Math 1000
Mathematical Literacy in Today’s World

Lecture 19

Winter 2010
Savings Models

- You put money in a savings account – that’s the principal amount
- You get interest at some rate added to your account periodically
Example

- At 4%, your $100 principal would earn 4% of the $100, which is (.04)*100 = $4, usually credited at the end of the year.
- This is $4 interest, computed by multiplying annual interest rate (4%) by the principal ($100)
- Then at the end of the year, your account would have $100 + $4 = $104 in it.
Then what happens?

- Let’s say you leave the money in the account for a second year.
- There are two different ways to pay interest in the second year.
- One way (simple interest) will still pay the same 4% of the principal $100, without noticing the extra $4 in your account.
Simple interest

- Simple interest will still pay the same 4% of the principal $100, without noticing the extra $4 in your account.

- So you will get \((.04) \times 100 = $4\) more at the end of the second year, increasing your account balance by $4 each year.

- Simple interest pays no interest “on interest”, only on principal.
Effect on your account

- Your $100 will grow by $4 each year
- After 5 years, you account balance will be $100 + $4 + $4 + $4 + $4 + $4 = $120
- After t years, your account balance will be $100 + 4t$
- This is a line equation, with slope $4/year$
Effect on your account

Balance

$0 $20 $40 $60 $80 $100 $120 $140 $160 $180

Balance

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
Linear growth

- Growth of the account balance like this is called **linear** growth, or **arithmetic** growth
- The balance goes up by the same amount each year
Formulas

- Usually the annual interest rate is written as \( r \)
- It’s a percent or fraction (like 4\% = .04)
- The principal amount is \( P \)
- The number of years is \( t \)
- The interest amount each year is \( P \times r \)
Formulas

- The amount of the interest on principal $P$ after $t$ years at interest rate $r$ is $I = P \times r \times t$
- And the account balance after $t$ years is $A = P + P \times r \times t = P(1 + r \times t)$
- This is the growth due to simple interest on principal $P$ during $t$ years at $r$ rate of interest
A different form of interest

- The other way to pay interest, says that after the first year you have a new principal of $104, rather than $100
- Then it pays 4% on that new principal
Compound Interest

- Compound interest says that after the first year you have a new principal of $104, rather than $100
- Then it pays 4% on that new principal 4% on $104 gives interest of $(.04)104 = $4.16 interest in the second year
- Not just the $4 of simple interest, but also interest “on the interest” from the first year
## Account Balances

<table>
<thead>
<tr>
<th>Time</th>
<th>Interest</th>
<th>Account Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beginning</td>
<td></td>
<td>$100.00</td>
</tr>
<tr>
<td>After 1 year</td>
<td>$4.00</td>
<td>$104.00</td>
</tr>
<tr>
<td>After 2 years</td>
<td>$4.16</td>
<td>$108.16</td>
</tr>
<tr>
<td>After 3 years</td>
<td>$4.33</td>
<td>$112.49</td>
</tr>
<tr>
<td>After 4 years</td>
<td>$4.50</td>
<td>$116.99</td>
</tr>
<tr>
<td>After 5 years</td>
<td>$4.68</td>
<td>$121.67</td>
</tr>
</tbody>
</table>
Language for this scenario

- This setup is called paying interest at 4% compounded annually.
- It also described as “paid annually”, since your balance in increased (paid) each year.
Formulas

- It’s best to keep track of the balance, instead of the interest
- At the beginning your balance is \( P \)
- After 1 year, your interest is \( P \times r \) and so your balance is \( P + P\times r = P(1 + r) \)
- Here we factor out the \( P \) from both terms
Compound interest multiplies

- Simple interest **adds** the same amount each year to your account
- Compound interest **multiplies** your account by the same factor each year
- The factor is 1+r, the “1” keeps the original principal, while the “r” adds the year’s interest
- New principal = old principal + interest
Effect on your account

- Simple
- Compound

Chart showing the effect on your account over time, with the y-axis representing dollars ranging from $0 to $200, and the x-axis representing time from 1 to 16.
Compound interest not linear growth

- The graph is a curved line, bending upward
- Account balances are accelerating (rate of growth is increasing, as larger and larger amounts of interest are added)
- This is called exponential growth, from the formula that we will see
Exponential growth

- After $t$ years at interest rate $r$ compounded annually on principal $P$, the account balance is $P(1+r)^t$

- The account balance $P$ has been multiplied by the increase factor $(1+r)$, $t$ times

- So the $t$ is the exponent used to computed the balance

- This is also called geometric growth, as compared with arithmetic growth (simple interest)
Example

