DIFFERENTIAL EQUATIONS AND THE PROJECTIVE PLANE

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1. Introduction

The analysis of differential equations can be enormously simplified by making appropriate coordinate changes. In this article, we examine three sets of coordinate transformations which are suggested by geometric considerations and by ideas of dynamical systems. These are

1. the coordinate changes associated to the usual charts in projective space,
2. blowing up points, and
3. rescaling time.

These can be used to

1. bring asymptotic questions into view in the finite part of the plane,
2. separate trajectories passing through a point at different slopes, and
3. remove singular sets,

respectively.

The remarkable thing is that while these are quite powerful tools, they are all given by simple transformations which can easily be taught to and used by undergraduates.

The paper is organized as follows. Our main focus is the existence of lines to which solutions to polynomial vector fields are asymptotic. We start with the example which led me to these investigations, and show how the techniques work in this particular case.

We then give an introduction to the projective plane and the coordinate transformations between the usual charts. The terms asymptotic slopes, intercepts and lines are made precise in the following section in terms of this geometry.

We then begin our analysis of asymptotic lines for polynomial differential equations by applying the coordinate transformations for the projective plane. This will result in a differential equation given by rational functions, and these will be singular where one or more denominators is zero. In the next section we show how to analyze behavior near the singular set by rescaling time to remove the singularities.

Combining these methods, we are able to determine the asymptotic slopes of solutions to a polynomial differential equation in the following section, proving a theorem which we briefly summarize as follows.

Theorem 1. The solutions of a polynomial vector field of degree \( N \) either have at most \( N + 1 \) asymptotic slopes or all but at most \( N - 1 \) asymptotic slopes.

The theorem is proved by exhibiting a polynomial of degree at most \( N + 1 \) which vanishes at the asymptotic slopes. If this polynomial is \( 0 \), we exhibit a second polynomial, among whose roots we will find any asymptotic slope which does not occur. This polynomial has degree at most \( N - 1 \).
We give some examples, including a common Riccati equation, to show these methods in action. For the Riccati equation, we will see that a large class of solutions have asymptotic slope $\infty$ and that all vertical lines occur as asymptotes. This is evident and elementary since the flow is especially simple in one of its transformed versions.

In the following section, we show how the technique of blowing up points completes our analysis of asymptotic lines, by allowing us to compute their intercepts in all cases. When the Jacobian is nonsingular at the relevant points, blowing up is not necessary, but the analysis using blowing up is far easier than that using the Jacobian, and works even when the Jacobian is singular. We use this method to prove our second main theorem.

**Theorem 2.** For each asymptotic slope $\alpha$ of a polynomial differential equation, there is a linear polynomial which must vanish at the intercept of any line of slope $\alpha$ asymptotic to a solution of the differential equation.

This means that in the generic case, in which this linear polynomial has a unique root, all solutions with the same asymptotic slope are asymptotic to just one line. In the non-generic case, the polynomial is constant, and then there will either be no (finite) asymptotic intercepts, or all asymptotic intercepts will occur, depending upon whether this constant is nonzero or zero, respectively.

The reader will note that we sometimes refer to vector fields and the flow given by their trajectories, and other times to (first order) differential equations and their solutions. Of course these are simply different ways of referring to the same objects.

I owe several people for teaching me various parts of this, including James Mauldon, Steve Williams, David Eisenbud, and Jaume Llibre, in temporal order of their assistance. More complete acknowledgements may be found at the end of the article.

The results here are not difficult and after finding them on my own, I discovered that they are not new. The generic part of the first theorem is alluded to in [1, I.6.2, p. 113], the embedding of a flow in $\mathbb{R}^2$ in the projective plane is referred to in [1, II.1.6, p. 163], the rescaling of time occurs repeatedly, viz. [1, II.1.4, p. 161], and the use of blowing-up goes back at least 100 years to Bendixson [2]. More recently, Dumortier [3] used blowing up to prove a very general theorem on resolution of singularities into elementary singularities. Results related to those here can be found in Llibre ([4] and [5]) and a survey of the field of polynomial vector fields can be found in Schlomiuk [6].

However, these results seem less well known than their simplicity might suggest. The techniques by which they are proven are quite useful in analysing differential equations, but are similarly less well known than they should be. Therefore, I offer this modest contribution as a means of popularizing them and the general technique of using coordinate changes to illuminate aspects of a differential equation one is studying.

