $\mathbb{C}_n$ (if $h \in \mathbb{C}_n$ then $\partial(h) \in \mathbb{C}_n$) and $\mathbb{C}_{(n)}$. If $\{f_i\} \subset \mathbb{C}_{(n)}$ then the Jacobian $\partial(h) = Jac(f_1, f_2, ..., f_{n-1}, h)$ is a derivation of $\mathbb{C}_{(n)}$.

Lemma 6. Let $\partial \in \text{Der}(\mathbb{C}_{(n)})$. If the transcendence degree of $\mathbb{C}_{(n)}^{\partial}$ is $n-1$ and $f_1, ..., f_{n-1}$ is a transcendence basis of $\mathbb{C}_{(n)}^{\partial}$ then there exists an $h \in \mathbb{C}_{(n)}$ such that $\partial(a) = hJac(f_1, ..., f_{n-1}, a)$ for all $a \in \mathbb{C}_{(n)}$. Here $Jac(f_1, ..., a)$ is the Jacobian relative to $x_1, ..., x_n$.

Proof. Any derivation of $\mathbb{C}_{(n)}$ is completely determined by its values on any n algebraically independent elements.

Indeed if we know ∂ on any transcendence basis $a_1, ..., a_n$ of a ring A then it extends uniquely on A. Let $a \in A$. Then there exists an algebraic dependence $P(a_1, ..., a_n, a) = 0$ which contains a since $a_1, ..., a_n$ are algebraically independent. So $0 = \partial(P) = \sum_i \frac{\partial P}{\partial a_i}\partial(a_i) + \frac{\partial P}{\partial a}\partial(a)$. (It is called the chain rule in calculus.) We can determine $\partial(a)$ from this equality.

Let $\epsilon(a) = Jac(f_1, ..., f_{n-1}, a)$. Then $\epsilon(f_i) = 0$ for $i = 1, ..., n-1$.
(Since a determinant with two identical columns is zero.)

Any element of $\mathbb{C}^{\partial}_{(n)}$ is algebraically dependent with $f_1, ..., f_{n-1}$ because it is a transcendence basis of $\mathbb{C}^{\partial}_{(n)}$. By Lemma 1 $\mathbb{C}^{\epsilon}_{(n)}$ is algebraically closed. So $\mathbb{C}^{\partial}_{(n)} \subset \mathbb{C}^{\epsilon}_{(n)}$. (It is another explanation why the Jacobian of algebraically dependent elements is zero.) If $\mathbb{C}^{\epsilon}_{(n)}$ is bigger then $\mathbb{C}^{\partial}_{(n)}$ then it contains n algebraically independent elements and ϵ is zero. But there exists an f which is algebraically independent over $\mathbb{C}^{\partial}_{(n)}$ (Lemma 4) and we checked that the Jacobian of algebraically independent elements is not zero. So $\epsilon(f) \neq 0$. Therefore $\mathbb{C}^{\partial}_{(n)} = \mathbb{C}^{\epsilon}_{(n)}$.

Let us take any g for which $\partial(g) \neq 0$.
As we just checked $\epsilon(g) \neq 0$.
Since $h\epsilon$ is a derivation for any $h \in \mathbb{C}(x_1, ..., x_n)$ it is sufficient to determine h by $h = \partial(g)(\epsilon(g))^{-1}$.

Lemma 7. Let us define a derivation ∂ by $\partial(h) = Jac(f_1, f_2, ..., f_{n-1}, h)$ where $f_1, ..., f_{n-1} \in \mathbb{C}_{(n)}$ are algebraically independent. If $g_1, g_2, ..., g_{n-1} \in \mathbb{C}^{\partial}_{(n)}$ then the derivation $\partial_{n-1}(h) = Jac(g_1, g_2, ..., g_{n-1}, h)$ is "proportional" to ∂ over $\mathbb{C}^{\partial}_{(n)}$ i. e. $a\partial = a_{n-1}\partial_{n-1}$ for some $a \in \mathbb{C}^{\partial}_{(n)}$ and $a_{n-1} \in \mathbb{C}^{\partial}_{(n)} \setminus 0$.

Proof. It is clear from the definition of ∂ that all $f_i \in \mathbb{C}^{\partial}_{(n)}$. We may assume that the $\{g_i\}$ are algebraically independent because otherwise ∂_{n-1} is zero. Under this assumption one of the standard proofs for the basis theorem in linear algebra can be used (the so called *replacement* proof).

Let us show that $a\partial = a_{n-1}\partial_{n-1}$ for some $a, a_{n-1} \in \mathbb{C}^{\partial}_{(n)} \setminus 0$. Since $\{f_i\}$ are algebraically independent it follows that the transcendence degree of $\mathbb{C}^{\partial}_{(n)}$ is $n-1$.

(It contains $n-1$ algebraically independent elements and cannot contain n algebraically independent elements.)
So any n elements of $\mathbb{C}^{\partial}_{(n)}$ are algebraically dependent.

Let P_1 be an irreducible polynomial from $\mathbb{C}_n$ for which $P_1(f_1, f_2, ..., f_{n-1}, g_1) = 0$. We may assume up to renumbering the $\{f_i\}$ that P_1 depends on f_1. Then $0 = Jac(P_1, f_2, ..., f_{n-1}, h) = Jac(f_1, f_2, ..., f_{n-1}, h)\frac{\partial P_1}{\partial f_1} + Jac(g_1, f_2, ..., f_{n-1}, h)\frac{\partial P_1}{\partial g_1}$ since $\frac{\partial P_1}{\partial x_j} = \sum_j \frac{\partial P_1}{\partial f_i}\frac{\partial f_i}{\partial x_j} + \frac{\partial P_1}{\partial g_1}\frac{\partial g_1}{\partial x_j}$.
So derivations ∂ and ∂_1 where $\partial_1(h) = Jac(g_1, f_2, ..., f_{n-1}, h)$ are proportional with coefficients from $\mathbb{C}^{\partial}_{(n)}$ which are not zeros with our choice of P_1. (Since $\frac{\partial P_1}{\partial f_1}(f_1, f_2, ..., f_{n-1}, g_1)$ and $\frac{\partial P_1}{\partial g_1}(f_1, f_2, ..., f_{n-1}, g_1)$ are in $\mathbb{C}^{\partial}_{(n)}$.)

Now we may assume that $\partial_i(h) = Jac(g_1, g_2, ..., g_i, f_{i+1}, ..., f_{n-1}, h)$ is a nonzero derivation and that ∂_i and ∂ are proportional over $\mathbb{C}^{\partial}_{(n)}$. Let us consider an irreducible polynomial P_{i+1} for which $P_{i+1}(g_1, g_2, ..., g_i, f_{i+1}, ..., f_{n-1}, g_{i+1}) = 0$. Since the elements $g_1, g_2, ..., g_i, f_{i+1}, ..., f_{n-1}$ are algebraically independent, (otherwise $\partial_i = 0$) such a polynomial depends on g_{i+1} and since the elements $\{g_i\}$ are algebraically independent this polynomial depends on at least one of the f's. So up to renumbering we can replace f_{i+1} by g_{i+1} and obtain ∂_{i+1} which is not zero and proportional to ∂_i over $\mathbb{C}^{\partial}_{(n)}$. This proves the lemma.

Remark. As we can see from the proof the derivations ∂ and ∂_{n-1} have the same constants and even induce the same degree function when ∂ is locally nilpotent derivation of $\mathbb{C}_n$, provided $g_1, g_2, ..., ..., g_{n-1}$ belong to $\mathbb{C}_n$ and are algebraically independent.

Let us call two locally nilpotent derivations *equivalent* if the corresponding degree functions are the same.

Lemma 8. If $\partial \in \mathrm{LND}(\mathbb{C}_n)$ then it is equivalent to a "Jacobian" derivation $\epsilon(a) = Jac(f_1, ..., f_{n-1}, a)$.

Proof. We can choose by Lemma 4 algebraically independent $f_1, ..., f_{n-1} \in \mathbb{C}_n^\partial$. Let $\epsilon(a) = Jac(f_1, ..., f_{n-1}, a)$. Let us find an $r \in \mathbb{C}_n$ of ∂-degree 1. Derivation $\epsilon(r)\partial - \partial(r)\epsilon$ is zero by Lemma 6. Therefore $\partial = h\epsilon$ where the numerator
$\partial(r)$ of h is in $\mathbb{C}_n^\partial$. If denominator q
of h
is in $\mathbb{C}_n^\partial$ then we are done because then $\epsilon(h) = 0$ and $\partial^n = h^n\epsilon^n$.

If not we may assume that q has an irreducible factor p which does not belong to $\mathbb{C}_n^\partial$. Since $\mathbb{C}_n^\partial$ is factorially closed the numerator of h and p are relatively prime.

Now $\partial(a) = h\epsilon(a)$. Therefore p divides $Jac(f_1, ..., f_{n-1}, a)$ (in $\mathbb{C}_n$) for any $a \in \mathbb{C}_n$.

By Lemma 7 for any $\{g_i\} \subset \mathbb{C}_{(n)}^\partial$ and for any $a \in \mathbb{C}_n$, $Jac(g_1, ..., g_{n-1}, a) = bJac(f_1, ..., f_{n-1}, a)$ where $b \in C_{(n)}^\partial$. Therefore for any $\{g_i\} \subset \mathbb{C}_{(n)}^\partial$ and any $a \in \mathbb{C}_n$, p divides $Jac(g_1, ..., g_{n-1}, a)$ in the ring A generated by rings $\mathbb{C}_n$ and $C_{(n)}^\partial$. By Lemma 4 all elements from A are fractions with the numerators in $\mathbb{C}_n$ and denominators in $\mathbb{C}_n^\partial$. So all elements from A have non-negative ∂ degree. We'll use it later.

Let us prove now that p divides $Jac(a_1, ..., a_n)$ in the ring A for any $\{a_i\} \subset A$.

