Math 1000
Mathematical Literacy in Today’s World

Lecture 23

Winter 2010
What about borrowing?

- You borrow some money (P) and pay back t years later
- This is like a savings account for the loaner
- In fact, the loaner can imagine that you are a bank and he has a savings account with you
Formulas

- For Borrowed principal P at start, annual interest rate r, for t years
  
  At simple interest your balance Owed is
  \[ A = P + P \times r \times t = P(1+r \times t) \]

- At compound interest your balance Owed is
  \[ A = P(1+r)^t \]

- For discounted loans \( P = \frac{A}{(1+r)^t} \)
Compounding more often than annually

- At rate $r$ compounded $m$ times per year
- The rate is $i = \frac{r}{m}$ per period
- So the Owed balance is multiplied by $1 + \frac{r}{m} = 1 + i$ each period
- At the end ($t$ years) you will owe $A = P(1+r/m)^{mt}$
- For discounted loans $P = A/(1+r/m)^{mt}$
Conventional loans

- The usual sort of loan is paid back not at the end of the loan period, but in equal periodic installments.
- These payments are called amortizing the loan, paying it back over time.
- The interest is computed on the amount of principal remaining, which is less, after payments.
Conventional loans

- As you make more and more payments, the principal decreases, so the interest amount decrease each period, so more of the principal gets paid off
- But you start paying very little of the principal (and then accelerate)
Conventional loans

- Imagine taking a $100,000 loan at 6%
- Monthly payments means ½% per month interest
- Old fashioned method was to compute an “amortization schedule” to see how the payments go
## Amortization Schedule

<table>
<thead>
<tr>
<th>End of Month</th>
<th>Remaining Principal</th>
<th>Interest - ½% per month</th>
<th>Payment</th>
<th>Amount to principal</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100,000.00</td>
<td>500.00</td>
<td>600.00</td>
<td>100.00</td>
</tr>
<tr>
<td>2</td>
<td>99,900.00</td>
<td>499.50</td>
<td>600.00</td>
<td>100.50</td>
</tr>
<tr>
<td>3</td>
<td>99,799.50</td>
<td>499.00</td>
<td>600.00</td>
<td>101.00</td>
</tr>
<tr>
<td>4</td>
<td>99,698.50</td>
<td>498.49</td>
<td>600.00</td>
<td>101.51</td>
</tr>
<tr>
<td>5</td>
<td>99,596.99</td>
<td>497.98</td>
<td>600.00</td>
<td>102.02</td>
</tr>
<tr>
<td>6</td>
<td>99,494.97</td>
<td>497.47</td>
<td>600.00</td>
<td>102.53</td>
</tr>
<tr>
<td>7</td>
<td>99,392.44</td>
<td>496.96</td>
<td>600.00</td>
<td>103.04</td>
</tr>
</tbody>
</table>
Amortization

- How long will it take to pay off this loan?
- Imagine what would happen if you paid this off in 15 years in one lump sum.
- It would of course be $100,000 at 6% compounded monthly (m=12) for 15 years.
- So $100,000 \times (1.005)^{180} = $245,409.40
A savings plan to pay it off

- So you want a savings plan in which you contribute \( d \) every month for 15 years, still at 6% compounded monthly.
- And at the end you should have \( $245,409.40 \) (\( = A \)).
- So we use the savings plan formula.
A savings plan to pay it off

- \( A = $245,409.40, \ r = .06, \ m = 12 \ , \ t = 15 \)
- \( A = d((1+r/m)^{(mt)}-1)/(r/m) \)
- So \( d = (A*r/m)/(1+r/m)^{(mt)}-1) \)

= \( (245409.40)*(0.005)/(1.005^{180}-1) \)

= \( 1227.05/(2.45409-1) \)

= \( 1227.05/1.45409 = $843.86 \)
A savings plan to pay it off

- So you want a savings plan in which you contribute $843.86 every month for 15 years, still at 6% compounded monthly.
- And at the end you will have $245,409.40 (A).
- And pay off the loan, which has now reached that amount.
Higher payment than for 30 years

- A 15-year mortgage on $100,000 is $843.86 per month, total of payments $180*843.86 = $151,894, total interest $51,894

- We did 30-year mortgage, it came out $599.55 per month, total payments $360*599.55 = $215,838, total interest $115,838
Formulas

- You borrow \( P \) at interest rate \( r \) compounded monthly \( (m=12) \) for 15 years \( (t=15) \), so interest \( i = \frac{r}{12} \) per period
- The loan balance at the end will be \( A = P(1+i)^{180} \)
- You pay it off by saving at rate \( r \), monthly amount \( d \)
Formulas

- The loan balance at the end will be $A = P(1+i)^{180}$
- You pay it off by saving at rate $r$, monthly amount $d$, so $A = d\left((1+i)^{180} - 1\right)/i$
- Thus $P(1+i)^{180} = A = d\left((1+i)^{180} - 1\right)/i$
- So $d = P(i)(1+i)^{180}/((1+i)^{180} - 1)$
In our case

