Math 1000
Mathematical Literacy in Today’s World

Lecture 20

Winter 2010
Interest per year

- With Principal P at rate r
- Simple interest adds $P^*r$ each year
- Compound interest multiplies by $1+r$ each year
Interest per compounding period

- With Principal P at rate r, m times each year
- Simple interest adds P*(r/m) each period
- Compound interest multiplies by 1+(r/m) each period
- So times (1+(r/m))^m each year
APY at 12% nominal (annual) rate

<table>
<thead>
<tr>
<th>Compounded</th>
<th>Rate per period</th>
<th>APY for year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually</td>
<td>12%</td>
<td>12%</td>
</tr>
<tr>
<td>Semiannually</td>
<td>6%</td>
<td>12.36%</td>
</tr>
<tr>
<td>Quarterly</td>
<td>3%</td>
<td>12.55%</td>
</tr>
<tr>
<td>Monthly</td>
<td>1%</td>
<td>12.68%</td>
</tr>
<tr>
<td>Daily</td>
<td>12%/365</td>
<td>12.75%</td>
</tr>
<tr>
<td>m times per year</td>
<td>12/m %</td>
<td>$(1 + .12/m)^m - 1$</td>
</tr>
</tbody>
</table>
To get balance after 1 year at a nominal (annual) rate

<table>
<thead>
<tr>
<th>Compounded</th>
<th>Rate per period</th>
<th>Multiply by</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually</td>
<td>$r$</td>
<td>$1 + r$</td>
</tr>
<tr>
<td>Semiannually</td>
<td>$r/2$</td>
<td>$(1 + r/2)^2$</td>
</tr>
<tr>
<td>Quarterly</td>
<td>$r/4$</td>
<td>$(1 + r/4)^4$</td>
</tr>
<tr>
<td>Monthly</td>
<td>$r/12$</td>
<td>$(1 + r/12)^{12}$</td>
</tr>
<tr>
<td>Daily</td>
<td>$r/365$</td>
<td>$(1 + r/365)^{365}$</td>
</tr>
<tr>
<td>m times per year</td>
<td>$r/m$</td>
<td>$(1+r/m)^m$</td>
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</table>
## APY at 12% nominal (annual) rate

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Rate per period</th>
<th>APY for year</th>
</tr>
</thead>
<tbody>
<tr>
<td>n times per year</td>
<td>12/n %</td>
<td>(1+.12/n)^n - 1</td>
</tr>
<tr>
<td>Daily</td>
<td>12%/365</td>
<td>12.7475%</td>
</tr>
<tr>
<td>Hourly</td>
<td>12%/(365*24)</td>
<td>12.7496%</td>
</tr>
<tr>
<td>Every Minute</td>
<td>12%/(365<em>24</em>60)</td>
<td>12.7497%</td>
</tr>
</tbody>
</table>
### APY at 12% nominal (annual) rate

<table>
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<tr>
<th>n times per year</th>
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</tr>
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<tr>
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<td>12/n %</td>
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<td>12.7497%</td>
</tr>
<tr>
<td>Continuously</td>
<td></td>
<td>12.7497%</td>
</tr>
</tbody>
</table>

\[= e^{0.12}\]
What is e?

- A messy (irrational) number, value approximately 2.71828
- On your calculator, labeled $e^x$
- Or maybe $(2^{nd} \text{ function}) \ln x$
- Or even $y^x$ key with $y = e = 2.71828$
What is $e^x$?

- $e$ raised to the $x$ power
- So $e^1 = e = 2.71828$
- Then for $r$ a nominal annual interest rate, $e^r$ is the continuous compounding factor which multiplies the account balance each year
- So formula is $A = Pe^{rt} = P(e^r)^t$ after $t$ years
Example

If your interest rate is 2% compounded continuously, say for 10 years on a principal of $100

The $P = 100$, $r = 0.02$, $t = 10$

So after ten years the balance is $A = Pe^{rt}$

$= 100e^{0.02\cdot 10} = 100e^{0.2} = 100\cdot 1.2214 = $122.14$
Again compounded daily

- If your interest rate is 2% compounded daily, say for 10 years on a principal of $100

  The $P = \$100$, $r = .02$, $m = 365$, $t = 10$

- So after ten years the balance is $A = P(1+.02/365)^{(365*10)} = 100*(1.0000548)^{3650} = 100*1.221396 = \$122.14$ when rounded to cents
So compounded daily is not that different

- Banks usually use a daily rate, but multiply by $e^{(r/365)}$, since $r/365$ is the daily interest rate.
- After 365 days, they have multiplied by $(e^{(r/365)})^{365} = e^r$, the factor to multiply by for annual interest.
Reversing the formula

- From $A = P \times e^{rt}$, can we get back the $rt$ (the exponent) from the $A$ and $P$?
- As before, we find the factor $A/P = e^{rt}$
- The we can get back the $rt$ by using the In key on the calculator
- So $rt = \ln(A/P)$
Reversing the formula

Suppose compounding continuously for 5 years increases the account balance from $100 to $200 – what is the interest rate?