### 2. The Motivating Example

One the way to my topology class one day, I decided I needed another example to illustrate the Poincaré-Hopf Theorem. Recall that this says that the winding number of a vector field along a curve equals the sum of the indices of the zeros of the vector field inside the curve. I chose the vector field

\[(1)\]

\[x' = x^2 - y^2,\]
\[y' = x^2 + y^2 - 1,\]
which is vertical along the lines \( y = \pm x \), horizontal along the circle \( x^2 + y^2 = 1 \), and has stationary points at their 4 intersections. To flesh out the example, I also roughly sketched for my class the trajectories of the flow. After class, curious how accurate my sketches had been, I set my little HP-48 to work graphing some trajectories. It soon became apparent that the unbounded trajectories all seemed to be heading off to \( \infty \) at the same slope, which I estimated to be \(-1.85\), and they all seemed to be asymptotic. (See Figure 1.)

To try and prove this, suppose that \((x(t), y(t))\) is an unbounded solution, and suppose further that \( |x(t)| \to \infty \), as seems to be the case on the unbounded trajectories. If there is a limiting slope, then it must satisfy

\[
m = \lim_{t \to \pm \infty} \frac{dy}{dx} = \lim_{t \to \pm \infty} \frac{x^2 + y^2 - 1}{x^2 - y^2} = \lim_{t \to \pm \infty} \frac{1 + \left(\frac{y}{x}\right)^2 + \left(\frac{1}{x}\right)^2}{1 - \left(\frac{y}{x}\right)^2} = \frac{1 + m^2 - 0}{1 - m^2}
\]

where we have used L'Hôpital's rule to replace \( \lim y/x \) by \( \lim dy/dx \). Therefore

\[
m(1 - m^2) = (1 + m^2)
\]

or,

\[m^3 + m^2 - m + 1 = 0.
\]

This cubic has exactly one real root, \( \alpha \simeq -1.839 \ldots \), in perfect accord with our experimental findings.

Note, however, we have had to assume that this limit exists. Further, we still do not know if the solutions are asymptotic or only parallel. Curious how to answer these questions, I asked my colleagues who know about such things how one goes about computing the asymptotic slopes of solutions to differential equations. Steve Williams said the magic word, 'compactify'. I went back to my office to see what the flow looked like in the projective plane. (Steve, it turns out, was thinking about the Riemann sphere, but, fortunately, I didn't know this.) If we let

\[(2) \quad w = \frac{y}{x} \quad z = \frac{1}{x}\]

then, as \( x \to \pm \infty \), \( z \) will go to 0 and \( w \) will approach the asymptotic slope, if there is one. Differentiating and using the inverse transformation

\[
x = \frac{1}{z} \quad y = \frac{w}{z},
\]

we find that

\[(3) \quad w' = \frac{w^3 + w^2 - w + 1}{z} - z \quad z' = w^2 - 1\]
so that $w'$ becomes infinite as $z \to 0$ unless $w$ approaches a root of $w^3 + w^2 - w + 1$ at the same time. So far, this is simply a different way to show that if an asymptotic slope exists, it must be $\alpha$. However, we can now study the flow in the $(w, z)$ plane (Figure 2) where points along the $w$-axis correspond to asymptotic slopes in the original $(x, y)$ plane.

Unfortunately, the flow is singular along the line $z = 0$. It is possible to cope with this and show directly that any trajectory on which $z \to 0$, i.e., on which $x \to \pm\infty$, must approach the point $(w, z) = (\alpha, 0)$. However, there is a much better way to do this. If we multiply the right hand sides of (3) by $z$, we will remove the singularities, and the trajectories off the line $z = 0$ will remain the same, except that time will have been rescaled. This is a standard technique in the study of dynamical systems, and I thank Jaume Llibre for bringing it to my attention. This means that we can now apply the standard theorems about existence and uniqueness to points along $z = 0$ in the new system, to make conclusions about these points in the old one. Our new system is

\begin{align*}
w' &= w^3 + w^2 - w + 1 - z^2 \\
z' &= z(w^2 - 1).
\end{align*}

Along $z = 0$, the flow has $z' = 0$ and $w' \neq 0$ except at $w = \alpha$. Thus, the points $(w, 0)$ with $w \neq \alpha$ are not stationary points and the only trajectories approaching them are the two lying inside the line $z = 0$ on either side of $(\alpha, 0)$. These lie in the circle at infinity for the $(x, y)$ plane so that trajectories in the $(x, y)$ plane satisfy $\lim y/x = \lim w = \alpha$ if $x \to \pm\infty$. This neatly solves our first problem: the asymptotic slope $\lim y/x$ does exist.
Further, the Jacobian of our new system (4) is
\[
\begin{pmatrix}
3w^2 + 2w - 1 & -2z \\
2wz & w^2 - 1
\end{pmatrix} = \begin{pmatrix}
3\alpha^2 + 2\alpha - 1 & 0 \\
0 & \alpha^2 - 1
\end{pmatrix} \approx \begin{pmatrix}
5.5 & 0 \\
0 & 2.4
\end{pmatrix}
\]
at \((\alpha, 0)\). This is nonsingular and shows that all trajectories near \((\alpha, 0)\) approach \((\alpha, 0)\) as \(t \to -\infty\), and further, that they do so vertically, since the vertical eigenvalue is the smaller of the two. This answers our second question: the unbounded trajectories are asymptotic to \(y = \alpha x\), rather than \(y = \alpha x + b\) for nonzero \(b\). This follows from the fact that the line \(y = \alpha x + b\) is transformed into the line \(w = \alpha + bz\) in \((w, z)\) coordinates, and we have found that our trajectories are all asymptotic to \(w = \alpha\) as they approach \((\alpha, 0)\).

