

Local Solutions to the Hamilton–Jacobi–Bellman Equation in Stochastic Problems of Optimal Control

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We suggest a method for solving control problems by using linear stochastic systems with functionals quadratic in the phase variable and constraints on the absolute values of control actions. Many problems of optimal control of mechanical systems under random perturbations can be written in the form

$$\begin{aligned} dx_i &= p_i ds, \\ dp_i &= (-2\alpha_i p_i - \omega_i^2 x_i + u_i) ds + \sigma_i(s) dw + \gamma_i d\nu, \quad (1) \\ t < s \leq T, \quad x_i(t) &= x_i^0, \quad p_i(t) = p_i^0. \end{aligned}$$

Here, the $w = w_i(s)$ are independent Wiener processes; the $\nu = \nu_i$ are independent Poisson processes with mathematical expectations λ_i ; the $\sigma_i(s)$ and γ_i are the intensities of the Wiener and Poisson processes, respectively; the α_i and ω_i are the oscillation frequency and friction coefficients, respectively; and the $u = u(s)$ are measurable functions satisfying almost everywhere the constraints

$$|u_i(s)| \leq R_i, \quad R_i = \text{const} > 0, \quad i = 1, 2, \dots, n. \quad (2)$$

The control purpose is to minimize the functional

$$\begin{aligned} J_{t,x,p}(\nu) &= \mathbb{E} \left\{ \frac{a}{2} \sum_{i=1}^n [\omega_i^2 x_i^2(T) + p_i^2(T)] \right. \\ &\quad \left. + \int_t^T \sum_{i=1}^n [\omega_i^2 x_i^2(s) + p_i^2(s)] ds \right\}. \quad (3) \end{aligned}$$

Here, \mathbb{E} denotes mathematical expectation and a and b are nonnegative constants. Functional (3) is the Boltz functional for the total energy of the system. Let $v(x, p, t)$, $x = (x_1, x_2, \dots, x_n)$, $p = (p_1, p_2, \dots, p_n)$ be the greatest lower bound of functional (3) over all admissible controls satisfying constraints (2) that can be attained under initial conditions (1):

$$\begin{aligned} v(t, x, p) &= \inf_{\{u_i\}_{i=1}^n} \{J_{t,x,p}(u): |u_i| \leq R_i\}, \\ u &= (u_1, u_2, \dots, u_n). \end{aligned} \quad (4)$$

Assuming that the function $v(t, x, p)$ exists and is sufficiently smooth, we can write the Hamilton–Jacobi–Bellman (HJB) equation [1, 2]:

$$\frac{\partial v}{\partial t} + Lv + \sum_{i=1}^n \inf_{|u_i| \leq R_i} \left\{ u_i \frac{\partial v}{\partial p_i} \right\} + P_{\lambda, \gamma}(v) + f = 0. \quad (5)$$

Here, L is a second-order linear differential operator,

$$\begin{aligned} Lv &= \sum_{i=1}^n \left[p_i \frac{\partial v}{\partial x_i} - (2\alpha_i p_i + \omega_i^2 x_i) \frac{\partial v}{\partial p_i} \right] \\ &\quad + \sum_{i=1}^n \frac{\sigma_i^2(t)}{2} \frac{\partial^2 v}{\partial p_i^2}, \end{aligned} \quad (6)$$

$$f(x, p) = \sum_{i=1}^n \frac{b}{2} (\omega_i^2 x_i^2 + p_i^2). \quad (7)$$

By $P_{\lambda, \gamma}(v)$ we denote the shift operator in the variables p_i generated by the Poisson process:

$$P_{\lambda, \gamma}(v) = \sum_{i=1}^n \lambda_i [v(x_i, p_i + \gamma_i, t) - v(x_i, p_i, t)]. \quad (8)$$

The function $v(x, p, t)$ satisfies the Cauchy condition:

$$v(x, p, t = T) = \frac{a}{2} \sum_{i=1}^n (\omega_i^2 x_i^2 + p_i^2). \quad (9)$$

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Calculating the greatest lower bound over all admissible control values, we obtain the following equalities containing the absolute values of partial derivatives with respect to p_i :

$$\inf_{|u_i| \leq R_i} \left\{ u_i \frac{\partial v}{\partial p_i} \right\} = -R_i \left| \frac{\partial v}{\partial p_i} \right|. \tag{10}$$

Solving Cauchy problem (5), (9) for a quasilinear degenerate parabolic equation in the entire phase space (x, p) is a fairly difficult problem [3]. In the presence of shift, this equation becomes generally functional-differential, which, naturally, complicates the problem.