From Lemma 4 we know that an element $a \in A$ can be presented as $a = \sum b_i r^i$ where $r \in \mathbb{C}_n$ and $b_i \in \mathbb{C}_{(n)}^\partial$. Using this presentation for a_i we see that it is sufficient to prove divisibility when all a_i' s are replaced by monomials $b_k r^k$. Since $Jac(a_1, ..., a_{n-1}, b_k r^k) = Jac(a_1, ..., a_{n-1}, b_k)r^k + Jac(a_1, ..., a_{n-1}, r^k)b_k$ and the order of a_i's is note essential we can reduce our question even further and assume that all a_i's are either elements of $\mathbb{C}_{(n)}^\partial$ or are equal to r^k. Further we may assume that just one of a_i's is equal to r^k since otherwise the Jacobian is zero. (If none of a_i's is equal to a power of r or two of a_i's are equal to powers of r the Jacobian will have algebraically dependent arguments.)

Since we already proved that p divides $Jac(g_1, ..., g_{n-1}, a)$ for any $\{g_i\} \subset \mathbb{C}^{\partial}_{(n)}$ and any $a \in \mathbb{C}_n$ we now proved that p divides $Jac(a_1, ..., a_n)$ for any $\{a_i\} \subset A$.

Therefore $1 = Jac(x_1, ..., x_{n-1}, x_n)$ should also be divisible by p in the ring A. Since $\deg_{\partial}(p^{-1}) < 0$ it is impossible. The lemma is finally proved.

Locally nilpotent derivations, a new ring invariant and applications

3. The group of automorphisms of the polynomial ring $\mathbb{C}[x, y]$

Let A be a ring of polynomials in two variables x and y. Let us describe LND(A). According to Lemma 8 $\partial \in \mathrm{LND}(A)$ is equivalent to $\epsilon(h) = Jac(f, h)$ for some $f \in A$. We would like to describe somehow polynomials f but it is too difficult. So instead we'll describe top homogeneous components of f.

To be more precise let ρ and σ be positive relatively prime integers. Let us assign x degree ρ and y degree σ. Then a monomial $x^i y^j$ has degree $\rho i + \sigma j$. Let $\overline{f}_{\rho,\sigma}$ be a polynomial consisting of all the monomials of f of maximal degree possible with their coefficients. If it is clear which ρ and σ we have in mind we will right just $\overline{f}$. We will call $\overline{f}$ top ρ, σ-homogeneous components of f. If $f = \overline{f}_{\rho,\sigma}$ we call f ρ, σ homogeneous.

Check that all properties of degree function are satisfied by ρ, σ degree. (See Lemma 5 for definition.)

It is rather obvious that $\overline{fg} = \overline{f}\overline{g}$. Similarly since $Jac(x^a y^b, x^c y^d) = (ad - bc)x^{a+c-1} y^{b+d-1}$ it is clear that $\overline{Jac(f, g)} = Jac(\overline{f}, \overline{g})$ if $Jac(\overline{f}, \overline{g}) \neq 0$.

From this we see that if $\partial(h) = Jac(f, h)$ is a locally nilpotent derivation then $\partial_1(h) = Jac(\overline{f}, h)$ is also locally nilpotent. Indeed $\partial_1^n(x)$ is homogeneous for all n and $\overline{\partial^n(x)} = \partial_1^n(x)$ if $\partial_1^n(x) \neq 0$. But $\partial^n(x) = 0$ for some n so $\partial_1^m(x) = 0$ for some $m \leq n$. The same is true for y, so $\partial_1^k(y) = 0$ and so ∂_1 is locally nilpotent. (We could also use Lemma 3 to prove it with A_i consisting of all $a \in A$ with $\deg_{\rho,\sigma}(a) \leq i$ and $k = \deg_{\rho,\sigma}(f) - \rho - \sigma$.)

If f is homogeneous then $f = \sum_{\rho p+\sigma q=d} f_{p,q} x^p y^q = cx^i y^j \prod_k (x^\sigma - \lambda_k y^\rho)$. Let $\overline{f} = cx^i y^j \prod_k (x^\sigma - \lambda_k y^\rho)$.

Since $\partial_1(\overline{f}) = 0$, all factors of $\overline{f}$ are ∂_1 constants. If we have two essentially different (algebraically independent) factors, say, $x^\sigma - \lambda y^\rho$ and $x^\sigma - \mu y^\rho$ where $\lambda \neq \mu$ then $\partial_1(x) = \partial_1(y) = 0$ and $\partial_1 = 0$ (see lemma 4). Since ∂_1 is not a zero derivation we proved that

Lemma 9. $\overline{f} = x^i$ or y^j or $(x^\sigma - \lambda y^\rho)^m$.

If $\overline{f} = (x^\sigma - \lambda y^\rho)^m$ then $\partial_1(h) = Jac((x^\sigma - \lambda y^\rho)^m, h)$ is locally nilpotent. Element $x^\sigma - \lambda y^\rho \in A^{\partial_1}$. So by Lemma 7 derivation $\partial_2(h) = Jac(x^\sigma - \lambda y^\rho, h)$ is also locally nilpotent (and even equivalent to ∂_1).

Now $\partial_2(x) = \lambda \rho y^{\rho-1}$ and $\partial_2(y) = \sigma x^{\sigma-1}$. If we consider degree relative to ∂_2 and denote by $d_x = \deg(x)$ and $d_y = \deg(y)$ then

$$d_x - 1 = (\rho - 1)d_y,\ d_y - 1 = (\sigma - 1)d_x.$$

So $-2 = (\sigma - 2)d_x + (\rho - 2)d_y$ which is possible only if either ρ or σ is 1. (ρ and σ are positive integers.)

If $\overline{f} = c(y - \mu x^\sigma)^m$ let us replace f by $\beta(f)$ where β is the automorphism which is given by $\beta(x) = x$, $\beta(y) = y + \mu x^\sigma$. If $\overline{f} = c(x - \lambda y^\rho)^n$ let us replace f by $\gamma(f)$ where γ is the automorphism which is given by $\gamma(x) = x + \lambda y^\rho$, $\gamma(y) = y$.

Then $\overline{\beta(f)} = cy^m$ and degree relative to x of $\beta(f)$ is smaller than n. Similarly $\overline{\gamma(f)} = cx^n$ and degree relative to y of $\gamma(f)$ is smaller than m.

Now we can use induction on, say, product of x and y degrees of f to see that the following statement is correct

Statement. If $\epsilon(h) = Jac(f, h)$ is a locally nilpotent derivation then there exists a sequence of automorphisms given by
$\Delta_i(x) = x$, $\Delta_i(y) = y + cx^{k_i}$ or
$\Delta_i(x) = x + cy^{k_i}$, $\Delta_i(y) = y$
such that for $\Delta = \Delta_1 ... \Delta_k$ we have
$\Delta(f) = p(x)$ or $\Delta(f) = p(y)$.

Therefore for any locally nilpotent derivation $A^\partial = \mathbb{C}[g]$ where g is the image of x or y under a composition of triangular automorphisms.

Now we are ready to finish the story.

Let α be an automorphism of A. Any element $h \in A$ can be written as a polynomial of $\alpha(x)$ and $\alpha(y)$. So we can associate with α a locally nilpotent derivation $\partial(h) = Jac_{\alpha(x),\alpha(y)}(\alpha(x), h)$. It is clear that $A^\partial = \mathbb{C}[\alpha(x)]$. So $\mathbb{C}[\alpha(x)] = \mathbb{C}[\Delta(x)]$ (or $\mathbb{C}[\alpha(x)] = \mathbb{C}[\Delta(y)]$). Therefore $\alpha(x) = a\Delta(x) + b$ where Δ is a composition of triangular automorphisms, $a \in \mathbb{C}^*$, and $b \in \mathbb{C}$ (or $\alpha(x) = a\Delta(y) + b$).

Hence $\Delta^{-1}\alpha(x) = ax + b$ or $\Delta^{-1}\alpha(x) = ay + b$. We can even make an additional triangular automorphism $\delta(x) = x - a^{-1}b$, $\delta(y) = y$ (or $\delta(x) = x$, $\delta(y) = y - a^{-1}b$) to eliminate b.

So there exists a composition of triangular automorphisms (let us denote it again by Δ) such that $\Delta^{-1}\alpha(x) = ax$ or $\Delta^{-1}\alpha(x) = ay$. Let us consider now the locally nilpotent derivation ∂ associated with $\Delta^{-1}\alpha(x)$. We know that $\partial(\Delta^{-1}\alpha(y)) = 1$. In the first case when $\Delta^{-1}\alpha(x) = ax$ this derivation is equivalent to $\epsilon(h) = Jac(x, h)$. Therefore in this case $\Delta^{-1}\alpha(y) = c(x)y + p(x)$. Similarly when $\Delta^{-1}\alpha(x) = ay$ we have $\Delta^{-1}\alpha(y) = c(y)x + p(y)$. In both cases we can assume that $c \in C^*$ since otherwise $\Delta^{-1}\alpha$ cannot be an automorphism.

If we proceed with reductions by talking further compositions with automorphism of Δ_i type then in the first case we obtain α_2 with
$\alpha_2(x) = ax$, $\alpha_2(y) = cy$
and in the second case $\alpha_2(x) = ay$, $\alpha_2(y) = cx$.

So all automorphisms of A are compositions of Δ_i type automorphisms (so called triangular or Jonquier automorphisms) and a linear automorphism of α_2 type.

Locally nilpotent derivations, a new ring invariant and applications

4. The groups of automorphisms of the surfaces $x^n y = P(z)$.

Today we shall find the group of automorphisms of a surface in $\mathbb{C}^3$ which is given by $x^n y = P(z)$ where $n > 1$ and $\deg(P) = d > 1$. (The case $n = 1$ was considered in [ML1] and [DG] and if $d = 1$ the surface is a plane.)