So \( 200 = 100 \times e^{5r} \)

So \( 2 = e^{5r} \)

So \( 5r = \ln(2) = 0.69 \) (approx.)

So \( r = \frac{\ln(2)}{5} = 0.138 \), or 13.8%
Now a savings program

- If you add some amount to your account each month, your principal will increase both because of interest and because of the extra principal.

- But interest will be paid for different lengths of time on your added principal amounts.
Now a savings program

- Say you get 6% interest compounded monthly
- And you put in $50 per month, at the end of each month
- Now 6% gives a monthly rate of \( \frac{6}{12} = \frac{1}{2}\% \) so we multiply each month by \( 1 + \frac{r}{12} \) = 1.005
Adding $50 each month at 6%

- After 1 month your account is $50
- After 2 months, your account is $50(1.005) + 50 = $100.25
- After 3 months, your account is $50(1.005)^2 + 50(1.005) + 50 = $150.75
- Each month we multiply the whole account by 1.005 and then add $50
Adding $50 each month at 6% 

- After 12 months your account will be 
  \[50(1.005)^{11} + 50(1.005)^{10} + 50(1.005)^{9} + 50(1.005)^{8} + 50(1.005)^{7} + 50(1.005)^{6} + 50(1.005)^{5} + 50(1.005)^{4} + 50(1.005)^{3} + 50(1.005)^{2} + 50(1.005)^{1} + 50,\] lots of work 

- Can we make this better?
A better formula?

In $50(1.005)^{11} + 50(1.005)^{10} + 50(1.005)^{9} + 50(1.005)^{8} + 50(1.005)^{7} + 50(1.005)^{6} + 50(1.005)^{5} + 50(1.005)^{4} + 50(1.005)^{3} + 50(1.005)^{2} + 50(1.005)^{1} + 50$, we can first factor out the 50
A better formula?

- Now $50[(1.005)^{11} + (1.005)^{10} + (1.005)^9 + (1.005)^8 + (1.005)^7 + (1.005)^6 + (1.005)^5 + (1.005)^4 + (1.005)^3 + (1.005)^2 + (1.005)^1 + 1]

- The form of this is $x^{11}+x^{10}+\ldots+x^1+x^0$ (where $x$ is the factor $1.005$), and the “…” represents the left-out terms.
A helpful trick

- Let’s see this trick for $x^3 + x^2 + x + 1$
- Call this formula $y = x^3 + x^2 + x + 1$
- Then $x*y = x^4 + x^3 + x^2 + x$
- Subtracting, $x*y - y = x^4 - 1$
- But $x*y - y = (x-1)*y = x^4 - 1$
- So divide to get $y = (x^4 - 1)/(x - 1)$
The geometric series trick

- Similarly, \( x^{(n-1)} + x^{(n-2)} + \ldots + x + 1 = \frac{x^n - 1}{x - 1} \)
- For our monthly savings plan, \( x = 1.005 \) and \( n = 12 \)
- So \( 50[(1.005)^{11} + (1.005)^{10} + \ldots (1.005)^2 + (1.005)^1 + 1] = 50*\frac{(1.005)^{12} - 1}{1.005 - 1} \)
The geometric series trick

- Account after this 1 year program is 
  \[ 50\times(1.005)^{12} - 1)/(1.005 - 1) = 
  50\times(1.06168 - 1)/(.005) = 50\times(.06168)/.005 
  = $616.80 \]

- This is 12 payments of $50, plus some interest
To approximate this answer

- If the $600 had been in all year, the ending account balance would have earned $600(1.005)^{12} = $637.01
- One approximation is to pretend that half of the $600 was on deposit for the whole year, and the other half put in at the end, so that $300(1.005)^{12} + 300 = $618.50
Formula for exact answer

- If an amount $d$ is deposited at the end of each compounding period with interest rate $r/m$ each period, then after $mt$ periods the account balance will be
  \[ A = d \times \frac{(1+r/m)^{(mt)} - 1}{(r/m)} \]

- Here $r$ is nominal annual interest rate, $t$ is number of years, and $m$ is number of compounding periods per year
Formula for exact answer

- If an amount $d$ is deposited at the end of each compounding period with interest rate $i$ each period, then after $n$ periods the account balance will be:
  \[ A = d \times \frac{(1+i)^n - 1}{i} \]

- Same formula with $i = \frac{r}{m}$ and $n = mt$
One more time

Say you deposit $10 per quarter into an account earning 4% compounded quarterly, for 10 years

\[ A = d\left(\frac{(1+r/m)^{(mt)} - 1}{r/m}\right) \]

So \( d = 10, \ r = .04, \ m = 4, \ t = 10 \)

\[ A = 10\left(\frac{(1.01)^{40} - 1}{.01}\right) = 10\left(1.48886 - 1\right)/.01 = $488.86 \]
Solving for d

- \[ A = d \times \left( \frac{(1+r/m)^{(mt)} - 1}{r/m} \right) = d \times \left( \frac{(1+i)^{n} - 1}{i} \right) \]

- So we can solve for d and get
  \[ d = \frac{A \times i}{(1+i)^{n} - 1} = \frac{A \times \left( \frac{r}{m} \right)}{(1+\frac{r}{m})^{mt} - 1} \]
Meaning for d

- So we can solve for d and get:
  \[ d = \frac{A \times i}{((1+i)^n - 1)} = \frac{A \times (r/m)/((1+r/m)^{mt} - 1)}{}} \]

- Interpretation: how much ( = d ) do we have to put in each period to get the account balance up to A by the end of n periods?
Example

- Jim wants to retire with $1,000,000 in his retirement account. He expects to work for 30 years and put money in his retirement account each month earning 5% per year.
- How much should he put in each month?
- Since interest is not compounded monthly, work with yearly contributions.
Example

- Jim wants to retire with $1,000,000 in his retirement account. He expects to work for 30 years and put money in his retirement account each month earning 5% per year.