As is known, control problems of stochastic linear systems with linear quadratic regulator have exact solutions. However, the optimal controls obtained by solving these problems are often inapplicable, because the controlling functions may take infinitely large values [4].

We suggest the following approach for solving the problem stated above. At the first stage, we solve problem (5), (9) in some unbounded domain of phase variables by the method of characteristics. In what follows, we refer to such a solution as a local solution to the HJB equation and to the domain in which it exists as the exterior domain. The complement to the exterior domain D^+ in the phase space is called the interior domain and denoted by D^- . The following theorem is valid.

Theorem 1. *The function*

$$\begin{aligned} v(x, p, t) = & \frac{1}{2} \sum_{i=1}^n \left\{ \left(p_i - \frac{R_i z_i - \lambda_i \gamma_i}{\omega_i} \sin \omega_i \tau \right)^2 \right. \\ & \left. + \left(\omega_i x_i - \frac{R_i z_i - \lambda_i \gamma_i}{\omega_i} (1 - \cos \omega_i \tau) \right)^2 \right. \\ & \left. + \int_0^\tau \sigma_i^2 (T - s) ds + \lambda_i \gamma_i^2 \tau \right\}, \tag{11} \\ & \tau = T - t, \quad z_i = \operatorname{sgn}(p_i), \end{aligned}$$

is a solution to Cauchy problem (5), (9) for the Mayer functional ($b = 0$) and $\alpha_i = 0$ ($i = 1, 2, \dots, n$) in the domain

$$\begin{aligned} D_R^+ = & \bigcup_{i=1}^n D_i^+, \\ D_i^+ = & \left\{ x, p, \tau: \frac{R_i - \lambda_i \gamma_i}{\omega_i} < p_i, \right. \end{aligned}$$

$$\left. p_i < \min \left[-\frac{R_i + \lambda_i \gamma_i}{\omega_i}, -\gamma_i \right] \right\} \quad \text{if } R_i - \lambda_i \gamma_i \geq 0; \tag{12}$$

$$D_R^+ = \bigcup_{i=1}^n D_i^+,$$

$$\begin{aligned} D_i^+ = & \left\{ x, p, \tau: 0 < p_i, \quad p_i < \min \left[-\frac{R_i + \lambda_i \gamma_i}{\omega_i}, -\gamma_i \right] \right\} \\ & \text{if } R_i - \lambda_i \gamma_i < 0. \end{aligned}$$

Note that if Eq. (1) contains no Poisson process, then the domains D_i^+ are the sets $|p_i| > \frac{R_i}{\omega_i}$ for $i = 1, 2, \dots, n$. The theorem can be generalized to the case of $\alpha_i \neq 0$. We do not present the generalization, because it contains cumbersome expressions.

If the functional in the initial problem is given in the form Lagrange ($a = 0$), then the following assertion is valid.

Statement. *Let $v(x, p, t)$ be a local solution to the Mayer problem of the form (11). Then, the function*

$$U(x, p, \tau) = \int_0^\tau v(x, p, \tau - s) ds$$

is a local solution to Eq. (5) for the problem with Lagrange functional.

The form of the exterior domain \bar{D}_R^+ in which this local solution exists differs from that of the corresponding domain D_R^+ for the Mayer problem. Thus, if Eq. (1) does not involve a Poisson process, then

$$\begin{aligned} \bar{D}^+ = & \bigcup_{i=1}^n \bar{D}_i^+, \\ \bar{D}_i^+ = & \left\{ x, p, \tau: |p_i| > \frac{R_i}{\tau \omega_i} (1 - \cos \omega_i \tau) \right\}. \end{aligned}$$

Take any positive numbers $\rho_i > R_i$, where $i = 1, 2, \dots, n$. Relations (12), which determine the domains D_i^+ , imply that each of these domains in the subspace (x_i, p_i) is the exterior of a strip along the x_i axis; the size of the strip may only increase under the replacement of R_i by ρ_i . Therefore, the local solution (11) is defined in the domain D_ρ^+ obtained from D_R^+ by formally changing R_i for ρ_i , and it remains a solution to problem (5), (9) in this domain.