3. THE PROJECTIVE PLANE

Intuitively, the projective plane \(\mathbb{RP}^2\) is obtained from the ordinary plane \(\mathbb{R}^2\) by adjoining a 'circle at infinity' containing one point for each possible slope. To obtain useful coordinates, we describe this description from a different definition. We define the real projective plane \(\mathbb{RP}^2\) to be the quotient of \(\mathbb{R}^3 - \{0\}\) by the equivalence relation \((x, y, z) \sim (rx, ry, rz)\) for each \(r \neq 0\). Write \([x, y, z]\) for the equivalence class of \((x, y, z)\) so that \([x, y, z] = [rx, ry, rz]\) for each real \(r \neq 0\). This quotient is a smooth manifold with three natural coordinate patches:

\[
\begin{align*}
U_x &= \{[x, y, z] \mid x \neq 0\} = \{[1, y, z]\} \\
U_y &= \{[x, y, z] \mid y \neq 0\} = \{[x, 1, z]\} \\
U_z &= \{[x, y, z] \mid z \neq 0\} = \{[x, y, 1]\}.
\end{align*}
\]
These cover \( \mathbb{R}P^2 \) and each is diffeomorphic to \( \mathbb{R}^2 \). (Using algebraic topology we can show that this is the minimum number of coordinate patches required to cover \( \mathbb{R}P^2 \).) There are convenient diffeomorphisms \( \mathbb{R}^2 \to U_x \) and \( \mathbb{R}^2 \to U_y \).

\[
(x, y) \mapsto [x, y, 1], \quad [x, y, z] = \begin{bmatrix} x & y \\ z & 1 \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

and similarly for \( U_x \) and \( U_y \).

Each of these patches is nearly onto, missing only a ‘circle at infinity’, e.g., \( \mathbb{R}P^2 - U_z = \{[x, y, 0]\} \). This circle (or projective line) represents the possible slopes in the following sense. The line \( y = mx + b \) in \( \mathbb{R}^2 \) becomes \( y/z = m(x/z) + b \), or \( y = mx + bz \) in \( U_z \). The solutions to the latter equation in \( \mathbb{R}P^2 \) consist of the original line in \( U_z \) together with the point \([1, m, 0]\). Similarly, a vertical line \( x = c \) in \( \mathbb{R}^2 \) becomes \( x/z = c \), or \( x = cz \) in \( U_z \), which intersects the circle at infinity \( z = 0 \) in the point \([0, 1, 0]\). Thus, the complement of \( U_z \) has one point for each possible slope in the plane \( U_z \), allowing us to think of the projective plane as the plane \( \mathbb{R}^2 \) together with a circle of possible slopes “at infinity”. Note further that the two sides of the “circle at infinity” near a slope \( m \) correspond to the two directions in which one can “go to infinity” along a line of slope \( m \).

There is a good way to visualize this which goes back to the origins of projective geometry in perspective drawing. Imagine a vantage point situated at \((0, 0, 0)\) in \( \mathbb{R}^3 \) from which we regard a curve in the plane \( z = 1 \), parameterized as \((x, y, 1)\). The image of the curve under the transformation (2) is the curve in the plane \( x = 1 \), parameterized by \((1, w, z)\), which would produce the same image to the eye, since the eye cannot distinguish points which lie along a line through \((0, 0, 0)\).