So let $Q = X^n Y - P(Z)$. We shall describe all nilpotent derivations of the ring $S = \mathbb{C}[X, Y, Z]/(Q)$ and show that $\mathrm{AK}(S) = \mathbb{C}[x]$ where x is the image of X. After that it will be an easy exercise to describe the group $\mathrm{Aut}(S)$.

Let us repeat two simple lemmas which we already discussed.

Lemma 1. Let A be a domain and let ∂ be a derivation of A. If $g \in A$ then $\partial(g) \neq 0$ if and only if g is algebraically independent over A^∂. If the transcendence degree of A^∂ is equal to the transcendence degree of A then $\partial = 0$.

Lemma 6. Let $\partial \in \mathrm{Der}(\mathbb{C}_{(n)})$. If the transcendence degree of $\mathbb{C}_{(n)}^\partial$ is $n-1$ and $f_1, ..., f_{n-1}$ is a transcendence basis of $\mathbb{C}_{(n)}^\partial$ then $\partial(a) = hJac(f_1, ..., f_{n-1}, a)$ for some $h \in \mathbb{C}_{(n)}$.

Locally nilpotent derivations of S.

Let x, y, z be the images of X, Y, Z in $S = \mathbb{C}[X, Y, Z]/(Q)$. We can make a linear substitution in z and assume without loss of generality that $P(z)$ is a monic polynomial with zero coefficient of z^{d-1}. So $P(z) = z^d + p_2 z^{d-2} + ... + p_d$. Let us consider S as a subring of $\mathbb{C}[x, x^{-1}, z]$.

Let ∂ be a locally nilpotent non-zero derivation of S and let $f \in S^{\partial} \setminus \mathbb{C}$. Such an f exists since $\mathrm{trdim}(S^{\partial}) = \mathrm{trdim}(S)-1$ by Lemma 4 and $\mathrm{trdim}(S) = 2$. So ∂ has a non-trivial kernel. Then $\partial(g) = hJac(f,g)$ by Lemma 6, where Jac is the Jacobian relative to x and z, and $h \in \mathbb{C}(x,z)$.

Lemma 10. $f \in \mathbb{C}[x,z]$.

Proof. Let us assume that $f \notin \mathbb{C}[x,z]$. Then $f = \sum c_{i,j,k}x^i y^j z^k$ where for some monomials $i - nj < 0$. Let us consider all monomials with $i - nj = -m$ the minimal possible and let $\underline{f}$ be a polynomial consisting of all these monomials of f with their coefficients. It is similar to the top homogeneous component we looked at in the previous section but is rather a bottom component. Of course you can look at it as a top component which corresponds to degree of x equal to minus one and degree of z equal to zero. $\underline{f} = \sum c_{i,j,k}x^i y^j z^k = x^{-m}\sum c_{j,k}(x^n y)^j z^k = x^{-m}\sum c_{j,k}(P(z))^j z^k$. Since here $i-nj = -m$ where i and j are non-negative integers we must have $j \geq mn^{-1}$. So $x^{-m}\sum c_{j,k}(P(z))^j z^k = x^{-m}P(z)^u \sum c_{j,k}(P(z))^{j-u} z^k$ where $u \geq mn^{-1}$ (and is an integer) and $j - u \geq 0$.

Let us now find a monomial of $x^{-m}P(z)^u \sum c_{j,k}(P(z))^{j-u}z^k$ with the maximal power of z possible. For $x^{-m}P(z)^u$ it is $x^{-m}z^{du} = (x^{-n}z^d)^u x^v$ where $v = nu - m$ is a non-negative integer. So for $x^{-m}P(z)^u \sum c_{j,k}(P(z))^{j-u}z^k$ such a monomial is $(x^{-n}z^d)^u x^v z^w$ where w is also a non-negative integer.

In other words we checked that in lexicographic order which is given by $x^{-1} >> z > 1$ maximal monomial of any element s from S is $(x^{-n}z^d)^i x^j z^k$ where i, j, k are non-negative integers.

In order to use Lemma 3 let us consider a filtration which is induced by a degree function given by $\deg(x) = -N$ and $\deg(z) = 1$ where N is a very big positive integer so that $\overline{f} = (x^{-n}z^d)^u x^v z^w$ and $\overline{h} = x^a z^b$.

Let ∂_1 be the locally nilpotent derivation of $\mathrm{Gr}(S)$ which is induced by ∂ (see Lemma 3). Let $\mathrm{gr}(y)$ denote $x^{-n}z^d$. Then $\mathrm{gr}(f) = x^v z^w \,\mathrm{gr}(y)^u$ where u is a positive integer. Since a product of elements from $\mathrm{Gr}(S)$ is a ∂_1-constant only if all multipliers are constants and ∂_1 is not a zero derivation it means that $v = w = 0$ (otherwise ∂_1 will be zero on two algebraically independent elements of $\mathrm{Gr}(S)$ and therefore zero by Lemma 1). So we may assume that $\mathrm{gr}(f) = \mathrm{gr}(y)^u$ and (using Lemma 7 or a trivial computation) that $\partial_2(g) = x^a z^b Jac(\mathrm{gr}(y), g)$ is a locally nilpotent derivation on $\mathrm{Gr}(S)$.

Conditions that $\partial_2(x)$ and $\partial_2(z)$ are in $\mathrm{Gr}(S)$ can be written as

$(a, b) + (-n, d-1) = \alpha_x(-n, d) + (\beta_x, \gamma_x)$ and

$(a, b) + (-n-1, d) = \alpha_z(-n, d) + (\beta_z, \gamma_z)$

correspondingly where all scalars in the right sides are non-negative integers. (We are using here the fact that all monomials in $\mathrm{Gr}(S)$ are given by $x^i \mathrm{gr}(y)^j z^k$.)

They may be rewritten as

$a + (\alpha_x - 1)n \geq 0,\;\; b - 1 - (\alpha_x - 1)d \geq 0,$

$a - 1 + (\alpha_z - 1)n \geq 0,\;\; b - (\alpha_z - 1)d \geq 0$ or

$\frac{b-1}{d} \geq (\alpha_x - 1) \geq \frac{-a}{n},$

$\frac{b}{d} \geq (\alpha_z - 1) \geq \frac{1-a}{n}.$

Let us use now the condition that ∂_2 is locally nilpotent. Let us denote ∂_2 degrees of x and z by p and q. Then

$np - dq = 0$

since the degree of $\mathrm{gr}(f)$ is zero and

$p - 1 = (a-n)p + (b+d-1)q$

since $\partial_2(x) = x^a z^b Jac(\mathrm{gr}(y), x)$.

So $p = d\Delta^{-1}$ and $q = n\Delta^{-1}$ where $\Delta = n(1-b) + d(1-a)$. Therefore $\Delta > 0$ (and Δ divides n and d).

Now the integers $\alpha_x - 1$ and $\alpha_z - 1$ both belong to the interval $[\frac{-a}{n}, \frac{b}{d}]$.

The length of this interval is $\frac{b}{d} + \frac{a}{n} = \frac{bn+ad}{dn} = \frac{n+d-\Delta}{nd} < \frac{n+d}{nd} \le 1$ since we assumed that $n > 1$ and $d > 1$. So $\alpha_x = \alpha_y$. But then $\frac{b-1}{d} \ge \frac{1-a}{n}$ which means that $0 \ge \Delta$. We reached a contradiction which proves the lemma.

Lemma 11. $f \in \mathbb{C}[x]$.

Proof. Let us now consider filtration which is induced by degree function given by $\deg(x) = 1$ and $\deg(z) = N$ where N is a big positive integer so that $\mathrm{gr}(f)$ and $\mathrm{gr}(h)$ are monomials which have the same degrees relative to z as f and h correspondingly.

Let ∂_1 be a locally nilpotent derivation of $\mathrm{Gr}(S)$ which is induced by ∂ (see Lemma 3). If $f \notin \mathbb{C}[x]$ then $\mathrm{gr}(f) = cx^i z^j$ where $j > 0$ and $c \in \mathbb{C}^*$. Let us divide f by c.

Since product of elements from $\mathrm{Gr}(S)$ is a ∂_1-constant only if all multipliers are constants and ∂_1 is not a zero derivation it means that $i = 0$. So $gr(f) = z^j$ and $\partial_1(g) = x^a z^b Jac(z^j, g)$ where $\mathrm{gr}(h) = x^a z^b$. Let us denote by deg the degree function induced by ∂_1 and let $d_x = \deg(x)$, $d_y = \deg(\mathrm{gr}(y))$, and $d_z = \deg(z)$. It is clear that $d_z = 0$ and that d_x, d_y should be non-negative integers. But $d_y = -nd_x$. So $d_x = d_y = 0$ which is also impossible since $\partial_1(x) = -jx^a z^{b+j-1} \ne 0$.

So we brought to a contradiction assumption $j > 0$ which means that $f \in \mathbb{C}[x]$.

From these two lemmas we see that $\mathrm{AK}(S) = \mathbb{C}[x]$ and that we can write ∂ as $\partial(g) = hJac(x, g)$.

So $\partial(z) = h \in S$ and $\partial(y) = hx^{-n}P'(z) \in S$ (here $P'(z)$ is the ordinary derivative relative to z). Since $\deg_\partial(\partial(z)) = \deg_\partial(z) - 1 = \deg_\partial(h)$ and $\deg_\partial(x) = 0$ we can see that h cannot contain any monomials with z and therefore $h \in \mathbb{C}[x]$. Even more, since the degree of P' relative to z is $d-1$ we know from the proof of Lemma 10 that hx^{-n} cannot contain negative degrees of x. So $h = x^n h_1$ where $h_1 \in \mathbb{C}[x]$.

We proved the following

Proposition. Derivation $\partial \in \mathrm{LND}(S)$ if and only if $\partial(g) = x^n h(x) Jac(x, g)$ or using the standard notation for partial derivatives $\mathrm{LND}(S) = x^n \mathbb{C}[x] \frac{\partial}{\partial z}$.

Let us use this information for description of automorphisms.

Automorphisms of S.