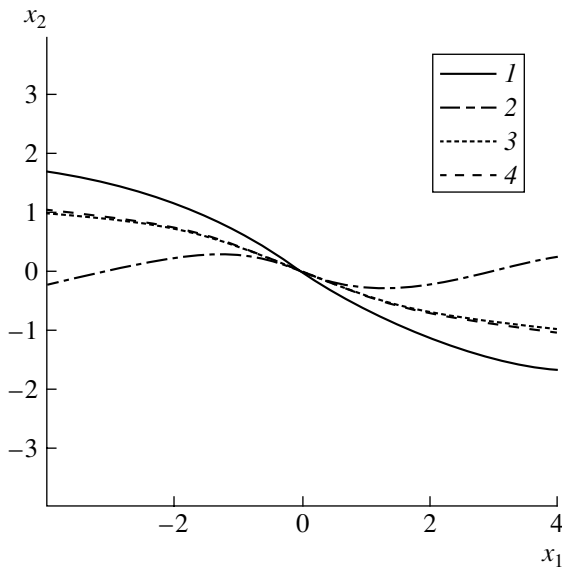


Fig. 1.

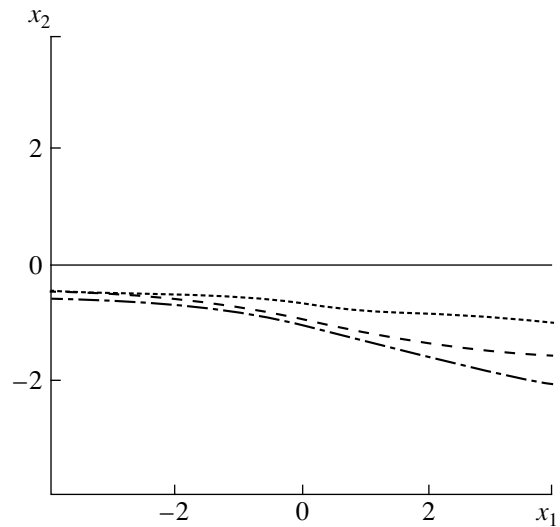


Fig. 2.

The following theorem shows that, for sufficiently large $\rho_i > 0$, the corresponding extension of a local solution from the exterior domain D_ρ^+ to the interior domain D_ρ^- is an asymptotics for the initial equation HJB (5).

Theorem 2. Let $v(x, p, t)$ (where $t = T - \tau$ for $0 < 0 \leq \tau \leq T$) be a solution to problem (5), (9) for the Mayer functional. Consider the functions

$$\begin{aligned} \tilde{z}_i(x_i, p_i, \sigma_i) = & \int_0^{\pi} \int_0^{\infty} (p_{\sigma_i}^+(\xi, r_i, \alpha, \varphi_i) \\ & - p_{\sigma_i}^-(\xi, r_i, \alpha, \varphi_i)) \xi d\xi d\alpha, \end{aligned} \tag{13}$$

where

$$\begin{aligned} & p_{\sigma_i}^\pm(\xi, r_i, \alpha, \varphi_i) \\ & = \frac{1}{4\pi\sigma_i} \exp\left(-\frac{\xi^2 - r_i^2 \mp 2r_i\xi \cos(\varphi_i - \alpha)}{4\sigma_i}\right). \end{aligned}$$

Here, $r_i = \sqrt{x_i^2 + p_i^2}$, $x_i = r_i \cos \varphi_i$, $p_i = r_i \sin \varphi_i$, and $\sigma_i > 0$. Let $\tilde{v}(x, p, t)$ denote the function obtained by replacing the functions $z_i = \text{sgn}(p_i)$ in the local solution (11) by the functions $\tilde{z}_i(x_i, p_i, \tau)$ defined by (13).

Then, there exists a set $(\sigma_1, \sigma_2, \dots, \sigma_n)$ such that, for sufficiently large ρ_i , the estimate

$$0 \leq \tilde{v}(x, p, \tau) - v(x, p, \tau) \leq K(\rho)(T - t) \tag{14}$$

holds for all $(x, p, \tau) \in D_\rho^+$. Here, $K(\rho)$ is a positive function depending on $\rho = (\rho_1, \rho_2, \dots, \rho_n)$ such that $K(\rho) \rightarrow 0$ as $\rho_i \rightarrow \infty$. If $(x, p, \tau) \in D_\rho^-$ where D_ρ^- is the interior domain complementary to D_ρ^+ , then, for fixed $\rho_i > 0$, inequality (14) with some fixed positive constant $K(\rho)$ holds.

A similar result is valid for the Lagrange problem.