Here, we write points in the plane \( U_z \) as \([1, w, z]\) to avoid confusion between different coordinates named \( y \). A point in the intersection \( U_z \cap U_x \) can then be written

\[
[x, y, 1] = [1, y, 1] = [1, w, z] = \begin{bmatrix} 1 & w \\ z & 1 \end{bmatrix}
\]

from which the coordinate transformations

\[
\begin{align*}
w &= y/x \\
z &= 1/x
\end{align*}
\]

(5)

are evident. Similarly, a point in the intersection \( U_z \cap U_y \) can be written

\[
[x, y, 1] = [x, 1, y] = [u, 1, v] = \begin{bmatrix} u & 1 \\ v & 1 \end{bmatrix}
\]

from which we obtain the coordinate transformations

\[
\begin{align*}
u &= x/y \\
v &= 1/y
\end{align*}
\]

(6)

To summarize, we write

\[
\begin{align*}
U_x &= \{[1, w, z]\} \\
U_y &= \{[u, 1, v]\} \\
U_z &= \{[x, y, 1]\}.
\end{align*}
\]

Then, in the intersections, these coordinates are related by the transformations (5) and (6).
4. Asymptotic slopes, intercepts and lines

The reader will no doubt have noticed a certain lack of precision in our statements about asymptotic slopes, such as \( \lim y/x = \alpha \) as \( x \to \pm \infty \). Is this the limit as the parameter \( t \) goes to \( \infty \), \( -\infty \), or some finite value? The answer is that it depends upon the parameterization, and, as we will see in Section 6, it will be very useful to alter the parameterization to remove singularities and to remove non-isolated stationary points. Thus, when we write \( \lim x \to \pm \infty \), we are making a statement about all trajectories on which \( |x| \) is unbounded.

There is also some possible ambiguity in the term *asymptotic slope*. We could mean \( \lim y/x \) or \( \lim dy/dx = \lim y'/x' \). Fortunately, L'Hôpital's rule assures us that the results agree.

Now, if \( \lim |x| = \infty \) and \( \lim y/x = \alpha \) for some curve \((x(t), y(t))\) in the \((x, y)\)-plane \(U_z\), then in \((w, z)\)-coordinates, we have \((w, z) \to (\alpha, 0)\). Thus, the curve \((x(t), y(t))\) has asymptotic slope \( \alpha \) if and only if the curve \((w(t), z(t))\) approaches \((\alpha, 0)\). This brings all finite asymptotic slopes in the \((x, y)\)-plane into the finite part of the \((w, z)\)-plane. To do the same with vertical asymptotes, we use the coordinates \(u = x/y\) and \(v = 1/y\). The curve \((x(t), y(t))\) has asymptotic reciprocal slope \( \lim x/y = \beta \) as \( y \to \infty \) if and only if the curve \((u(t), v(t))\) approaches \((\beta, 0)\). This brings all asymptotic behavior into view in either the \((w, z)\) or \((u, v)\) planes.

To study asymptotic lines, first note that the line \( y = mx + b \) transforms into \( w = m + bz \). Thus, if the tangent to \((w(t), z(t))\) at \((\alpha, 0)\) is the line \( w = \alpha + bz \), it makes sense to say that \( y = \alpha x + b \) is an asymptotic line to the curve \((x(t), y(t))\), since it is a better approximation to the curve than any other straight line. In this case we also call \( b \) the asymptotic intercept of \((x(t), y(t))\).

If the tangent is \( z = 0 \) then the asymptotic intercept is infinite. For example, the curve \( x = y^2 \) transforms into \( z = w^2 \). When \( z = 0 \), \( w \) is also 0, corresponding to the fact that the asymptotic slope \( \lim y/x \) of \( x = y^2 \) is 0. At \((w, z) = (0, 0)\) the tangent line to \( z = w^2 \) is \( z = 0 \), which cannot be written in the form \( w = \alpha + bz \). In the \((x, y)\) plane this corresponds to the fact that the \( y > 0 \) branch of \( x = y^2 \) passes above every horizontal line \( y = b \) as \( y \to \infty \), and similarly on the \( y < 0 \) branch. This is what it means for the asymptotic intercept to be infinite.

5. Application to polynomial vector fields

Let us apply these transformations to a general polynomial vector field

\[
\begin{align*}
x'(t) &= f(x, y) \\
y'(t) &= g(x, y)
\end{align*}
\]

If \( n = \deg f \) and \( m = \deg g \), let \( N = \max(n, m) \) and let us write \( f \) and \( g \) as sums of homogeneous polynomials

\[
\begin{align*}
f(x, y) &= \sum_{i=0}^{n} f_i(x, y) \\
g(x, y) &= \sum_{i=0}^{m} g_i(x, y)
\end{align*}
\]
where \( f_i \) and \( g_i \) are homogeneous of degree \( i \). We regard this as a flow in the patch \( U_z \) in \( \mathbb{RP}^2 \). To see what it looks like in \( U_z \), we note that

\[
\begin{align*}
  f\left(\frac{1}{z}, \frac{w}{z}\right) &= \sum_{i=0}^{n} \frac{f_i(1, w)}{z^i} \\
  g\left(\frac{1}{z}, \frac{w}{z}\right) &= \sum_{i=0}^{m} \frac{g_i(1, w)}{z^i}.
\end{align*}
\]