Lemma 12. Let $\alpha \in \mathrm{Aut}(S)$. Then $\alpha(x) = c_1 x$ and $\alpha(z) = c_2 z + b(x)$ where $c_1, c_2 \in \mathbb{C}^*$, $b(x) \in \mathbb{C}[x]$, $b(x) \equiv 0 \pmod{x^n}$, and $P(c_2 z) = c_2^d P(z)$.

Proof. Since α induces an automorphism of $\mathrm{Aut}(\mathrm{AK}(S)) = \mathbb{C}[x]$ we see that $\alpha(x) = c_1 x + b_1$ where $c_1 \in \mathbb{C}^*$ and $b_1 \in \mathbb{C}$. Since α acts on $\mathrm{LND}(S)$ we see that $(c_1 x + b_1)^n \equiv 0 \pmod{x^n}$. These two observations prove that $\alpha(x) = c_1 x$. Now $\partial^2(z) = 0$ for any $\partial \in \mathrm{LND}(S)$. Therefore $\partial^2(\alpha(z)) = 0$ for any $\partial \in \mathrm{LND}(S)$ and $\alpha(z) = c_2 z + b$ where $c_2, b \in \mathbb{C}[x]$. Since α is invertible we see that $c_2 \in \mathbb{C}^*$.

Next, $\alpha(x^n y) = c_1^n x^n \alpha(y) = P(c_2 z + b) = c_2^d P(z) + \Delta(x, z)$ where $\deg_z(\Delta) < d$. So $c_1^n \alpha(y) = x^{-n}(c_2^d P(z) + \Delta(x, z)) = c_2^d y + \Delta(x, z) x^{-n}$. Since $\deg_z(\Delta) < d$ it means that $\Delta \equiv 0 \pmod{x^n}$. Since $\Delta = P(c_2 z + b) - c_2^d P(z) = d c_2^{d-1} z^{d-1} b + \delta$ where $\deg_z \delta < d - 1$ we see that $b \equiv 0 \pmod{x^n}$. Therefore $P(c_2 z + b) \equiv P(c_2 z) \pmod{x^n}$ and $\Delta \equiv P(c_2 z) - c_2^d P(z) \equiv 0 \pmod{x^n}$ which is possible only if $P(c_2 z) - c_2^d P(z) = 0$.

Now we are ready to check the following

Theorem 1. The group $\mathrm{Aut}(S)$ is generated by the following automorphisms.
(a) $H(x) = \lambda x$, $H(y) = \lambda^{-n} y$, $H(z) = z$ where $\lambda \in \mathbb{C}^*$.
(b) $T(x) = x$, $T(y) = y + [P(z + x^n f(x)) - P(z)]x^{-n}$, $T(z) = z + x^n f(x)$; where $f(x) \in \mathbb{C}[x]$.
(c) If $P(z) = z^d$ then the automorphisms $R(x) = x$, $R(y) = \lambda^d y$, $R(z) = \lambda z$ where $\lambda \in \mathbb{C}^*$ should be added.
(d) If $P(z) = z^i p(z^m)$ then the automorphisms $S(x) = x$, $S(y) = \mu^i y$, $S(z) = \mu z$ where $\mu \in \mathbb{C}$ and $\mu^m = 1$ should be added.

Proof. It is clear that all of these transformations are automorphisms. It is also clear from Lemma 12 that any automorphism is a composition of an automorphism H, an automorphism T, and an automorphism α for which $\alpha(x) = x$ and $\alpha(z) = cz$. For a general polynomial $P(c_2 z) - c_2^d P(z) \neq 0$ and α is identical automorphism. Cases (c) and (d) describe all polynomials for which $P(c_2 z) - c_2^d P(z) = 0$ is possible.

Remark. Triangular automorphisms T form a normal subgroup and the group $\mathrm{Aut}(S)$ is a semidirect product of T and L where L is a subgroup of linear automorphisms generated by automorphisms from (a), (c), and (d). Group $\mathrm{Aut}(S)$ is a metabelian group.

Isomorphisms of S.

Let S_1 and S_2 be two algebras which correspond to $Q_1 = X_1^{n_1} Y_1 - P_1(Z_1)$

and $Q_2 = X_2^{n_2}Y_2 - P_2(Z_2)$ where n_1, n_2, d_1, and d_2 are all larger then 1.

Theorem 2. $S_1 \cong S_2$ if and only if $n_1 = n_2 = n$, $d_1 = d_2 = d$, and $P_2(z) = \lambda^{-d}P_1(\lambda z)$ where $\lambda \in \mathbb{C}^*$.

Proof. Let α be an isomorphism of these algebras. We know that $\mathrm{AK}(S_i) = \mathbb{C}[x_i]$ and that $\mathrm{LND}(S_i) = x_i^{n_i}\mathbb{C}[x_i]\frac{\partial}{\partial z_i}$. So as in Lemma 12 we can conclude that $\alpha(x_1) = c_1x_2$ and $\alpha(z_1) = c_2z_2 + b(x_2)$ where $c_1, c_2 \in \mathbb{C}^*$, $b(x) \in \mathbb{C}[x]$. We may even assume using Theorem 1 that $c_1 = 1$. We may also assume without loss of generality that $d_1 \leq d_2$ because we can switch S_1 and S_2.

Let us assume that $d_1 < d_2$. Then $\alpha(y_1) = x_2^{-n_1}P_1(c_2z_2 + b) \notin S_2$ since its bottom component is a monomial $x_2^{-n_1}z_2^{d_1}$ and we showed in the proof of Lemma 10 that the bottom components of the elements from S_2 with negative power of x_2 should contain z_2 in the power d_2 at least. So $d_1 = d_2 = d$. Similarly, an assumption that $n_1 > n_2$ brings us to a contradiction. Therefore $n_1 = n_2 = n$.

Now we can see that $b(x) \equiv 0 \pmod{x^n}$, so using Theorem 1 again we may assume that $\alpha(z_1) = cz_2$. Finally, $P_1(cz) = c^dP_2(z)$ as in Lemma 12. So $P_2(z) = c^{-d}P_1(cz)$.

Locally nilpotent derivations, a new ring invariant and applications, part 2

Further properties

Let A be an algebra over $\mathbb{C}$ and let $a \in A$. Let us denote by $C(a)$ the algebraic closure of a in A, i. e., the subalgebra of all elements algebraically dependent with a over $\mathbb{C}$. Then

Lemma 13. Let A be a domain over $\mathbb{C}$ and let $a \in A \setminus \mathbb{C}$. Then $\operatorname{GK}\dim(C(a)) = 1$.

Proof. Let $x, y \in C(a)$. We want to show that they are algebraically dependent. Let us assume that x and y are algebraically independent. Then subalgebra B of A which is generated by a, x, and y contains polynomial ring $\mathbb{C}[x, y]$ and the field F of fractions of B contains the field $E = \mathbb{C}(x, y)$. Let us consider a derivation ∂ on F which is given by $\partial(x) = 1$ and $\partial(y) = 0$. It can be uniquely extended on $F = E[a]$ since a is algebraically dependent over E and if we consider corresponding irreducible polynomial $P(a, x, y) = 0$ then $\partial(P) = \partial(a)P_1 + \partial(x)P_2 + \partial(y)P_3 = 0$ and we can find $\partial(a)$. Now we know that there exist polynomials p and q in two variables over $\mathbb{C}$ such that $p(a, x) = q(a, y) = 0$ and we may assume that they are of the smallest possible degree. Then $\partial(a)p_1 + \partial(x)p_2 = 0$ and $\partial(a)q_1 + \partial(y)q_2 = 0$. From the first equality $\partial(a) \neq 0$ and from the second $\partial(a) = 0$. So we have a contradictions which shows that x and y must be algebraically dependent.

Remark. Similar consideration show that algebraic dependence between pairs of elements of a domain is a transitive relation.

Lemma 14. If R is a domain then for any $r \in R \setminus \mathrm{AK}(R)$ the ring $C(r)$ is isomorphic to a subring of a polynomial ring with one generator.

Proof. Let $\partial \in \mathrm{LND}(R)$ be a derivation which is not zero on r. Such a ∂ exists because otherwise $r \in \mathrm{AK}(R)$. As we know R^{∂} is algebraically closed in R. So $C(r) \cap R^{\partial} = \mathbb{C}$. (If $C(r) \cap R^{\partial} \ni s \notin \mathbb{C}$ then r and s are algebraically dependent and $\partial(r) = 0$.)

First we show that $C(r)$ is finitely generated. Let us denote $\deg_{\partial}$ by deg and consider the set $\deg(C(r) \setminus \mathbb{C})$. Since it is a semigroup of positive integers, this semigroup is finitely generated.

Prove it.

Let us choose a finite generating set of this semigroup and a set of elements $r_1, ..., r_m \in C(r)$ with the corresponding "generating" degrees. Let A be a subalgebra generated by these elements. Let $b \in C(r) \setminus \mathbb{C}$. By the definition of algebra A there exists an element $a \in A$ with $\deg(a) = \deg(b)$.

Let us show that there exists a $c \in \mathbb{C}$ such that $\deg(b - ca) < \deg(b)$. By Lemma 13 we can find an algebraic dependence Q between b and a over $\mathbb{C}$. Let P be the top homogeneous component of Q (relative to the degree deg). Then $P(a, b) = \prod_{i=1}^{n}(\lambda_i a - \mu_i b)$ because $\deg(a) = \deg(b)$. If $\deg(\lambda_i a - \mu_i b) = \deg(b)$ for all i then $\deg(P(a, b)) = \deg(Q(a, b)) = n \deg(b)$, and $Q(a, b) \neq 0$. So indeed $\deg(b - ca) < \deg(b)$ for some $c \in \mathbb{C}$. Now we can use induction on degree of b to observe that $C(r) = A$.

