Math 1000
Mathematical Literacy in Today’s World

Lecture 22

Winter 2010
Interest per year

- With Principal P at rate r
- Simple interest adds $P \times r$ each year
- Compound interest multiplies by $1+r$ each year
What about borrowing?

- Suppose you borrow $1000, to be paid back in 10 years at 5% simple interest.
- This is like a savings account for the loaner.
- In fact, the loaner can imagine that you are a bank and he has a savings account with you.
Borrowing reverses saving

- Suppose you borrow $1000, to be paid back in 10 years at 5% simple interest.
- That means $5\%(1000) = $50 per year.
- Total interest would then be $10($50) = $500, so you would need to pay back $1000 + $500 = $1500.
- Formula $A = P(1+rt) = 1000(1+.05*10)$.
Borrowing at compound interest

- Suppose you borrow $1000, to be paid back in 10 years at 5% interest compounded annually.
- The loaned principal is $P = 1000$.
- Amount needing to be repaid in 10 years is $P(1+r)^{10} = 1000(1.05)^{10} = $1628.89$. 
Borrowing at compound interest

- Suppose you borrow $1000, to be paid back in 10 years at 5% interest compounded monthly
- Now $m = 12$ so $i = .05/12 = .004167$
- Amount needing to be repaid in 10 years is $P(1+.004167)^{(10*12)} = 1000(1.004167)^{120} = $1646.98$
Credit Card Interest

- Often 1.5% per month
- A nominal rate (a rate for specified period)
- The compounded rate for one year would be \((1+.015)^{12} = 1.1956 = 1 + .1956\)
- Equivalent to 19.56% for the year
- “Effective Annual Rate” (computed like APY)
EAR versus APR

- Effective annual rate is, like APY, the annual rate which gives the same interest for one year as the nominal rate.
- APR (Annual Percentage Rate) is simply the nominal rate times the number of compounding periods per year (our m).
- APR for credit cards is $1.5\% \times 12 = 18\%$. 
Suppose the nominal rate is $i$, per period, with $m =$ number of periods per year.

- The APR = $m \times i$
- But EAR = $(1+i)^m - 1$ (like APY)
Discounted loans

- You may wish to borrow and intend to pay back $1000 in 10 years at 5%
- The amount you receive (as the loan) is a smaller amount (the difference being the interest)
- How much less?
Discounted loans

- You may wish to borrow and intend to pay back $1000 in 10 years at 5%
- So $A = 1000$ and $P$ is unknown
- $1000 = P(1.05)^{10}$
- So $P = \frac{1000}{(1.05)^{10}} = \frac{1000}{1.62889} = $613.91 (discounted loan proceeds)
Formulas

- For Borrowed principal $P$ at start, annual interest rate $r$, for $t$ years
  
  - At simple interest your balance Owed is
    \[ A = P + Prt = P(1+r*t) \]
  
  - At compound interest your balance Owed is
    \[ A = P(1+r)^t \]
Compounding more often than annually

- At rate r compounded m times per year
- The rate is $i = \frac{r}{m}$ per period
- So the Owed balance is multiplied by $1 + \frac{r}{m} = 1 + i$ each period
- At the end (t years) you will owe $A = P(1+\frac{r}{m})^{mt}$
- For discounted loans $P = \frac{A}{(1+\frac{r}{m})^{mt}}$
Stafford Student Loans

- Unsubsidized Stafford loans carry 6.8% simple interest during the "grace period" (until six months after you finish)
- You can pay back just the interest during the grace period
- And then pay back all the principal immediately thereafter
Or you can delay paying the interest
When your grace period ends (or you start paying back), the interest amount is “capitalized”, or added to principal
In this case there is a compounding effect, since interest will now be due on interest (so not quite simple interest in this case)
Conventional loans

- The usual sort of loan is paid back not at the end of the loan period, but in equal periodic installments.
- These payments are called amortizing the loan, paying it back over time.
- The interest is computed on the amount of principal remaining, which is less, after payments.
Conventional loans

- As you make more and more payments, the principal decreases, so the interest amount decrease each period, so more of the principal gets paid off
- But you start paying very little of the principal (and then accelerate)
Conventional loans

- Imagine taking a $100,000 loan at 6%
- Monthly payments means \( \frac{1}{2}\% \) per month interest
- Old fashioned method was to compute an “amortization schedule” to see how the payments go
## Amortization Schedule