Therefore,

\[
\begin{align*}
  w' &= \frac{y'x - yx'}{x^2} = \sum_{i=0}^{N} \frac{g_i(1, w) - w f_i(1, w)}{z^{i-1}} \\
  z' &= -\frac{x'}{x^2} = -\sum_{i=0}^{N} \frac{f_i(1, w)}{z^{i-2}}.
\end{align*}
\]

Similarly, in \( U_y \), we find that

\[
\begin{align*}
  u' &= \frac{x'y - xy'}{y^2} = \sum_{i=0}^{N} \frac{f_i(u, 1) - u g_i(u, 1)}{v^{i-1}} \\
  v' &= -\frac{y'}{y^2} = -\sum_{i=0}^{m} \frac{g_i(u, 1)}{v^{i-2}}.
\end{align*}
\]

In the vast majority of cases, these vector fields in the \( U_x \) and \( U_y \) planes will be singular exactly where we wish to study them, so we now examine the effect of removing these singularities.

6. Rescaling time to remove singularities

Suppose \( (x(t), y(t)) \) is a trajectory of the vector field

\[
\begin{align*}
  x' &= f(x, y) \\
  y' &= g(x, y)
\end{align*}
\]

defined in some open set \( \Omega \) in the plane, and let \( h(x, y) \neq 0 \) in \( \Omega \). Suppose that \( (x(t), y(t)) \) is defined at \( t = t_0 \), and let

\[
\tau(t) = \int_{t_0}^{t} \frac{1}{h(x(s), y(s))} ds.
\]

It is then immediate that

\[
\begin{align*}
  \frac{dx}{d\tau} &= \frac{dx}{dt} \frac{dt}{d\tau} = f(x, y)h(x, y) \quad \text{and} \\
  \frac{dy}{d\tau} &= \frac{dy}{dt} \frac{dt}{d\tau} = g(x, y)h(x, y),
\end{align*}
\]

so that \( (x(\tau), y(\tau)) \) is a trajectory of this new vector field. We describe this by saying that the vector field has been rescaled by \( h \). The key observation is that the trajectories of the new differential equation are the same as those of the old one, but are travelled at
a different speed. This rescaling of time can have the effect of breaking a trajectory into several, separated by fixed points, or the reverse, when we include points where \( h = 0 \).

This technique can be used in two distinct ways. The first is the one we have already mentioned: if \( f \) and \( g \) are rational functions and we let \( h \) be the least common multiple of their denominators, the new vector field will be given by polynomials and is therefore defined everywhere. The trajectories of the original vector field will continue to be trajectories of the new vector field, except that the rescaling may allow old trajectories to join at points which were singularities for the original flow.

The second use is to remove nonisolated stationary points. If \( f \) and \( g \) have a common factor, then every point on the curve along which this common factor is zero will be a stationary point for the flow. Dividing by this common factor will not change the trajectories off this curve, but will make the flow nonzero at all but isolated points (assuming we divide by the greatest common divisor of \( f \) and \( g \)).

The virtue of this technique is that we can study how the original flow approaches its singular points by rescaling to remove the singularities so that the standard existence and uniqueness theorems now apply at the formerly singular points. This is what we shall use it for.

7. Determining Asymptotic Slopes

We can now continue the analysis of our polynomial vector field

\[
\begin{align*}
x' &= f(x, y) \\
y' &= g(x, y).
\end{align*}
\]

We retain the notation \( n = \deg(f) \), \( m = \deg(g) \) and \( N = \max(n, m) \), and the decomposition (8) of \( f \) and \( g \) into homogeneous parts from Section 5.

Theorem 3. The finite asymptotic slopes \( \lim y/x \) of a polynomial flow (7) of degree \( N \) are roots of the polynomial \( g_N(1, w) - wf_N(1, w) \), and the finite asymptotic reciprocal slopes \( \lim x/y \) are roots of the polynomial \( f_N(u, 1) - ug_N(u, 1) \).

There are at most \( N + 1 \) asymptotic slopes if \( g_N(1, w) - wf_N(1, w) \) is nonzero. Otherwise, the flow is a radial flow,

\[
\begin{align*}
x' &= xp(x, y) \\
y' &= yp(x, y)
\end{align*}
\]

up to terms of lower degree, and all but at most \( N - 1 \) asymptotic slopes occur. Those which do not occur are roots of \( f_N(1, w) \) and their reciprocals are roots of \( g_N(u, 1) \).