As we saw in the proof of Lemma 2 there exists a $t \in S = \mathrm{Frac}(R)$ for which $\partial(t) = 1$ and $R \subset S^{\partial}[t]$. Let us look at r_i's as polynomials in t with coefficients in S^{∂} and let us consider a subfield E of S^{∂} which is generated by the coefficients of all r_i's. Let $t_1, ..., t_k$ be a basis of transcendence of E over

$\mathbb{C}$. Then $E = \mathbb{C}(t_1, ..., t_k)[\theta]$ where θ is algebraic over $\mathbb{C}[t_1, ..., t_k]$. Clearly t is transcendental over E and $C(r) \subset E[t]$. So we have an embedding of $C(r)$ into a polynomial ring but over too big a field.

We want now to construct an embedding of $C(r)$ into $\mathbb{C}[t]$. In fact, it is sufficient to find a homomorphism φ of $C(r)$ into $\mathbb{C}[t]$ such that its image contains a polynomial of non-zero degree. Let us check that such φ has zero kernel. Indeed, let $\varphi(s) = 0$ where $s \in C(r)^*$. If $s \in \mathbb{C}^*$ then φ is identically zero. So let $s \in C(r) \setminus \mathbb{C}$. By Lemma 13 for any element $p \in C(r)$ there exists an irreducible polynomial dependence Q between s and p. Therefore $Q(s, p) = 0$ and $\varphi(Q(s, p)) = Q(\varphi(s), \varphi(p)) = Q(0, \varphi(p)) = 0$ which implies that $\varphi(p) \in \mathbb{C}$. But then $\varphi(C(r)) \subset \mathbb{C}$ contrary to our assumption.

We will find such a homomorphism with "specialization" trick. Let $P(\theta)$ be an irreducible polynomial for θ over $\mathbb{C}[t_1, ..., t_k]$. Suppose that $c_1, ..., c_k$ are some numbers in $\mathbb{C}$. Let us replace all t_i by c_i in $P(\theta)$. It may happen that all the coefficients of P with the exception of the free term will become zeros and this specialized polynomial will not define a specialization of θ. But if it not the case we can use any root λ of specialized polynomial as a specialization of θ. After that we can replace all t_i by c_i and θ by λ in all elements of $C(r)$. What may happen though is that some of these substitutions will give us zero denominators and do not define anything sensible. But if not then we will get a homomorphism of $C(r)$ in $\mathbb{C}[t]$.

So we want to find such numbers $\{c_i\}$ that none of these bad things happen and that the image of the corresponding specialization contain a non-constant monomial. Here we are going to use the fact that $C(r)$ is finitely generated as algebra. (Even countably generated algebra will be acceptable but, say, for a ring $\mathbb{C}(t_1)[t]$ it would be impossible to find a "good" specialization.)

Let us consider all the coefficients $r_{i,j}$ of all r_i as "rational functions" from E with denominators $f_{i,j} \in \mathbb{C}[t_1, ..., t_k]$. Further let $h = g[\theta]f^{-1}$ be the leading coefficient of r_1 and and let $P[\theta]$ be an irreducible polynomial for θ. Since $g \neq 0$ polynomials g and P are relatively prime over the field $\mathbb{C}(t_1, ..., t_k)$. So we can find $u, v \in \mathbb{C}[t_1, ..., t_k][\theta]$ such that $ug + vP = w$ where $w \in C[t_1, ..., t_k]^*$.

We also can find $c_1, ..., c_k \in \mathbb{C}$ such that all $f_{i,j}(c_1, ..., c_k)$ and $w(c_1, ..., c_k)$ are not equal to zero as well as, say, leading coefficient of P. It is obviously possible. (We can consider the product Π of all these polynomials and then use the fact that $\Pi = 0$ cannot be satisfied by all points in $\mathbb{C}^k$.) After that we can find specialization of θ which will satisfy the specialization of polynomial P. As the result we obtained a homomorphism φ which is defined on all $C(r)$ and for which $\varphi(h) \neq 0$. (h is the leading coefficient of r_1.) So the image of $C(r)$ in $\mathbb{C}[t]$ contains a non-constant polynomial $\varphi(r_1)$.

As we saw above this proves the lemma.

Question. Let $C(r_1, r_2)$ be the algebraic closure of a subalgebra which is generated by r_1 and r_2. Is it true that $C(r_1, r_2)$ is isomorphic to a subring of a polynomial ring with two generators if $C(r_1, r_2) \cap \mathrm{AK}(R) = \mathbb{C}$?

Small rings with small invariant

Let us talk now about "small" rings with $\mathrm{AK} = \mathbb{C}$.

Lemma 15. If $\mathrm{GK}\dim(R) = 1$ where R is a domain then either $\mathrm{AK}(R) = R$ or $\mathrm{AK}(R) = \mathbb{C}$. If $\mathrm{AK}(R) = \mathbb{C}$ then R is isomorphic to a polynomial ring $\mathbb{C}[x]$.

Proof. If $\mathrm{LND}(R) = 0$ then $\mathrm{AK}(R) = R$. Otherwise let us take a nonzero $\partial \in \mathrm{LND}(R)$. As we already saw $R \cap R^\partial = \mathbb{C}$ since R^∂ is algebraically closed in R and any two elements of R are algebraically dependent. So $R^\partial = \mathbb{C}$ and $\mathrm{AK}(R) = \mathbb{C}$. Next, $R^\partial \cap \partial(R) \neq 0$ (otherwise ∂ is not locally nilpotent on R^∂) and so there is an element $r \in R$ such that $\partial(r) = c \in \mathbb{C}^*$. Therefore $\partial(t) = 1$ for $t = c^{-1}r$.

Straightforward induction on $\deg_\partial$ like one we used in the proof of Lemma 4 shows that t generates R. So $R = \mathbb{C}[t]$ is a ring of polynomials.

Remark. Let Γ be an affine curve. (It simply means that the ring of regular functions on Γ is a domain $O(\Gamma)$ for which Gelfand-Kirillov dimension is one.) From the lemma we see that Γ is isomorphic to the line if and only if $\mathrm{AK}(O(\Gamma)) = \mathbb{C}$.

Locally nilpotent derivations, a new ring invariant and applications, part 2

Rings with small invariant

Let us look now at the case $\operatorname{GK}\dim(R) = 2$ and $\operatorname{AK}(R) = \mathbb{C}$. Here unlike one-dimensional case, it is not true that R is isomorphic to the ring $\mathbb{C}[x, y]$.

For example if R is generated by x^2, xy and y^2 then $\partial_1(h) = xJac(x, h)$ and $\partial_2(h) = yJac(y, h)$ are locally nilpotent derivations of R and $\operatorname{AK}(R) = \mathbb{C}$.

One more example. Let R_1 be generated by x^2, x^3, xy, x^2y, xy^2, y^2, and y^3. Then ∂_1 and ∂_2 are locally nilpotent derivations of R_1 as well and $\operatorname{AK}(R_1) = \mathbb{C}$.

Prove that R and R_1 are not isomorphic to $\mathbb{C}[x, y]$.

Nevertheless the following is true.

Theorem. If R is a domain with $\operatorname{GK}\dim(R) = 2$ and $\operatorname{AK}(R) = \mathbb{C}$ then R is isomorphic to a finitely generated subring S of the ring $\mathbb{C}[u, v]$ such that $S \cap \mathbb{C}[u] \neq \mathbb{C}$ and $\partial(h) = Jac(u, h)$ is a locally nilpotent derivation of S.

We will prove it in three lemmas.

Lemma 16. Let R be a domain with $\operatorname{GK}\dim(R) = 2$. If $\operatorname{AK}(R) = \mathbb{C}$ then R is isomorphic to a subring of the ring $\mathbb{C}(x)[y]$.

Proof. We know by Lemma 4 that $\operatorname{GK}\dim(R^{\partial}) = 1$ for any non-zero $\partial \in \operatorname{LND}(R)$. So there exist non-zero $\partial_1, \partial_2 \in \operatorname{LND}(R)$ such that $R^{\partial_1} \neq R^{\partial_2}$. (Otherwise $\operatorname{GK}\dim(\operatorname{AK}(R)) = 1$.)

Let $r \in (R^{\partial_1} \setminus R^{\partial_2})$. Then $\partial_2(r) \neq 0$ and $C(r)$ is a subring of a polynomial ring with one generator by Lemma 14. It is clear that $C(r) \subset R^{\partial_1}$ since R^{∂_1} is algebraically closed by Lemma 1. But in fact $R^{\partial_1} = C(r)$ since $\operatorname{GK}\dim(C(r)) = 1$. (If $s \in R^{\partial_1} \setminus C(r)$ then r and s are algebraically independent and $\operatorname{GK}\dim(R^{\partial})$ is at least two.) So R^{∂_1} is a subring of a polynomial ring with one generator. From Remark to Lemma 2 we know that $R \subset \operatorname{Frac}(R^{\partial_1})[t_1]$ where $t_1 \in \operatorname{Frac}(R)$ and $\partial_1(t_1) = 1$, and lemma is proved.

Remark. In fact we proved that if we have one locally nilpotent derivation for which the ring of constants is a subring of the field of rational functions with one variable then R is isomorphic to a subring of the ring $\mathbb{C}(x)[y]$.

It is also clear that either $R^{\partial_1} \cap R^{\partial_2} = \mathbb{C}$ or $R^{\partial_1} = R^{\partial_2}$. Indeed if $R^{\partial_1} \cap R^{\partial_2} \ni s \notin \mathbb{C}$ then $\partial_1(s) = \partial_2(s) = 0$, $\partial_1(C(s)) = \partial_2(C(s)) = 0$, and $R^{\partial_1} = R^{\partial_2} = C(s)$.

Let us prove now that

Lemma 17. Under the assumptions of Lemma 16 ring R is finitely generated.