- Suppose you put $250 into a savings account earning 3% per year?
- How much will be in the account after 10 years:
  - At simple interest?
  - At interest compounded annually?
Simple interest

- 3% of $250 is $250(0.03) = $7.50
- So the account will earn $7.50 each year
- After 10 years it will have earned $7.50*10 = $75
- So the account balance will be $250 + $75 = $325
- Or use \( P(1+rt) = 250(1 + .03*10) = 250(1.3) = $325 \)
Compound interest

- Here we multiply the account by \((1+.03) = 1.03\) every year (compounded annually).
- After 10 years the account will have \(250(1.03)^{10}\) in it.
- For this we need the \(y^x\) key on the calculator.
- First \((1.03)^{10} = 1.344\).
- Then \(250*1.344 = $335.98\) after 10 years.
- Formula \(P(1+r)^t = 250(1.03)^{10} = $335.98\).
Compounding more often

- Banks can compound the interest more often than annually
- Let’s use 12% interest rate for this example
- Compounding annually, after 1 year your balance is $100(1+.12) = $112
- The words are “12% compounded annually gives an Annual Percentage Yield (APY) of 12%”
Let’s use 12% interest rate for this example
Try compounding twice a year
But not 12% both times but half of it (6%) every six months
Start with $100, after six months you have $100(1.06) = $106
After the next six months, you have $106(1.06) = $112.36 = 100(1.06)^2
Compounding twice a year

- At 6% every six months, after 1 year you have $112.36 = 100(1.06)^2$
- As compared to compounding annually, for which after 1 year your balance would have been $100(1+.12) = $112$
- The extra 36¢ is interest on the interest from the first six months
Compounding twice a year

- So compounding twice a year results in multiplying your account balance by \((1.06)^2 = 1.1236\)
- Just like getting 12.36% interest for the whole year (multiplying by \((1+.1236)\)), the “effective rate”
- The words are “12% compounded twice a year (semiannually) gives an Annual Percentage Yield (APY) of 12.36%”
Compounding four times a year

- Nominal (annual) rate still 12%, compounded quarterly
- You get $12/4 = 3\%$ per quarter
- After 1 quarter, your account is $100 \times 1.03 = $103$
- After 2 quarters, your account is $103 \times 1.03 = 106.09 = 100(1.03)^2$
- After 3 quarters, your account is $100(1.03)^3 = 109.27$
Compounding four times a year

- After 4 quarters (1 year), your account is
  \[ 100(1.03)^4 = $112.55 \]
- So for the whole year your account has been multiplied by \( (1.03)^4 = 1.1255 = 1 + .1255 \)
- “12% compounded four times a year (quarterly) gives an Annual Percentage Yield (APY) of 12.55%”
Compounding monthly

- Nominal (annual) rate still 12%, compounded monthly
- You get 12/12 = 1% per month
- After 1 month, your account is 100*1.01 = $101
- After 2 months, your account is 101*1.01 = $102.01 = 100(1.01)^2
- After 3 months, your account is 100(1.01)^3 = $103.03
Compounding monthly

- After 12 months (1 year), your account is $100 \times (1.01)^{12} = $112.68$

- So for the whole year your account has been multiplied by $(1.01)^{12} = 1.1268 = 1 + 0.1268$

- “12% compounded twelve times a year (monthly) gives an Annual Percentage Yield (APY) of 12.68%”
Compounding daily

- Nominal (annual) rate still 12%, compounded daily
- You get \( \frac{12}{365} = 0.03288\% \) per day
- After 1 day, your account is \( 100 \times (1.0003288) \)
- After 2 days, your account is \( 100 \times (1.0003288)^2 \)
- After 3 days, your account is \( 100 \times (1.0003288)^3 \)
Compounding daily

- After 365 days (1 year), your account is
  \[100(1.0003288)^{365} = \$112.75\]
- So for the whole year your account has been multiplied by \((1.0003288)^{365} = 1.1275 = 1 + .1275\)
- “12% compounded daily gives an Annual Percentage Yield (APY) of 12.75%”
# APY at 12% nominal (annual) rate

<table>
<thead>
<tr>
<th>Compounded</th>
<th>Rate per period</th>
<th>APY for year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually</td>
<td>12%</td>
<td>12%</td>
</tr>
<tr>
<td>Semiannually</td>
<td>6%</td>
<td>12.36%</td>
</tr>
<tr>
<td>Quarterly</td>
<td>3%</td>
<td>12.55%</td>
</tr>
<tr>
<td>Monthly</td>
<td>1%</td>
<td>12.68%</td>
</tr>
<tr>
<td>Daily</td>
<td>12%/365</td>
<td>12.75%</td>
</tr>
<tr>
<td>m times per year</td>
<td>12/m %</td>
<td>((1 + \frac{.12}{m})^m - 1)</td>
</tr>
</tbody>
</table>
To get balance after 1 year at a nominal (annual) rate