Proof: We begin by proving that the asymptotic slopes are roots of the polynomial \( g_N(1, w) - wf_N(1, w) \). In the \((w, z)\) plane the flow is given by (9). There are two cases to consider.

First, suppose that \( g_N(1, w) - wf_N(1, w) \neq 0 \). Then we rescale (9) by multiplying by \( z^{N-1} \), which gives the vector field

\[
\begin{align*}
w' &= g_N(1, w) - wf_N(1, w) + \sum_{i=0}^{N-1} z^{N-i}(g_i(1, w) - wf_i(1, w)) \\
z' &= -\sum_{i=0}^{N} z^{N-i+1} f_i(1, w).
\end{align*}
\]
Therefore, along $z = 0$, we have $z' = 0$ and $w' = g_N(1, w) - wf_N(1, w)$. If $w_0$ is not a root of this latter polynomial, then the flow through $(w_0, 0)$ lies in the $w$-axis, and no trajectory passing through a point $(w_1, z_1)$ with $z_1 \neq 0$ will approach $(w_0, 0)$. Thus, no solutions to (7) will have asymptotic slope $w_0$. It follows that the asymptotic slopes will all be roots of the polynomial $g_N(1, w) - wf_N(1, w)$.

Second, suppose that $g_N(1, w) - wf_N(1, w) = 0$. Note that the polynomials $f_n(1, w)$ and $g_m(1, w)$ are nonzero since $f_n(x, y)$ and $g_m(x, y)$ are. It therefore follows from $g_N(1, w) - wf_N(1, w) = 0$ that $n = m = N$. If we write
\begin{align*}
g_N(x, y) &= c_0 x^N + c_1 x^{N-1} y + \cdots + c_N y^N \\
f_N(x, y) &= d_0 x^N + d_1 x^{N-1} y + \cdots + d_N y^N
\end{align*}
then $g_N(1, w) = wf_N(1, w)$ implies that $c_0 = d_N = 0$ and otherwise $d_i = c_{i+1}$. It follows that $f_N = x p_{N-1}$ and $g_N = y p_{N-1}$, where
\begin{align*}
p_{N-1}(x, y) &= c_1 x^{N-1} + \cdots + c_N y^{N-1}.
\end{align*}

In this case we rescale by multiplying by $z^{N-2}$ to obtain
\begin{align*}
w' &= g_{N-1}(1, w) - wf_{N-1}(1, w) + \sum_{i=0}^{N-2} z^{N-i-1}(g_i(1, w) - wf_i(1, w))
\end{align*}
\begin{align*}
z' &= -f_N(1, w) - \sum_{i=0}^{N-1} z^{N-i} f_i(1, w),
\end{align*}
so, along $z = 0$, we have $z' \neq 0$ except at the roots of the polynomial $f_N(1, w)$. Thus, there are trajectories passing through all but these points of the $w$-axis, and all but the corresponding asymptotic slopes occur. Since every real number is a root of the zero polynomial, these asymptotic slopes are trivially roots of $g_N(1, w) - wf_N(1, w)$ in this case as well.

The analysis in the $(u, v)$ plane is obviously the same.

Now, to count the number of possible asymptotic slopes, first note that the condition that $g_N(1, w) - wf_N(1, w)$ be nonzero is the same as the condition that $f_N(u, 1) - ug_N(u, 1)$ be nonzero. Suppose first that this condition holds. Simply combining the results on asymptotic slopes and asymptotic reciprocal slopes, it might appear that we could have $N + 1$ finite asymptotic slopes together with asymptotic slope $\infty$, for a total of $N + 2$. But, if $\infty$ occurs, then 0 is a root of $f_N(u, 1) - ug_N(u, 1)$ and hence of $f_N(u, 1)$. Therefore the coefficient $d_N$ in (12) is 0 and $f_N(1, w)$ has degree at most $N - 1$. Therefore $g_N(1, w) - wf_N(1, w)$ has degree at most $N$ and there can be at most $N$ finite asymptotic slopes. Hence, there are at most $N + 1$ asymptotic slopes when the polynomial $g_N(1, w) - wf_N(1, w)$ is nonzero.

If $g_N(1, w) - wf_N(1, w) = 0$, we saw above that $f_N = x p_{N-1}$ and $g_N = y p_{N-1}$, so that the highest degree part of the flow (7) is simply a rescaled version of the radial outward flow
\begin{align*}
x' &= x \\
y' &= y
\end{align*}
which clearly has trajectories of every slope. We also saw that $f_N(1, w)$ is nonzero and of degree at most $N - 1$ since $d_N = 0$. Since the finite asymptotic slopes which do not occur must be roots of $f_N(1, w)$, there are at most $N - 1$ of them. As in the generic case, however, if asymptotic slope $\infty$ does not occur, then $g_N(0, 1) = 0$, so that $d_{N-1} = c_N = 0$ and $f_N(1, w)$ has degree $N - 2$. Thus, there are at most $N - 1$ asymptotic slopes omitted. Example 5
shows that this phenomenon, omitting a finite number of asymptotic slopes, can actually occur, and Example 6 shows that the potentially omitted slopes are not necessarily omitted.