Proof. We will use the following well known fact. If ring A has $\mathbb{Z}^+$ filtration and $\operatorname{Gr}(A)$ is finitely generated then A is finitely generated.
(Prove it)
Let ∂_1 and ∂_2 from $\operatorname{LND}(R)$ be the derivations from the proof of Lemma 16. As we know $R \subset F[t_1]$ where $F = \operatorname{Frac}(R^{\partial_1})$ and $\partial_1(t_1) = 1$. Let us consider filtration on R which is given by $\deg_{\partial_1}$ or, which is just the same, by t_1-degree of elements. Let us denote $\operatorname{Gr}(R)$ by A and let us denote by gr the natural mapping from R to A. As we know from Lemma 16 the ring R^{∂_1} is a subring

of a polynomial ring. Let us denote this polynomial ring by $\mathbb{C}[x_1]$ and let us denote $\mathrm{gr}(x_1)$ by x and $\mathrm{gr}(t_1)$ by t. Then $A \subset \mathbb{C}[x,t]$ where $\mathbb{C}[x] \supset \mathrm{gr}(R^{\partial_1})$. Indeed if $r \in R$ and $\deg_{t_1}(r) = k$ then $\partial_1^k(r) = k!r_k \in R^{\partial_1} \subset \mathbb{C}[x_1]$. So $r = r_k t_1^k + \delta$ where $r_k \in \mathbb{C}[x_1]$ and $\deg_{t_1}(\delta) < k$.

Let us denote by ∂_3 and ∂_4 locally nilpotent derivations of A which are induced by ∂_1 and ∂_2 (see Lemma 3). (Check that conditions of Lemma 3 are satisfied.) It is clear that $A^{\partial_3} = A_0 = \mathrm{gr}(R^{\partial_1})$.

Now $\partial_1(r) = Jac(x_1, r)$ and $\partial_2(r) = hJac(f, r)$ where $r \in R$, $f \in R^{\partial_2}$ and $h \in \mathbb{C}(x_1, t_1)$. (See Lemma 6. It is clear that we can extend ∂_2 to a derivation of $\mathbb{C}(x_1, t_1)$.) So $\partial_3(a) = Jac(x, a)$ and $\partial_4(a) = \mathrm{gr}(h)Jac(\mathrm{gr}(f), a)$. If $\mathrm{gr}(f) \in A_0$ then $f \in R^{\partial_1}$ and $R^{\partial_2} = R^{\partial_1}$ which contradicts our assumptions. So $\mathrm{gr}(f) = t^n g(x)$ where $n > 0$ and $\partial_4(A_0) \neq 0$. Therefore $\mathrm{AK}(A) = \mathbb{C}$. (If $Jac(t^n g(x), p(x)) = 0$ then $p(x) \in \mathbb{C}$.)

Let us consider now on A a filtration which is given by degree relative to x. Let us denote by B the corresponding graded ring and by gr_1 the natural mapping from A to B. Let ∂_5 and ∂_6 be locally nilpotent derivations of B which are induced by ∂_3 and ∂_4. Let us denote by u and v the images of x and t in B. Then $\partial_5(b) = Jac(u, b)$ and $\partial_6(b) = u^a v^b Jac(u^m v^n, b)$. We know that $\partial_5, \partial_6 \in \mathrm{LND}(B)$. We also know that $B \supset \mathrm{gr}_1(A_0)$ and $B \ni u^m v^n$ where $n > 0$ and that B has a basis (over $\mathbb{C}$) consisting of monomials.

It remains to show that B is finitely generated. Since we can identify monomials in B with integer vectors in the first quadrant we want to show that the corresponding semigroup of vectors is finitely generated.

It is not true in general that every subsemigroup of $\mathbb{Z}^+ \times \mathbb{Z}^+$ is finitely generated.

(Give an example of a subsemigroup of $\mathbb{Z}^+ \times \mathbb{Z}^+$ which is not finitely generated.)

But there is additional information about the semigroup S which corresponds to B. Namely S contains vectors $(k, 0)$ and (m, n) and $(i, j) \in S$ only if $i, j \geq 0$ (since $B \subset \mathbb{C}[u, v]$) and $in - jm \geq 0$. (Here is an explanation why $in - jm \geq 0$. Let us denote ∂_6-degree on B by deg. Then $\deg(u^k) > 0$ and $\deg(u^m v^n) = 0$. Therefore $\deg(u^i v^j) \geq 0$ only if $in - jm \geq 0$.)

Prove that a subsemigroup of $\mathbb{Z} \times \mathbb{Z}$ which satisfies these additional conditions is finitely generated.

Now we can make the next step.

Lemma 18. Under the assumptions of Lemma 16 ring R is a subring of a polynomial ring with two generators.

Proof. Let us recall that the exponent of a derivation from $\mathrm{LND}(R)$ is an automorphism of R.

Let us look at R as a subring of $R[u, v]$ where u and v are two variables. Let ∂_1 and ∂_2 be the derivations from the proof of Lemma 16. As we know $R^{\partial_1} \cap R^{\partial_2} = \mathbb{C}$.

We can extend ∂_1 and ∂_2 on $R[u, v]$ by $\partial_i(u) = \partial_i(v) = 0$ and these extended derivations are locally nilpotent on $R[u, v]$. Let us denote these derivations again by ∂_1 and ∂_2. Let us define φ on $R[u, v]$ by $\varphi(p) = \exp(u\partial_1)\exp(v\partial_2)(p)$. As we recalled φ is an automorphism of $R[u, v]$ and, of course, $\varphi(R)$ is isomorphic to R.

According to Lemmas 16 and 17 ring R is a subring of $\mathbb{C}(x_1)[t_1]$ which is finitely generated. Let $r_1, ..., r_k$ be a generating set of R.

Let us use the specialization trick again. Let us chose complex values for x_1 and t_1 in such a way that the resulting specialization ψ of $\varphi(R)$ into $\mathbb{C}[u, v]$ is an embedding.

We can assume without loss of generality that $r_1 \in (R^{\partial_1} \setminus \mathbb{C})$ and $r_2 \in (R^{\partial_2}) \setminus \mathbb{C})$. Then $\varphi(r_1) = r_1 + v\partial_2(r_1) + \delta_1$ and $\varphi(r_2) = r_2 + u\partial_1(r_2) + \delta_2$ where $\partial_2(r_1) \neq 0$, $\partial_2(r_1) \neq 0$ and δ_i consist of monomials of u and v with degree larger then one. It is clear that r_1 and r_2 are algebraically independent. (If r_1 and r_2 are algebraically dependent then $\partial_1(r_2) = \partial_2(r_1) = 0$.) Therefore $\varphi(r_1)$ and $\varphi(r_2)$ are algebraically independent.

Can you give a different proof that $\varphi(r_1)$ and $\varphi(r_2)$ are algebraically independent?

Let us chose $c_1, c_2 \in \mathbb{C}$ so that c_1 is not a root of denominators of all r_i and that $\psi(\partial_1(r_2)) \neq 0$ and $\psi(\partial_2(r_1)) \neq 0$. Let us denote by α the composition of ψ and φ. Then α is defined on all R and $\alpha(R) \subset \mathbb{C}[u, v]$. It remains to show that the kernel of α is zero.

Let us assume that $r \in R^*$ and $\alpha(r) = 0$. We know that r_1 and r_2 are algebraically independent and it is easy to show that $\alpha(r_1)$ and $\alpha(r_2)$ are algebraically independent.
Do it.
Since $\operatorname{GK}\dim(R) = 2$ there exists an irreducible algebraic dependence $Q(r, r_1, r_2) = 0$. So $0 = \alpha(Q(r, r_1, r_2)) = Q(0, \alpha(r_1), \alpha(r_2))$ and $\alpha(r_1)$ and $\alpha(r_2)$ are algebraically dependent.

We reached a contradiction which proves the lemma.

Remark. It is clear that $\varphi(r) \in R[u]$ if $r \in R^{\partial_2}$ and it is easy to verify that $\alpha(\partial_2(r)) = \frac{\partial}{\partial v}(\alpha(r))$.

So we proved the Theorem

As we saw from examples it is not true in general that R is isomorphic to the ring $\mathbb{C}[x, y]$. Nevertheless the following is true.

Lemma 19. Let R be a unique factorization domain with $\operatorname{GK}\dim(R) = 2$. If $\operatorname{AK}(R) = \mathbb{C}$ then R is isomorphic to a polynomial ring $\mathbb{C}[x, y]$.
(This lemma is similar to a theorem of Miyanishi, see J. of Algebra, 173(1995) 144-165 t. 2.6)

Proof. We'll give here a proof which does not depend on Lemmas 15 - 18. Try to give a shorter proof using these Lemmas.

Since R^∂ is factorially closed for any $\partial \in \operatorname{LND}(R)$ we see that R^∂ is also a UFD. As we already know R^∂ is isomorphic to a subring of a polynomial ring with one variable for any non-zero $\partial \in \operatorname{LND}(R)$.

A subring of a polynomial ring with one variable is UFD only if it is isomorphic to a polynomial ring. Indeed UFD is integrally closed and by Luroth theorem [VderW] we may assume that $R^\partial \subset \mathbb{C}[t]$ where $t \in \operatorname{Frac}(R^\partial)$. (Here is a proof without Luroth theorem. Let us assume that $p, q \in \mathbb{C}[x]$ are irreducible and that p has the minimal degree possible. It is easy to show that p and q are algebraically dependent, that is that $\operatorname{GK}\dim(\mathbb{C}[x]) = 1$. Let $P(p, q) = 0$ be an irreducible dependence between them. Then $P(p, q) = P(p, 0) + qQ(p, q)$. So $P(p, 0) = \prod(p - \lambda_i)$ is divisible by q. Elements $p - \lambda$ are irreducible for any $\lambda \in \mathbb{C}^*$ because otherwise we will have an irreducible element with the degree smaller then the degree of p. Since q is irreducible it implies that $q = c(p - \lambda)$ for some $c \in \mathbb{C}^*$ and $\lambda \in \mathbb{C}$. Since each element of our subring is a product of irreducible elements this subring should be $\mathbb{C}[p]$.)

