Kinds of loans

- You borrow some money \((P)\) and:
- (1) pay back all of it \(t\) years later, or
- (2) pay back just the interest each period, then pay back all of \(P\), \(t\) years later
- (3) pay back the interest each period, plus some of the principal \(P\)
(1) Pay back all at once

- This is a “one-payment” loan
- At compound interest rate $r$ for $t$ years, the principal has grown to $P(1+i)^n$, where $i = \frac{r}{m}$, the interest rate per period, $n = mt$, the total number of periods in $t$ years, and $m =$ number of compounding periods per year
(2) Interest-only Loan

- If you pay only interest each period, this is often called a “balloon” loan, since the last payment is the entire principal P

- At compound interest rate r for t years, the periodic interest is $P \times i$, where $i = \frac{r}{m}$, and the principal P remains the same
(3) Conventional Loan

- Here you pay interest each period, plus some towards the principal.

- To get a formula for this, we imagined putting payment $d$ each period into a savings account at interest rate $i = \frac{r}{m}$ each period, in order to pay off the imagined “one-payment” loan at the end.
Accumulating to pay off loan

- At rate $r$ compounded $m$ times per year
- The rate is $i = \frac{r}{m}$ per period
- After $t$ years the payoff balance will be $A = P\,(1+i)^n$ , for $n = mt$
- And you will have accumulated $A = d\,((1+i)^n-1)/i$ , from the savings plan formula
Accumulating to pay off loan

- $A = P(1+i)^n = d*((1+i)^n-1)/i$
- So $d = P*i*(1+i)^n/((1+i)^n-1)$
- Again, we can interpret this by noting that the $P*i$ factor is the interest per period (as in (2)) and the extra factor $(1+i)^n/((1+i)^n-1)$ makes the payment slightly bigger to pay off the principal
Conventional loans

- The periodic payments amortize the loan by paying the interest each period plus some of the principal.
- The amount of principal decreases, slowly at first, then faster, since the interest is computed on the smaller principal each period.
Example

Imagine taking a $50,000 loan at 8%, paying quarterly, for 10 years.

So \( P = 50000 \), \( r = 0.08 \), \( m = 4 \), \( t = 10 \).

Then \( i = \frac{r}{m} = 0.02 \) and \( n = mt = 40 \).

Payment is \( d = \frac{P \cdot i \cdot (1+i)^n}{((1+i)^n-1)} = \frac{50000 \cdot 0.02 \cdot (1.02)^{40}}{(1.02^{40}-1)} = \frac{1000 \cdot 2.0804}{1.0804} = $1925.58 \) quarterly.
Example

- Imagine taking a $50,000 loan at 8%, paying quarterly, for 10 years
- \( d = $1925.58 \) every quarter
- Total of payments is \( 40(1925.58) = $77023.32 \), so total interest is \( $27023.32 \)
Adjust this example

- A $50,000 loan at 8%, paying quarterly, for 10 years, costs $1925.58 per quarter
- What if you paid $2000 each quarter instead, to reduce the principal faster?
- Back to accumulation, at $2000 per quarter, \( A = 2000(1+i)^{n-1}/i \), but now \( n \) is not known
Adjust this example

- A = 2000(1.02)^{n-1}/.02, but now n is not known
- And A = 50000(1.02)^n for payoff balance
- So 2000((1.02)^n-1)/.02 = 50000(1.02)^n and 100000((1.02)^n-1) = 50000(1.02)^n
- so 2((1.02)^n-1) = (1.02)^n, 1.02^n = 2
- Turns out you can get done in 35 quarters
Adjust this example

- Versus paying $1925.58 quarterly for 40 quarters
- Paying $2000 each quarter reduces the term of the loan to 35 quarters
- That is more than 10% reduction in term for only a 4% increase in payment
Determining equity

- What if you sell the house after some number of months – how much do you still owe?
- Equity – that’s the amount you don’t still owe - the amount you have paid off - the reduction in the principal
Determining equity

- Say you had a $100,000 30-year mortgage (monthly payment was $599.55 at 6%)
- What is your equity after 10 years? (=120 months)
- That is, how much has the $100,000 been reduced by 10 years of payments?
Determining equity

That is, how much has the $100,000 been reduced by 120 months of paying $599.55 per month?

Imagine what loan you could get for paying $599.55 per month for the remaining 240 months.

That would be $120,000 reduced by 120 months of paying $599.55 per month.

So \( d = 599.55 \), \( i = r/m = .005 \), \( n = 240 \)
Determining equity

- So $d = 599.55$, $i = \frac{r}{m} = 0.005$, $n = 240$
- Now we want $P$ from $d = P \cdot i \cdot (1+i)^n/(1+i)^n-1$
- So figure first $(1+i)^n = (1.005)^{240} = 3.3102$
- Then $599.55 = P \cdot 0.005 \cdot (3.3102)/(2.3102)$
- So $P = 599.55 \cdot (2.3102)/0.005/3.3102 = 83685.60$
Determining equity

- So paying $599.55 per month for 120 months would pay off a loan of $P = 83,685.60$
- Thus we still owe $83,685.60 on this $100,000 loan
- So our equity is only $16,314.40 (about $136 per month)
Determining equity

- How did we do this again?
- We determined the loan amount we could pay off with the remaining payment schedule, and that gave us what we still have to pay
- The rest of the loan is paid off, so it is the equity
Equity after \( k \) months

- That means there are still \( n-k \) payments of \( d \) remaining.

- So imagined loan amount is \( \text{newP} = d \times (\frac{(1+i)^{(n-k)}-1}{i(1+i)^{(n-k)}}) \)

- The rest of the loan is now \( \text{oldP} - \text{newP} \), so that is the equity.
Adjustable Rate Mortgages

- This means that the interest rate can be adjusted at certain times
- Usually determined by the standard mortgage rate at the various times
- Often the rate adjusts every one to five years or so
- Abbreviation is ARMs, vs. Fixed rates
Why Adjustable Rates?