**Exercise 4.** Apply the theorem to the linear system
\[
\begin{align*}
x' &= ax + by \\
y' &= cx + dy
\end{align*}
\]
to determine the asymptotic slopes of the solutions as functions of \(a, b, c,\) and \(d.\) Deduce the eigenvectors of
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
directly from this without first determining the eigenvalues.

**Example 5.** Consider
\[
\begin{align*}
x' &= xy \\
y' &= 1 + y^2.
\end{align*}
\]
We have \(f_3(x, y) = xy, g_2(x, y) = y^2, f_1 = g_1 = f_0 = 0,\) and \(g_0 = 1.\) Thus we are in the exceptional case \(g_2(1, w) - wf_2(1, w) = 0\) and all asymptotic slopes will occur except possibly slope 0, the unique root of \(f_2(1, w) = w.\) Since the solutions all lie on the hyperbolas \(x^2 = c(1 + y^2),\) there is no solution of asymptotic slope 0.

This system can easily be solved explicitly, but the transformed system
\[
\begin{align*}
w' &= z \\
z' &= -w
\end{align*}
\]
can be solved even more easily, and clearly satisfies \(w^2 + z^2\) constant, from which it trivially follows that \(x^2/(1 + y^2)\) is constant.

**Example 6.** The exceptional slopes may occur. Consider
\[
\begin{align*}
x' &= -xy^2 \\
y' &= -x - y^3
\end{align*}
\]
We have \(f_3(x, y) = -xy^2, g_3(x, y) = -y^3,\) and \(g_1(x, y) = -x,\) with all other terms zero. Again, the top degree term, \(g_3(1, w) - wf_3(1, w) = 0,\) and the exceptional slope is 0, the root of \(f_3(1, w) = -w^2.\) However, in this case, the transformed system
\[
\begin{align*}
w' &= -z \\
z' &= w^2
\end{align*}
\]
has two trajectories approaching the exceptional slope \(w = 0,\) one as \(t \to \infty\) and one as \(t \to -\infty,\) and correspondingly, the original system has a trajectory \(y^3 = -\frac{3}{2}x\) with asymptotic slope 0 as \(t \to \pm \infty.\) In general, the trajectories are given by \(y^3 = Cx^3 - \frac{3}{2}x,\) which has asymptotic slope \(C,\) and \(x = 0\) which has asymptotic slope \(\infty.\) Thus, all asymptotic slopes occur. As in the preceding example, this is most easily deduced by first noting that \(z^2 + \frac{2}{3}w^3\) is constant along solutions of the transformed system.
Example 7. The Riccati equation \( y' = y^2 - t \) can, of course, be solved explicitly in terms of Airy functions by the usual device of converting it into a second order linear equation. Our goal is more modest. Converting it into the autonomous system

\[
\begin{align*}
x' &= 1 \\
y' &= y^2 - x,
\end{align*}
\]

and applying the theorem, we see that the asymptotic slopes are roots of \( g_2(1, w) = w^2 \) and reciprocals of asymptotic slopes are roots of \(-ug_2(u, 1) = -u\). That is, the asymptotic slopes are 0 and \( \infty \). (See Figure 3.)

Let us examine the trajectories whose asymptotic slope is \( \infty \) more carefully. In the \((u, v)\)-plane, the desingularized flow is

\[
\begin{align*}
u' &= v^2 - u + u^2v \\
v' &= uv^2 - v.
\end{align*}
\]

The point \((u, v) = (0, 0)\) corresponds to infinite asymptotic slope, so we examine the flow in the vicinity of \((0, 0)\) in the \((u, v)\)-plane. The Jacobian is

\[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\]

so there are trajectories tangent to \( u = bv \) for every \( b \). These transform back to \( x = b \), so we have two trajectories asymptotic to \( x = b \) for every \( b \), one as \( t \) and \( y \to \infty \), and another as \( t \) and \( y \to -\infty \).
Example 8. More generally, any non-autonomous 1st order polynomial differential equation \( y' = g(t, y) \) can be converted to an autonomous 1st order system
\[
\begin{align*}
x' &= 1 \\
y' &= g(x, y).
\end{align*}
\]
If we assume that \( g \) is not a constant, then the theorem says that infinite slope is always possible, and that the possible finite asymptotic slopes are the roots of \( g_N(1, w) \).