Let $R^\partial = \mathbb{C}[x]$. The subring $R^\partial \cap \partial(R)$ is an ideal of R^∂ since $R^\partial \partial(R) \subset$

$\partial(R)$. Since in our case $R^\partial = \mathbb{C}[x]$ is a principal ideal domain it means that $R^\partial \cap \partial(R) = p\mathbb{C}[x]$. So there exists $y \in R$ for which $\partial(y) = p(x)$. Assume that $z \in R \setminus \mathbb{C}[x, y]$ and has the minimal ∂-degree possible, say, $k+1$. Then $\mathbb{C}[x, y] \ni \partial(z) = q_0 y^k + ... + q_k$ where $q_i \in \mathbb{C}[x]$ since $\deg_\partial(\partial(z)) = k$. Let us assume that $\deg_x(q_0)$ is also minimal possible. Since $\partial(y^{k+1}) = (k+1)py^k$ it implies that $q_0|p$, otherwise we can take a linear combination of z and y^{k+1} over $\mathbb{C}[x]$ and replace q_0 by $gkd(p, q_0)$. So $p = q_0 p_1$ and $\partial((k+1)p_1 z - y^{k+1}) = f$ where $\deg_\partial(f) < k$. Hence $\deg_\partial((k+1)p_1 z - y^{k+1}) < k+1$ and $(k+1)p_1 z = y^{k+1} + h$ where $h \in \mathbb{C}[x, y]$ and $\deg_\partial(h) < k+1$.

Let λ be a root of p_1. Then $p_1 = (x - \lambda)p_2$ and $(k+1)(x-\lambda)p_2 z = y^{k+1} + h = [y^{k+1} + h(\lambda, y)] + [h(x, y) - h(\lambda, y)] = \prod_{i=1}^{k+1}(y - \mu_i) + (x - \lambda)h_1(x, y)$. Since R is UFD this equality implies that one of the factors $y - \mu$ is divisible by $x - \lambda$. (It is clear that $x - \lambda$ is irreducible for any $\lambda \in \mathbb{C}$ since it is irreducible in $\mathbb{C}[x] = R^\partial$ and R^∂ is factorially closed.) Therefore $\frac{y-\mu}{x-\lambda} \in R$ and $\partial(\frac{y-\mu}{x-\lambda}) = \frac{p-\mu}{x-\lambda} \in R^\partial$. Since we assumed that $R^\partial = p\mathbb{C}[x]$ it is impossible. So p_1 does not have roots and is a complex number which means that $z \in \mathbb{C}[x, y]$.

Problem. Describe all two-dimensional domains R with $\mathrm{AK}(R) = \mathbb{C}$.

Let R be a three-dimensional domain given by $xv - yu = 1$. It is possible to show that it is a UFD with $\mathrm{AK}(R) = \mathbb{C}$ which is not isomorphic to $\mathbb{C}[x, y, z]$.

Problem. Find an algebraic condition such that a three-dimensional domain with $\mathrm{AK}(R) = \mathbb{C}$ which satisfies this condition is isomorphic to $\mathbb{C}[x, y, z]$.

Locally nilpotent derivations, a new ring invariant and applications, part 2

Applications to cancellation questions

Let V be an affine algebraic variety (over $\mathbb{C}$) and let $O(V)$ be its coordinate ring (ring of regular functions). In algebraic language $O(V)$ is a factor ring of a polynomial ring.

(**GC**) The general cancellation question may be formulated as follows: If $V_1 \times \mathbb{C}^k$ and $V_2 \times \mathbb{C}^k$ are isomorphic does it follow that V_1 and V_2 are isomorphic?

(**ZC**) The most famous problem of this type which is still open in general is Zariski cancellation problem in which $V_2 \simeq \mathbb{C}^n$.

For $n = 2$ there is a positive answer for (ZC) (see [Fu], [MS], and [Su]). For (GC) there is a positive answer for $n = 1$ (see [AEH]) and a negative answer for $n = 2$ (see [Ho], [Dan], [Kr]). (ZC) is open for $n > 2$.

For an algebraist these problems may be reformulated as

(**GC**) If polynomial rings $R_1[x_1, ..., x_k]$ and $R_2[x_1, ..., x_k]$ are isomorphic does it follow that $R_1 \simeq R_2$?

(**ZC**) If $R_1[x_1, ..., x_k] \simeq \mathbb{C}[x_1, ..., x_{n+k}]$ does it follow that $R_1 \simeq \mathbb{C}[x_1, ..., x_n]$?

It seems that the invariant AK is very appropriate for obtaining information in this type of question.

Let us start with the following

Lemma 20. Let R be a domain. If $\mathrm{GK}\dim(R) = 1$ and $\mathrm{Der}(R)$ form a one-dimensional free R module then $\mathrm{AK}(R[x_1, ..., x_n]) = \mathrm{AK}(R)$. If $\mathrm{AK}(R[x_1, ..., x_n]) = \mathbb{C}$ then R is isomorphic to a polynomial ring $\mathbb{C}[x]$.

Proof. Let us denote $R[x_1, ..., x_n]$ by R_n. It is clear that $\mathrm{AK}(R_n) \subset R$ since all partial derivatives belong to $\mathrm{LND}(R_n)$. Even more, $\mathrm{AK}(R_n) \subset \mathrm{AK}(R)$ because we can extend any locally nilpotent derivation $\partial \in \mathrm{LND}(R)$ to a locally nilpotent derivation of R_n by putting $\partial(x_i) = 0$ for all x_i.

If $\partial(R) = 0$ for any $\partial \in \mathrm{LND}(R_n)$ then $\mathrm{AK}(R_n) = R$. In this case $\mathrm{AK}(R) = R$ as well since $\mathrm{AK}(R) \supset \mathrm{AK}(R_n) = R$.

Otherwise let us take $\epsilon \in \mathrm{LND}(R_n)$ for which $\epsilon(R) \neq 0$. The restriction ϵ_R of ϵ on R can be written as $\epsilon_R = \sum \mathbf{x}^{\mathbf{i}} \partial_{\mathbf{i}}$ where ∂_i are derivations on R and $\mathbf{x}^{\mathbf{i}}$ are monomials in $x_1, ..., x_n$.

Since all derivations ∂_i are in a one-dimensional free module we can rewrite ϵ_R as $\epsilon_R = h\partial$ where ∂ is a basis of this module and $h \in R_n^*$. So $\epsilon(r) = h\partial(r)$ for $r \in R$. Therefore $\deg_\epsilon(r) - 1 = \deg_\epsilon(h) + \deg_\epsilon(\partial(r))$ which means that $\deg_\epsilon(\partial(r)) < \deg_\epsilon(r)$ and, since degrees of non-zero elements cannot become negative, this means that ∂ is a non-zero locally nilpotent derivation of R.

Now we can use Lemma 15. As we know from this Lemma $\mathrm{AK}(R)$ is either R or $\mathbb{C}$. So if we have a non-zero $\partial \in \mathrm{LND}(R)$ then $\mathrm{AK}(R) \neq R$ and $\mathrm{AK}(R) = \mathbb{C}$. Therefore $\mathrm{AK}(R_n) \subset \mathrm{AK}(R)$ is also $\mathbb{C}$.

We proved that $\mathrm{AK}(R_n) = \mathrm{AK}(R)$. Remaining claim that $R \simeq \mathbb{C}[x]$ when $\mathrm{AK}(R_n) = \mathrm{AK}(R) = \mathbb{C}$ was also proved in Lemma 15.

Remark. Since $\mathrm{GK}\dim(O(\Gamma)) = 1$ when Γ is a curve this lemma means that if Γ is a smooth curve then $\mathrm{AK}(\Gamma \times \mathbb{C}^n)$ completely determines Γ: if $\mathrm{AK}(\Gamma \times \mathbb{C}^n) = \mathbb{C}$ then Γ is a line; if $\mathrm{AK}(\Gamma \times \mathbb{C}^n) \neq \mathbb{C}$ then $O(\Gamma) = \mathrm{AK}(\Gamma \times \mathbb{C}^n)$. (In algebraic terms smoothness means that the $O(\Gamma)$-module of derivations of $O(\Gamma)$ is free.)

We will see later that in fact it is true for any curve.

In the next lemma we know only how to handle "one-dimensional" cylinders.

Lemma 21. Let A be a commutative domain of finite GK dimension. If $\mathrm{AK}(A) = A$ then $\mathrm{AK}(A[x]) = \mathrm{AK}(A)$.

Proof. Let us assume that ∂ is a non-zero locally nilpotent derivation of $A[x]$ which is not identically zero on A. Let $\mathrm{def}(r) = \deg_x(\partial(r)) - \deg_x(r)$. Let $m = \max(\mathrm{def}(r)|r \in A)$. Then $m < \infty$. Indeed since $\mathrm{GK}\dim(A) < \infty$ there exists a finite transcendence bases of A. Let $y_1, ..., y_k$ be such a transcendence bases. For $r \in A$ there exists an algebraic dependence between r and $y_1, ..., y_k$. Let $p(r, y_1, ..., y_n) = 0$ be such an irreducible polynomial for r. Since $\partial(r)p_0 + \sum_i \partial(y_i)p_i = 0$ where all "partial derivatives" p_i of p are elements of A and $p_0 \neq 0$ we can conclude that $\deg_x(\partial(r)) \leq \max(\deg_x(\partial(y_i)))$.

Since $\mathrm{def}(r) = \deg(r)$ for $r \in A$, it is clear that $m \geq 0$.

Let us consider a filtration which is given by x-degree and the induced locally nilpotent derivation ∂_1.

If $\mathrm{def}(x) < m$ then $\partial_1(x) = 0$ and $\partial_1(a) = \partial_2(a)x^m$ for all $a \in A$ where ∂_2 is a non-zero locally nilpotent derivation on A (contrary to our assumption).