- If the loaner lends at (say) 3% for 30 years, then after some time the standard rates may rise (say) to 6%
- At that point the loaner feels he is losing money on this loan, since you are still paying 3%
- If the loaner could, he would adjust the rate
Why Adjustable Rates?

- But without an ARM he cannot
- Because of the risk of having interest rates rise during the term of the loan, usually loaners charge higher rates on fixed rate mortgages
- So if you agree to allow the loaner to adjust the rate during the term, he may be willing to charge a lower interest rate
Dangers of ARMs

- If interest rates rise during the term of the loan, the rate charged on the loan will rise.
- Then the payment amount will be refigured (recalculated) using the new rate and the current principal (value of the loan).
- So the monthly payment will rise.
Reducing these dangers

- There may be limits on how much the rate can change
- There may be limits on how often
- You could expect to pay off the loan early by selling (or before the rate changes)
- You can arrange to refinance (pay off and get a new loan) before rate changes
Annuities

- An Annuity is a specific number of (usually equal) periodic payments
- Like a pension for a certain number of years
- A “Life Annuity” is periodic payments for the life of the recipient (and so has an unknown number of payments)
Calculating Annuities

- Like a savings plan, except that you take the money out instead of depositing
- Think of you as being the bank in the conventional loan scenario
- You put a principal amount in some account, then the account pays you monthly for some period
Calculating Annuities

- So it’s the same formula again, loan amount vs. payment amount
- \[ P = d \times \frac{(1+i)^n - 1}{i} \times \frac{1}{(1+i)^n} \]
- Say you want a 30-year annuity paying $1000 per month and you can get 3% compounded monthly
- So \( d = 1000 \), \( i = r/m = .03/12 = .0025 \)
Calculating Annuities

- \[ P = d \times \frac{(1+i)^n - 1}{i} \times \frac{1}{(1+i)^n} \]

Say you want a 30-year annuity paying $1000 per month and you can get 3% compounded monthly.

So \( d = 1000 \), \( i = \frac{r}{m} = \frac{.03}{12} = .0025 \), \( n = 12 \times 30 = 360 \) months.

- \( (1+i)^n = (1.0025)^{360} = 2.45684 \)
Calculating Annuities

- \[ P = d \times \frac{(1+i)^n - 1}{i/(1+i)^n} \]
- So \( d = 1000 \), \( i = \frac{r}{m} = \frac{.04}{12} = .0025 \), \( n = 12 \times 30 = 360 \) months
- \( (1+i)^n = (1.0025)^{360} = 2.45684 \)
- \[ P = 1000 \times \frac{1.45684}{.0025}/2.45684 = \$237,189.23 \]
- Needed in the bank for this annuity
Calculating Annuities

Here $237,189.23 will do the work of $360,000 (= $1000*360 months) because of the 3% interest over a long time
In Reverse

- Suppose you have $100,000 in the bank when you retire, and you can get 3% compounded monthly.
- How much can you receive each month for 25 years?
- So \( P = 100000 \), \( i = \frac{r}{m} = \frac{0.03}{12} = 0.0025 \), \( t = 25 \), \( n = 25 \times 12 = 300 \)
In Reverse

- So \( P = 100000 \), \( i = \frac{r}{m} = \frac{.03}{12} = .0025 \), \( t = 25 \), \( n = 25 \times 12 = 300 \)
- \((1+i)^n = (1.0025)^{300} = 2.11502\)
- \(d = (P \times i) \times \frac{(1+i)^n}{((1+i)^n)-1} = (100000 \times .0025) \times (2.11502)/(1.11502) = (250 \times 1.896845) = $474.21\)
- You can get $474.21 monthly for 25 years
In Reverse

- If you have $100,000 in the bank when you retire, and you can get 3% compounded monthly, then for 25 years you can get $474.21 monthly

- \( P\times i = $250 \) is interest and the other $224.21 is reducing the principal
Interest Only

- If you have $100,000 in the bank when you retire, and you can get 3% compounded monthly, then forever(!) you can get $250 monthly
- Now we never use up the principal
- This is called a “perpetuity”, since it is perpetual (never ends)
Life Annuity

- If you need payments the whole time you live, the annuity does not have a known number of payments.
- Usually the number of payments is determined by the “life expectancy” of the recipient; how many more years he/she is expected to live at the time of the annuity.
Life Annuity

- Companies paying life annuities use “mortality tables” to determine life expectancy
- Some people live longer than their expectancies, some shorter
- Usually only insurance companies offer life annuities, to average it out
The formulas for Amortization

- A (imagined debt after t years) = \( P(1+i)^n \)
  where \( i = \frac{r}{m} = \text{interest per period} \), \( r = \text{annual rate} \), \( m = \text{number of times compounded per year} \) and \( t = \text{number of years} \)

- Also \( A = \frac{d((1+i)^n-1)}{i} \)

- So \( d = \frac{P*i*(1+i)^n \div (1+i)^n-1)}{(1+i)^n} \)
The formulas for Amortization

- For home equity after k payments, we use \( d = \frac{\text{oldP} \times i \times (1+i)^n}{(1+i)^n - 1} \) to get the periodic payment, then use \( \text{newP} = \frac{d((1+i)^{(n-k)}-1)/i}{(1+i)^{(n-k)}} \) to get new principal, which is the loan amount still owing.

- Then Equity = oldP - newP.
The End for Today

Quiz on Chapter 22 is Thursday April 22

Final exam is Thursday April 29 in this room, 10:40-1:10