<table>
<thead>
<tr>
<th>End of Month</th>
<th>Remaining Principal</th>
<th>Interest - ½% per month</th>
<th>Payment</th>
<th>Amount to principal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100,000.00</td>
<td>500.00</td>
<td>600.00</td>
<td>100.00</td>
</tr>
<tr>
<td>2</td>
<td>99,900.00</td>
<td>499.50</td>
<td>600.00</td>
<td>100.50</td>
</tr>
<tr>
<td>3</td>
<td>99799.50</td>
<td>499.00</td>
<td>600.00</td>
<td>101.00</td>
</tr>
<tr>
<td>4</td>
<td>99698.50</td>
<td>498.49</td>
<td>600.00</td>
<td>101.51</td>
</tr>
<tr>
<td>5</td>
<td>99596.99</td>
<td>497.98</td>
<td>600.00</td>
<td>102.02</td>
</tr>
<tr>
<td>6</td>
<td>99494.97</td>
<td>497.47</td>
<td>600.00</td>
<td>102.53</td>
</tr>
<tr>
<td>7</td>
<td>99392.44</td>
<td>496.96</td>
<td>600.00</td>
<td>103.04</td>
</tr>
</tbody>
</table>
Amortization

- How long will it take to pay off this loan?
- Imagine what would happen if you paid this off in 30 years in one lump sum
- It would of course be $100,000(P) at 6%(r) compounded monthly (m=12) for 30 years(t)
- So $100000(1.005)^{360} = $602,257.52
A savings plan to pay it off

- So you want a savings plan in which you contribute \( d \) every month for 30 years, still at 6% compounded monthly.
- And at the end you should have $602,257.52 (A).
- So we use the savings plan formula.
A savings plan to pay it off

- $A = \$602,257.52$, $r = .06$, $m = 12$, $t = 30$
- \[ A = d \left( \left(1 + \frac{r}{m}\right)^{(mt)} - 1 \right) \left( \frac{r}{m} \right) \]
- So \[ d = \frac{(A \times r/m)}{(1 + r/m)^{(mt)} - 1} \]
  \[ = \frac{(602257.52 \times .005)}{(1.005^{360}) - 1} \]
  \[ = \frac{3011.2876}{(6.022575) - 1} \]
  \[ = 3011.2876 / 5.022575 = \$599.55 \]
A savings plan to pay it off

So you want a savings plan in which you contribute $599.55 every month for 30 years, still at 6% compounded monthly

And at the end you will have $602,257.52 (A)

And pay off the loan, which has now reached that amount
A lot of interest

- You put in $599.55 each month for 360 months, that’s $599.55*360 = $215,838
- But you paid $602,257.52
- So that’s $386,419.48 in interest
- (With no interest 100000/360 = $277.77 per month instead of $599.55 )
Formulas

- You borrow $P$ at interest rate $r$ compounded monthly ($m=12$) for 30 years ($t=30$), so interest $i = r/12$ per period.
- The loan balance at the end will be $A = P(1+i)^{360}$.
- You pay it off by saving at rate $r$, monthly amount $d$. 
Formulas

- The loan balance at the end will be $A = P(1+i)^{360}$
- You pay it off by saving at rate $r$, monthly amount $d$, so $A = d((1+i)^{360}-1)/(i)$
- Thus $P(1+i)^{360} = A = d((1+i)^{360}-1)/(i)$
- So $d = P(i)(1+i)^{360}/((1+i)^{360}-1)$
In our case

- \( d = P(i)(1+i)^{360}/((1+i)^{360}-1) \)
- \( P = 100000, \ i = \frac{.06}{12} = .005 \)
- So \( d = 100000(.005)(1.005)^{360}/(1.005^{360}-1) = 500(6.02257)/(5.02257) = 599.55 \)
Formula for varying m and t

- For principal P at start, annual interest rate r, for t years, compounded m times per year, and payment d each period
- \[ A = P(1+r/m)^{(mt)} = d\left(\frac{(1+r/m)^{(mt)}-1}{r/m}\right) \]
- \[ d = P\left(\frac{(1+r/m)^{(mt)}}{(1+r/m)^{(mt)}-1}\right) \]
Buying over 3 years

- If you borrow $5000 for a cheap car over three years at 10% APR, what will be your monthly payment?
- $P = 5000$, $r = .1$, $m = 12$, $t = 3$
- So $(1+r/m) = 1+.1/12 = 1.00833$
- And $(1+r/m)^{(mt)} = 1.00833^{36} = 1.348$
Buying over 3 years