- \( d = P(i)(1+i)^{180}/((1+i)^{180}-1) \)
- \( P = 100000, \ i = .06/12 = .005 \)
- So \( d = \)
  \[
  100000(.005)(1.005)^{180}/(1.005^{180}-1) = 500(2.454)/(1.454) = 843.86
  \]
Formula for varying m and t

- For principal P at start, annual interest rate r, for t years, compounded m times per year, and payment d each period

\[
A = P(1+r/m)^{(mt)} = d\left(\frac{(1+r/m)^{(mt)}-1}{(r/m)}\right)
\]

\[
d = P\frac{(r/m)((1+r/m)^{(mt)}-1)}{((1+r/m)^{(mt)}-1)} = P*i*(1+i)^n/((1+i)^n-1)
\]
Interpretation

- \[ d = P\left(\frac{r}{m}\right)\left(\frac{(1+\frac{r}{m})^{mt}}{(1+\frac{r}{m})^{mt}-1}\right) = P^*i^*\frac{(1+i)^n}{((1+i)^n-1)} \]

- The factor \( P^*i \) gives the interest on the principal \( P \) for the first period

- The other factor \( \frac{(1+i)^n}{((1+i)^n-1)} \) increases the \( d \) by enough to pay off the principal in \( n \) periods as well as the interest
Determining equity

- What if you sell the house after some number of months – how much do you still owe?
- Equity – that’s the amount you don’t still owe - the amount you have paid off - the reduction in the principal
Determining equity

- Say you had a $100,000 15-year mortgage (monthly payment was $843.86 at 6%)
- What is your equity after 9 years? (=108 months)
- That is, how much has the $100,000 been reduced by 9 years of payments?
Determining equity

That is, how much has the $100,000 been reduced by 108 months of paying $843.86 per month?

Imagine what loan you could get for paying $843.86 per month for the remaining 72 months

So $d = 843.86$, $i = r/m = .005$, $n = 72$
Determining equity

- So \( d = 843.86, \ i = \frac{r}{m} = 0.005, \ n = 72 \)
- Now we want \( P \) from \( d = \frac{P \times i \times (1+i)^n}{(1+i)^n - 1} \)
- So figure first \((1+i)^n = (1.005)^{72} = 1.432\)
- Then \( 843.86 = P \times 0.005 \times 1.432 / 0.432 \)
- So \( P = 843.86 \times 0.432 / 0.005 / 1.432 = 50,914.46 \)
Determining equity

- So paying $843.86 per month for 72 months would pay off a loan of $50,914.46.
- Thus we still owe $50,914.46 on this $100,000 loan.
- So our equity is $49,085.54.
Determining equity

- How did we do this again?
- We determined the loan amount we could pay off with the remaining payment schedule, and that gave us what we still have to pay
- The rest of the loan is paid off, so it is the equity
Equity after k months

- That means there are still $n-k$ payments of $d$ remaining.

- So loan amount $P = d \times ((1+i)^{(n-k)}-1)/(i \times (1+i)^{(n-k)})$.

- The rest of the loan is now (original loan amount – $P$), so that is the equity.
Suppose you finance a $20,000 car over 4 years at 10%, but then you want to pay it off after 3 years.

First figure monthly payment $d$ using $P = 20000$, $r = .1$, $m = 12$, $t = 4$, so $i = .00833$ and $n = 48$

$$d = P \times i \times (1+i)^n/((1+i)^n-1) = 20000 \times (.00833) \times 1.489/.489 = 507.48$$
One more time

- Suppose you finance a $20,000 car over 4 years at 10%, paying $507.48 per month, but then you want to pay it off after 3 years.

- 12 payments left at $507.48, so corresponding $P = d*((1+i)^{12}-1)/i/(1+i)^{12}$

- $P = 507.48*(.10473)/(.00833)/(1.10473) = $5777.74, still left to pay off
Annuities

- An Annuity is a specific number of (usually equal) periodic payments
- Like a pension for a certain number of years
- A “Life Annuity” is periodic payments for the life of the recipient (and so has an unknown number of payments)
Calculating Annuities

- Like a savings plan, except that you take the money out instead of depositing
- Think of you as being the bank in the conventional loan scenario
- You put a principal amount in some account, then the account pays you monthly for some period
Calculating Annuities

- So it’s the same formula again, loan amount vs. payment amount
- \( P = d \times \frac{(1+i)^n-1}{i(1+i)^n} \)
- Say you want a 25-year annuity paying $1000 per month and you can get 4% compounded monthly
- So \( d = 1000 \), \( i = \frac{r}{m} = \frac{.04}{12} = .0033 \)
Calculating Annuities

- \( P = \frac{d*((1+i)^n-1)/i}{(1+i)^n} \)

- Say you want a 25-year annuity paying $1000 per month and you can get 4% compounded monthly

- So \( d = 1000, \ i = \frac{r}{m} = \frac{.04}{12} = .0033, \ n = 12*25 = 300 \) months