If $\mathrm{def}(x) = n \geq m$ then $\partial_1(x) = bx^{n+1}$ where $b \in A^*$. But this is also impossible since by taking ∂_1-degree we obtain that $\deg(x) - 1 = (n + 1)\deg(x) + \deg(b)$ which does not admit non-negative ∂_1-degrees for x and b

if n is non-negative.

Therefore any locally nilpotent derivation must be identically zero on A which proves the claim.

Remark. This means that if $A_1[x] \simeq A_2[x]$ and $A_1 \not\simeq A_2$ then $\mathrm{LND}(A_i) \neq 0$.

Since the exponential of a locally nilpotent derivation gives an automorphism we see that if V_1 and V_2 are non-isomorphic algebraic varieties and $V_1 \times \mathbb{C} \simeq V_2 \times \mathbb{C}$ then V_1 and V_2 should admit non-trivial $\mathbb{C}$ actions, which is a rather strong restriction.

Today we compute the invariant AK for a class of rings. This gives a new proof of the cancellation theorem of Abhyankar-Eakin-Heinzer in the case of characteristic zero.

Notations, assumptions, and the claim

Let R be a commutative domain over the field $\mathbb{C}$. Let us also assume that $\mathrm{GK}\dim(R) = 1$, i. e., that any two elements of $R \setminus \mathbb{C}$ are algebraically dependent.

Let us denote the polynomial ring $R[x_1, ..., x_n]$ by R_n. Let $S = \mathrm{Frac}(R)$ and $S_n = \mathrm{Frac}(R_n)$.

Any $\partial \in \mathrm{Der}(R_n)$ can be uniquely extended to a derivation of S_n. Let us denote this extension also by ∂.

The following will be proved:

Theorem. $\mathrm{AK}(R_n) = \mathrm{AK}(R)$.

Proofs

Lemma 22. Either $\mathrm{AK}(R_n) = R$ or $\mathrm{AK}(R_n) = \mathbb{C}$.

Proof. We already saw that $\mathrm{AK}(R_n) \subset \mathrm{AK}(R)$. Now, we proved in Lemma 1 that R_n^∂ is algebraically closed in R_n. Since $\mathrm{GK}\dim(R) = 1$ it implies that either $R \cap R_n^\partial = \mathbb{C}$ or $R \subset R_n^\partial$. So either all $\partial \in \mathrm{LND}(R_n)$ are identically zero on R and $\mathrm{AK}(R_n) = R$ or there exists $\partial \in \mathrm{LND}(R_n)$ which is not identically zero on R and $\mathrm{AK}(R_n) = \mathbb{C}$.

Lemma 23. If $\partial \in \mathrm{LND}(R_n)$ and $\partial \neq 0$ then there exists $t \in S_n$ such that $\partial(t) = 1$ and $R_n \subset S_n^\partial[t]$.

Proof. It was proved in Lemma 2.

Lemma 24. If $r_1, r_2 \in R_n^\partial \setminus 0$ and $r_1 r_2 \in R_n^\partial$ then $r_1, r_2 \in R_n^\partial$.

Proof. We already proved it. On the other hand by Lemma 23 any element of R_n can be presented as a polynomial of t. So the lemma just states that the product of two polynomials is a constant only if both multipliers are constants.

Let us assume now that $\mathrm{AK}(R_n) = \mathbb{C}$. As we know from Lemma 14 the ring R is then a subring of a polynomial ring in one variable. By Luroth theorem [VderW] we may assume even that $R \subset \mathbb{C}[t]$ where $t \in \mathrm{Frac}(R)$. We are going to use this assumption in Lemmas 25, 26, 27 and 28.

Lemma 25. Let $\partial \in \mathrm{Der}(R)$. Let us denote the extension of ∂ to a derivation of $\mathbb{C}(t) = S$ also by ∂. Then $\partial(\mathbb{C}[t]) \subset \mathbb{C}[t]$.

Proof. Let $\partial(t) \in \mathbb{C}(t) \setminus \mathbb{C}[t]$. Then up to a change of variable we may assume that $\partial(t)$ has a pole of order $m-1$ at $t=0$. Let us present $r \in R$ as $r = r(0) + t^i s$ where $s \in \mathbb{C}[t]$ and $s(0) \neq 0$. Now, $\partial(r) = it^{i-1}\partial(t)s + t^i\partial(s) = t^{i-m}s_1$ where $s_1 \in \mathbb{C}[t]$ and $s_1(0) \neq 0$. If i is not divisible by m then several applications of ∂ to r will take r out of $\mathbb{C}[t]$ and so out of R. Since it is impossible, i must be divisible by m. But if $m > 1$ then $\mathrm{Frac}(R) \not\ni t$ contrary to our assumption on R. (R should contain a and ta for some a.)

Lemma 26. There exists an $r \in R$ such that $r\mathbb{C}[t] \subset R$.

Proof. By the assumptions on R there exist two elements $f, g \in R$ such that $g = tf$. Let $\deg_t(f) = k+1$ and let us assume that f is monic. Elements $g^i f^{k-i} = t^i f^k \in R$ for $i = 0, ..., k$. Now, if $i > k$ then $t^i = t^{i-k-1}f + r_i(t)$ with $\deg(r_i) < i$. So we can use induction on i to observe that $t^i f^k \in R$ for all i.

Lemma 27. There exists an $h \in R$ such that $h\mathbb{C}[t] \subset R$ and $\partial(h) \in h\mathbb{C}[t]$ for any $\partial \in \mathrm{Der}(R)$.

Proof. By Lemma 26 there are non-zero ideals of $\mathbb{C}[t]$ which belong to R. Let us take an ideal of $\mathbb{C}[t]$ which is maximal among the ideals belonging to R. It is uniquely defined. Let us denote by h its generator. Now if $\partial \in \mathrm{Der}(R)$ then $R \supset \partial(h\mathbb{C}[t]) = \partial(h)\mathbb{C}[t] + h\partial(\mathbb{C}[t])$. Since $\partial(\mathbb{C}[t]) \subset \mathbb{C}[t]$ by Lemma 25, it implies that the ideal $\partial(h)\mathbb{C}[t] \subset R$ and thus $\partial(h) \in h\mathbb{C}[t]$.

Lemma 28. Let R be a domain such that $\mathrm{GK}\dim(R) = 1$ and $\mathrm{AK}(R[x_1, ..., x_n]) = \mathbb{C}$. Then R is isomorphic to a polynomial ring with one generator.

Proof. As we know $R \subset \mathbb{C}[t]$ where $t \in \text{Frac}(R)$. We want to show that $t \in R$. It was shown in the Lemma 27 that there exists a non-zero $h \in R$ such that $h\mathbb{C}[t] \subset R$ and $\epsilon(h)$ is divisible by h in the ring $\mathbb{C}[t]$ for any $\epsilon \in \text{Der}(R)$. If $h \in \mathbb{C}$ then $t \in R$ and the lemma is proved. So let us assume that $h \notin \mathbb{C}$. Then $h \notin \text{AK}(R_n)$ by the assumptions of the lemma and there exists a $\partial \in \text{LND}(R_n)$ such that $\partial(h) \neq 0$.

The restriction of ∂ on R can be presented as $\sum \mathbf{m_i}\epsilon_\mathbf{i}$ where $\mathbf{m_i}$ are monomials from $\mathbb{C}[x_1, ..., x_n]$ and $\epsilon_\mathbf{i} \in \text{Der}(R)$. So $\partial(h)$ is divisible by h in the ring $\mathbb{C}[t, x_1, ..., x_n]$.

Let us denote $\mathbb{C}[t, x_1, ..., x_n]$ by A and let $S = \text{Frac}(R_n) = \text{Frac}(A)$. Let us extend ∂ on S and denote this extension again by ∂. Then $\partial(A) \subset A$ by Lemma 25 applied to $\epsilon_\mathbf{i}$ since it is sufficient to show that $\partial(t) \in A$. It is clear from Lemma 27 that $hA \subset R_n$.

Now let $k = \deg_\partial(h)$ and let $g = \partial^k(h)$. Then $g \neq 0$ and $g = hg_1 \in R_n^\partial$ where $g_1 \in A$. Therefore $g^2 = h^2g_1^2 = h(hg_1^2) \in R_n^\partial$.

Since both h and hg_1^2 are in R_n, by Lemma 24 they are in R_n^∂ and $\partial(h) = 0$. This contradiction proves the claim.

This finishes the proof of Theorem. As we already observed $\text{AK}(R_n) \subset \text{AK}(R)$ since all partial derivatives belong to $\text{LND}(R_n)$ and we can extend any locally nilpotent derivation $\partial \in \text{LND}(R)$ to a locally nilpotent derivation of R_n by putting $\partial(x_i) = 0$ for all x_i. So the only possibility for $\text{AK}(R_n) \neq \text{AK}(R)$ is $\text{AK}(R_n) = \mathbb{C}$ and $\text{AK}(R) = R$. But we showed in Lemma 28 that it is impossible.

Corollary. Let Γ_1 and Γ_2 be two curves. If $O(\Gamma_1 \times \mathbb{C}^n) \simeq O(\Gamma_2 \times \mathbb{C}^n)$ then

$O(\Gamma_1) \simeq O(\Gamma_2)$. (This is a theorem of Abhyankar-Eakin-Heinzer, see [AEH]).

Proof. By the Theorem if $\mathrm{AK}(O(\Gamma\times\mathbb{C}^n)) \neq O(\Gamma)$ then $\mathrm{AK}(O(\Gamma\times\mathbb{C}^n)) = \mathbb{C}$ and $O(\Gamma)$ is isomorphic to a polynomial ring $\mathbb{C}[t]$. In both cases we know $O(\Gamma)$.

Lemma 20 and the Theorem lead us to the following

Conjecture. $\mathrm{AK}(R_n) = \mathrm{AK}(R)$ for any domain R. Of course in this generality it is sufficient to prove that $\mathrm{AK}(R[x]) = \mathrm{AK}(R)$ just for one variable.

Any progress with this conjecture should give us results on Zariski cancellation problem!

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