Theorem 2 suggests the following approach to solving HJB equations in the entire space. On the interface Γ_ρ between the exterior and interior domains D_ρ^+ and D_ρ^- , we define a boundary condition by using the function $\tilde{v}(x, p, t)$; after this, the problem can be solved by the grid method in the domain D_ρ^- , because this domain is bounded with respect to the variables p_i . The solution to the HJB equation thus obtained is analytic on the exterior domain, numerical in the interior domain, and continuous (by construction) on Γ_ρ . However, it may have discontinuous first derivatives; in this case, the interior domain must be extended by increasing the corresponding values of ρ_i . Under this procedure, the boundary Γ_ρ moves inside the exterior domain, until the first derivatives coincide. This is possible, because, by Theorem 2, the function \tilde{v} is an asymptotics for the solution to the HJB equation for sufficiently large ρ_i . The numerical-analytic solution thus obtained makes it possible to completely synthesize an optimal control law. A similar approach (without mathematical substantiation) has been applied earlier to solve a number of problems of stochastic optimal control in [5–7].

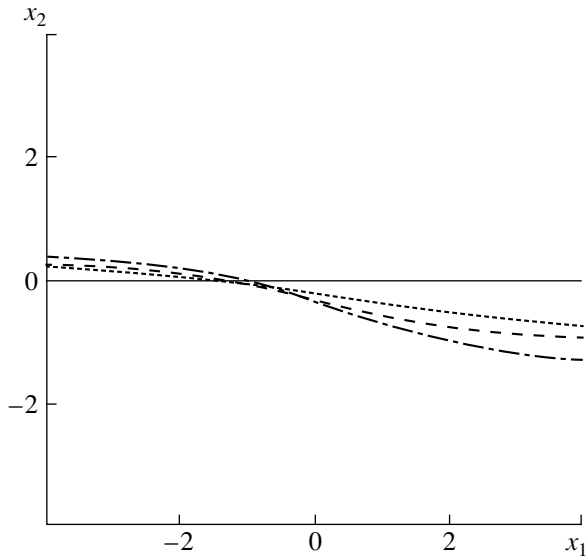


Fig. 3.

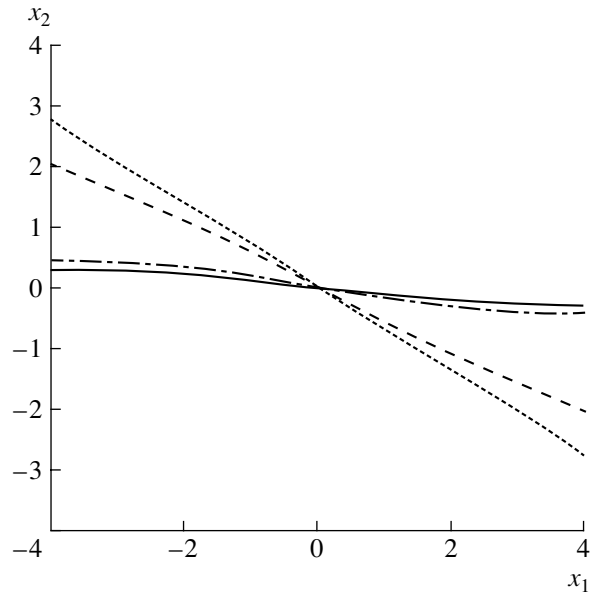


Fig. 4.

Figures 1–4 show the numerical calculation results obtained by the method described above for $n = 1$. Figure 1 presents a comparison of the switching lines for the Mayer (1, 2) and Lagrange (3, 4) problems at the moments of time $\tau = \frac{\pi}{2}$ (1, 3) and $\tau = \pi$ (2, 4). Note that the results of these calculations have been confirmed by independent numerical calculations based on the method of cell mappings performed by these authors in [8].

Figures 2 and 3 show the switching lines for a system with Gaussian and Poisson noises for $R - \lambda\gamma < 0$ and $R - \lambda\gamma > 0$, respectively (the notation in Figs. 2–4 is the same as in Fig. 1). These results were obtained for the following parameter values: $\omega = 1, \sigma^2 = 2, \gamma = 1, R = 1, \lambda = \frac{3}{2}$ (in Fig. 2), and $\lambda = \frac{1}{2}$ (in Fig. 3).

Figure 4 shows the numerical calculation results for a system without Poisson noise but with viscous friction. The numerical results obtained for $\alpha \approx \alpha_{eq} = \frac{16R^2}{(3\pi^2\sigma^2)}$ showed that viscous friction substantially

affects the position of the switching lines, while in the case of $\alpha \ll \alpha_{eq}$, it can be neglected.

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