- So $A = 1,000,000$, and $m = 1$ and $t = 30$ and $r = .05$ (or $i = .05$ and $n = 30$).
Example

- So $A = 1,000,000$, and $m = 1$ and $t = 30$ and $r = .05$ (or $i = .05$ and $n = 30$)

- $d = A(.05)/(1.05^{30}-1) = 1000000(.05)/(4.322-1) = 50000/3.322 = 15051$ per year

- So Jim must put in $15,051$ per year, or $1254.28$ per month
Change to monthly compounding

- So $A = 1,000,000$, and $m = 12$ and $t = 30$ and $r = .05$ (or $i = .05/12$ and $n = 360$)

- $d = A(.05/12)/(1.00417)^{360}-1 = 1000000(.00417)/(4.4675-1) = 4167/3.4675 = 1201.60$ per month

- A little less, compared with annual compounding’s $1254.28$ per month
Sinking Fund

- Just means you sink the same amount of money into the fund each period in order to accumulate a certain amount by a certain date
- Just like Jim’s example
Present Value

- If you had $100 now, it would become worth more in 10 years because you could get interest between now and then.
- Similarly, if you imagine $100 in 10 years, it is not worth as much now because it would take less money now to earn interest bringing it up to the $100 later.
Example of Present Value

- If you had $100 now, it is a principal $P$ and would become $A = P(1+r/m)^{(10m)}$ after 10 years at annual rate $r$ compounded $m$ times a year.

- Similarly, $100$ in 10 years is the $A$, and the corresponding present value $P$ can be solved for, $P = A/(1+r/m)^{(10m)}$. 
Example of Present Value

- If you want $100 in 10 years, how much should you put in now?
- Say interest is 3% compounded continuously (factor is \( e^{(.03)^{10}} \)), so that \( A = Pe^{(rt)} = Pe^{.3} \)
- Then you need to put in \( P = A/e^{(.3)} = 100/1.3499= \$74.08 \)
Example of Present Value

- This $74.08 is the “Present Value” of $100 in 10 years at 3%.
- The $100 is the “Future Value” of $74.08.
- In general, money later is worth less than money now.
Inflation

- What does inflation do to Present Value?

- The value of $100 now is less later; it can buy less because prices are higher

- If inflation is 3% per year, then an item which costs $1.00 now will cost $1.03 in one year

- The future cost is 1(1.03), usual formula
Inflation

- If 3% inflation continues, after the second year the item will cost $1.03(1.03) = 1(1.03)^2
- The same formula applies, future cost after n years = (present cost)(1.03)^n
Inflation

- If 3% inflation continues, we think that dollars now are worth more than dollars later, according to the $P = A/(1+i)^n$ formula – BUT here the $i$ is rate of inflation not interest

- Instead of $i$, most use $a$ for the rate of inflation (still $i$ for interest)
Inflation

- If 3% inflation continues, then the value of 1 dollar one year from now is $1/(1.03) = 0.9709$

- At rate $a$ of inflation, the value of 1 dollar in 1 year has become $1/(1+a)$

- After $n$ years, the value of 1 dollar is $1/(1+a)^n$
Inflation

- So we see that inflation reduces the value of a dollar by multiplying it by $1/(1+a)$, which is less than 1.
- After $n$ years, the value of 1 dollar has been multiplied by $1/(1+a)^n$ times, so multiplied by $1/(1+a)^n$. 
Depreciation

- In depreciation the value of some asset reduces over time, because it gets old or obsolete.
- If it reduces by 10% per year, we can use the same method as for inflation, and say that each year the value is multiplied by $\frac{1}{1 + 0.10} = \frac{1}{1.1}$, or divided by 1.1.
How to overcome inflation?

- Inflation makes dollars worth less but interest makes accounts increase.
- If there is inflation of 3% per year, but you get 3% interest, these overcome each other.
- Your dollars are worth less but you have more of them.
How to overcome inflation?

- So $A = P(1+i)^n/(1+a)^n$

- Here we increase the account balance with interest rate $i$, but decrease the purchasing power by inflation rate $a$

- Your dollars are worth less but you have more of them
How to overcome inflation?

- So \( A = P(1+i)^n/(1+a)^n \)
- If \( i \) is bigger than \( a \), then your purchasing power is growing (a positive “real” rate of growth)
- Otherwise shrinking (a negative “real” rate of growth)
The End for Today