<table>
<thead>
<tr>
<th>Compounded</th>
<th>Rate per period</th>
<th>Multiply by</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually</td>
<td>r</td>
<td>$1 + r$</td>
</tr>
<tr>
<td>Semiannually</td>
<td>r/2</td>
<td>$(1 + r/2)^2$</td>
</tr>
<tr>
<td>Quarterly</td>
<td>r/4</td>
<td>$(1 + r/4)^4$</td>
</tr>
<tr>
<td>Monthly</td>
<td>r/12</td>
<td>$(1 + r/12)^{12}$</td>
</tr>
<tr>
<td>Daily</td>
<td>r/365</td>
<td>$(1 + r/365)^{365}$</td>
</tr>
<tr>
<td>m times per year</td>
<td>r/m</td>
<td>$(1 + r/m)^{m}$</td>
</tr>
</tbody>
</table>
Example

- Start with $100, in an account earning 4% annual interest, compounded quarterly
- What will the balance be in 5 years?
- Each quarter the account will be multiplied by $1 + 4%/4 = 1 + .01 = 1.01$
- There will be 5*4 quarters in 5 years, so the balance will be multiplied by $(1.01)^{20} = 1.22$
- The balance after 5 years will be $100*1.22 = $122
Or by Formula

- Start with $100, in an account earning 4% annual interest, compounded quarterly.
- What will the balance be in 5 years?
- \[ A = P \times (1 + \frac{r}{m})^{mt} \] with \( P = $100 \), \( r = 0.04 \), \( m = 4 \), \( t = 5 \)
- So \[ A = 100 \times (1.01)^{20} = 100 \times 1.22 = $122 \]
- The balance after 5 years will be $122
Check your answer!

- Start with $100, in an account earning 4% annual interest, compounded quarterly
- What will the balance be in 5 years if interest was simple?
  - You would get 4% of $100 ( = $4) each year, for total interest of $20
  - Your balance would then be $120
  - Only a little less than the $122 we got compounding
Reversing the formula

- Suppose we started with $100, and after 6 years, the account balance had become $200.
- What sort of interest rate is this?
- Maybe it is $r$, an unknown annual rate.
- If so, we have multiplied by $1+r$ six times, so that $100 \times (1+r)^6 = 200$.
- Dividing by 100 gives $(1+r)^6 = 2$.
- So multiplying by $1+r$ six times resulted in multiplying by 2.
Reversing the formula

- From \((1+r)^6 = 2\)
- Taking sixth root gives \(1+r = 2^{(1/6)} = 1.12246\)
- Or raise both sides to \(1/6\) power, getting
  \[ ((1+r)^6)^{(1/6)} = (1+r)^{(6*1/6)} = (1+r)^1 = 1+r = 2^{(1/6)} = 1.12246 \]
- Here again we use the \(y^x\) key on the calculator
- So \(r = 0.12246\), or \(12.246\%\) annual percentage yield
Reversing the formula

- From $A = P*(1+r)^t$, we can solve
- First $A/P = (1+r)^t$
- The take $t$-th root, $(A/P)^{(1/t)} = 1+r$
- So $r = (A/P)^{(1/t)} - 1$
- In our previous example $A = $200, $P = $100, and $t = 6$
- And $r = (200/100)^{(1/6)} - 1 = 2^{(1/6)} - 1 = 1.12246 - 1 = .12246 \approx 12.246\%$
The End for Today
<table>
<thead>
<tr>
<th>Frequency</th>
<th>Rate per period</th>
<th>APY for year</th>
</tr>
</thead>
<tbody>
<tr>
<td>n times per year</td>
<td>12/n %</td>
<td>$(1+.12/n)^n - 1$</td>
</tr>
<tr>
<td>Daily</td>
<td>12%/365</td>
<td>12.7475%</td>
</tr>
<tr>
<td>Hourly</td>
<td>12%/(365*24)</td>
<td>12.7496%</td>
</tr>
<tr>
<td>Every Minute</td>
<td>12%/(365<em>24</em>60)</td>
<td>12.7497%</td>
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<td>12.7497%</td>
</tr>
<tr>
<td>Continuously</td>
<td></td>
<td>12.7497%</td>
</tr>
</tbody>
</table>

\[(1+.12/n)^n - 1\]

= \[e^{0.12}\]
What is e?