- \((1+i)^n = (1.0033)^{300} = 2.7138\)
Calculating Annuities

- \[ P = d * ((1 + i)^n - 1) / i / (1 + i)^n \]
- So \( d = 1000 \), \( i = r/m = 0.04/12 = 0.0033 \), \( n = 12 \times 25 = 300 \) months
- \( (1 + i)^n = (1.0033)^{300} = 2.7138 \)
- \[ P = 1000 (1.7138) / (0.0033) / (2.7138) = $191367.57 \]
- Accuracy not so good here, use \( 0.00333333 \)
Calculating Annuities

- $P = d \cdot \frac{(1+i)^n - 1}{i(1+i)^n}$

So $d = 1000$, $i = \frac{r}{m} = \frac{0.04}{12} = 0.0033$, $n = 12 \times 25 = 300$ months

$(1+i)^n = (1.0033)^{300} = 2.7138$

$P = 1000 \cdot \frac{1.7138}{0.0033} / 2.7138 = 191367.57$

Accuracy not so good here, use 0.0033333
Calculating Annuities

- $P = d \times \frac{((1+i)^n-1)}{i/(1+i)^n}$
- So $d = 1000$, $i = r/m = 0.04/12 = 0.0033333$, $n = 12 \times 25 = 300$ months
- $(1+i)^n = (1.0033333)^{300} = 2.7138$
- $P = 1000 \times (1.7138)/(0.0033333)/(2.7138) = 189,452.48$
- This made a $2000$ difference
Calculating Annuities

- So to receive 25 years of $1000 monthly payments (at 4%) requires $189,452.48 in the bank now.
- Notice that the total received is $300,000.
- The extra is interest.
In Reverse

- Suppose you have $100,000 in the bank when you retire, and you can get 4% compounded monthly.
- How much can you receive each month for 35 years?
- So $P = 100000$, $i = r/m = .04/12 = .00333333$, $t = 35$, $n = 35 \times 12 = 420$
In Reverse

- So $P = 100000$, $i = r/m = .04/12 = .0033333$, $t = 35$, $n = 35 \times 12 = 420$
- $(1+i)^n = (1.0033333)^{420} = 4.04577$
- $d = \frac{P \cdot i \cdot (1+i)^n}{((1+i)^n)-1} = \frac{100000 \times .0033333 \times (4.04577)/(3.04577)}{(333.33)(1.32832)} = $442.77$
- You can get $442.77$ monthly for 35 years
In Reverse

- If you have $100,000 in the bank when you retire, and you can get 4% compounded monthly, then for 35 years you can get $442.77 monthly.
- The factor $P \times i = 100000(0.0033333) = $333.33$ is the interest per month.
- The extra factor “uses up” the principal.
Interest Only

- If you have $100,000 in the bank when you retire, and you can get 4% compounded monthly, then forever(!) you can get $333.33 monthly.
- The factor $P \times i = 100000 \times 0.0033333 = 333.33$ is the interest per month.
- Now we never use up the principal.
Interest Only

- If you take out only the interest, then the principal stays the same, so this can go on forever.
- This is called a “perpetuity”, since it is perpetual (never ends).
Life Annuity

- If you need payments the whole time you live, the annuity does not have a known number of payments.
- Usually the number of payments is determined by the “life expectancy” of the recipient; how many more years he/she is expected to live at the time of the annuity.
Life Annuity

- Companies paying life annuities use “mortality tables” to determine life expectancy
- Some people live longer than their expectancies, some shorter
- Usually only insurance companies offer life annuities, to average it out
Life Annuity

- Suppose you want to retire at age 60 and expect a conservative interest rate of 3% compounded monthly.
- According to life expectancy tables you can expect to live 22.2 more years.
- So \( i = \frac{r}{m} = \frac{0.03}{4} = 0.0025 \) and \( n = mt = 12 \times 22.2 = 266 \) months.
Life Annuity

- So \( i = \frac{r}{m} = \frac{.03}{4} = .0025 \) and \( n = mt = 12 \times 22.2 = 266 \) months, so \((1+i)^n = 1.9429\)

- If you have $200,000 in the bank at the time you are 60, you can get \( d = \frac{P \times i \times (1+i)^n}{((1+i)^n-1)} = \frac{200000(.0025)(1.9429)/(.9429) = $1030.28}{\text{per month for your expected life}}\)
Life Annuity

- If you have $200,000 in the bank at the time you are 60, you can get $d = 1030.28 per month for your expected life of 22.2 years.
- What if you live longer?
- Then you probably should have bought a life annuity from an insurance company with the money.
The formulas for Amortization

- A (imagined debt after t years) = \( P(1+i)^n \), where \( i = \frac{r}{m} \) = interest per period, \( r \) = annual rate, \( m \) = number of times compounded per year and \( t \) = number of years

- Also \( A = \frac{d((1+i)^n-1)}{i} \)

- So \( d = \frac{P*i*(1+i)^n}{(1+i)^n-1} \)
The formulas for Amortization

- From \( d = P \cdot i \cdot (1+i)^n / ( (1+i)^n - 1) \)
- The book divides top and bottom of this fraction to get the traditional formula \( d = P \cdot i / (1 - (1+i)^{-n}) \), where (as usual) a negative exponent \( f^{-n} \) means \( 1/(f^n) \)
- Slightly better since \( (1+i)^n \) appears only once (but you should be good at this to use it)
The End for Today