- \[ d = P\left(\frac{r}{m}\right)\left(\frac{(1 + \frac{r}{m})^{mt}}{(1 + \frac{r}{m})^{mt} - 1}\right) \]
- So \[ d = 5000\left(\frac{.00833}{1.348}/(1.348-1)\right) = 41.65(1.348/.348) = $161.33 \text{ per month} \]
- (If no interest \[ 5000/36 = $138.88 \])
- There is much less difference here, because it’s only for three years, not 30
Interest over 3 years

- Original P was $5000
- \( d = \$161.33 \) per month
- Total payments = \( 36 \times 161.33 = \$5807.88 \)
- Interest paid is $807.88
Buying over 4 years

If you borrow $5000 for a cheap car over four years at 10% APR, what will be your monthly payment?

P=5000, r = .1, m = 12, t = 4

So \((1+r/m) = 1+.1/12 = 1.00833\)

And \((1+r/m)^{(mt)} = 1.00833^{48} = 1.489\)
Over 4 years

- \[ d = P \left( \frac{r/m}{(1+r/m)^{mt}} \right) \frac{((1+r/m)^{mt})}{(1+r/m)^{mt}-1} \]
- So \[ d = 5000(0.00833)(1.489)/(1.489-1) \]
  \[ = 41.65(1.489/0.489) = $126.82 \text{ per month} \]
- (If no interest \( 5000/48 = $104.16 \))
- Lower than \$161.33, but you have to pay \[ 48(126.82) = $6087.36 \text{ instead of} \]
  \[ 36(161.33) = $5807.88 \]
Buying over 7 years

- If you borrow $5000 for a cheap car over seven years at 10% APR, what will be your monthly payment?
  - \( P=5000, \ r = .1, \ m = 12, \ t = 7 \)
  - So \( (1+r/m) = 1+.1/12 = 1.00833 \)
  - And \( (1+r/m)^{(mt)} = 1.00833^{84} = 2.00792 \)
Over 7 years

- \( d = P\left(\frac{r}{m}\right)\left(\frac{(1+\frac{r}{m})^\left(mt\right)}{(1+\frac{r}{m})^\left(mt\right)-1}\right) \)

- \( d = \frac{5000(0.00833)(2.00792)}{(2.00792 - 1)} = 41.65(2.00792 / 1.00792) = $82.97 \) per month

- (If no interest \( 5000/84 = $59.52 \))

- Lower than \$161.33, but you have to pay \( 84(82.97) = $6969.48 \), now nearly \$2000 in interest
Another way to think

- Pretend you are the loaner
- You will be getting d per month, but later
- The present value of the first payment is $d/(1+i)$, for $i =$ rate per month
- The present value of the second payment (after two months) is $d/(1+i)^2$
- Etc.
Another way to think

- The total present value of all 360 payments is \( \frac{d}{1+i} + \frac{d}{(1+i)^2} + \frac{d}{(1+i)^3} + \ldots + \frac{d}{(1+i)^{360}} \)

- That should be what you are willing to pay for that “payment program”, so it is \( P \)

- Add up using geometric series trick

- Get \( P = \frac{d((1+i)^n-1)}{((1+i)^n, \text{ similar form}} \)
The formulas for Amortization

- $A$ (imagined debt after $t$ years) = $P(1+i)^n$, where $i = r/m =$ interest per period, $r =$ annual rate, $m =$ number of times compounded per year and $t =$ number of years

- Also $A = d((1+i)^n-1)/i$

- So $d = P*i*(1+i)^n / (1+i)^n-1)$
The formulas for Amortization

- From \( d = P \cdot i \cdot (1+i)^n / (1+i)^n - 1 \)
- The book divides top and bottom of this fraction to get the traditional formula \( d = P \cdot i / (1 - (1+i)^{-n}) \), where (as usual) a negative exponent \( f^{-6} \) means \( 1/(f^6) \)
- Slightly better since \( (1+i)^n \) appears only once (but you should be good at this to use it)
Quiz on Chapter 21 will be Tuesday

- These Formulas will be available:
  - $A = P(1+r/m)^{mt} = P(1+i)^n$
  - $A = d((1+r/m)^{mt} - 1)/(r/m) = d((1+i)^n - 1)/i$
  - $A = Pe^{rt}$
  - $rt = \ln(A/P)$
The End for Today