Example 9. In a similar vein, consider a general autonomous 2nd order polynomial differential equation \( x'' = g(x, x') \). Converting to a 1st order system, we have
\[
\begin{align*}
x' &= y \\
y' &= g(x, y)
\end{align*}
\]
If \( g \) is not linear, the asymptotic slopes will be infinity and the roots of \( g_N(1, w) \) just as in the previous example. If \( g \) is linear, then the asymptotic slopes will be the roots of \( g_1(1, w) - w^2 \).

8. Determining asymptotic intercepts

Now that we have determined the asymptotic slopes which can occur, we wish to determine the asymptotic intercepts as well, to find all lines to which solutions are asymptotic. As we have seen in Section 4, this amounts to determining the slopes at which trajectories approach points on the \( w \)-axis in the \((w, z)\)-plane, or the \( u \)-axis in the \((u, v)\)-plane.

For the Riccati equation, Example 7, we were able to see that every vertical line is an asymptote of a solution, because the flow in the vicinity of \((u, v) = (0, 0)\) was especially simple, with trajectories approaching \((0, 0)\) in each direction.

To determine asymptotic intercepts in general, first note that for the non-generic case, in which
\[
g_N(1, w) - w f_N(1, w) = 0,
\]
the solution is quite easy. In the \((w, z)\)-plane there are trajectories passing through \((w, z) = (\alpha, 0)\) as long as \( \alpha \) is not a root of \( f_N(1, w) \), and in the appropriately rescaled flow \((13)\) we have
\[
\begin{align*}
w' &= g_{N-1}(1, \alpha) - \alpha f_{N-1}(1, \alpha) \\
z' &= f_N(1, \alpha)
\end{align*}
\]
there. Thus, the two trajectories in the \((x, y)\)-plane with asymptotic slope \( \alpha \) (as \( t \to \pm \infty \)) have asymptotic intercept
\[
\begin{align*}
w' &= g_{N-1}(1, \alpha) - \alpha f_{N-1}(1, \alpha) \\
z' &= f_N(1, \alpha)
\end{align*}
\]
In the generic case, there may be many trajectories approaching \((\alpha, 0)\). Blowing up this point will enable us to deal with this quite easily.
9. Blowing up

In its simplest form, blowing up replaces a point by the line of possible slopes at which one can pass through that point. The transformation is very simple. To blow up the origin, map \( \mathbb{R}^2 \to \mathbb{R}^2 \) by \((r, s) \mapsto (x, y) = (s, rs)\). Under this mapping, the entire \( r \)-axis, \( s = 0 \) is mapped to the origin, while its complement \( s \neq 0 \) is mapped diffeomorphically to \( x \neq 0 \). The line \( y = mx \) is the image of the line \( r = m \). Thus, a curve passing through \((0, 0)\) at slope \( m \) in the \((x, y)\)-plane passes through \((m, 0)\) in the \((r, s)\)-plane. As a consequence, curves passing through \((0, 0)\) at differing slopes no longer intersect in the \((r, s)\)-plane. For example, the curve \( y^2 = x^3 + x^2 \), which has a double point at the origin, becomes \( r^2 s^2 = s^3 + s^2 \), which, upon removing the common factor \( s^2 \), is \( r^2 = s + 1 \). This passes though the \( r \)-axis at \( r = \pm 1 \), corresponding to the slopes \( \pm 1 \) of the two branches of \( y^2 = x^3 + x^2 \) at the origin.

Since the line \( y = \alpha x + b \) transforms into \( w = \alpha + bz \), we shall blow up the point \((w, z) = (\alpha, 0)\) in such a way that \( b \) becomes the ‘blown-up’ coordinate. Thus, we let

\[
\begin{align*}
\alpha &= w = \alpha + rs \\
\beta &= s/z \\
\gamma &= s = z.
\end{align*}
\]

Then we are interested in studying the flow in the \((r, s)\)-plane in the vicinity of \((r, s) = (b, 0)\). Let us retain the notations of Sections 5 and 7. If \( \alpha \) is an asymptotic slope of our system, then Theorem 3 says that

\[
g_N(1, \alpha) - \alpha f_N(1, \alpha) = 0
\]

so we may factor

\[
g_N(1, w) - w f_N(1, w) = (w - \alpha) p_N(w)
\]

where \( p_N \) is a polynomial of degree at most \( N \). Similarly if \( \alpha \) is a reciprocal asymptotic slope of our system, then we may factor

\[
f_N(\alpha, 1) - u g_N(u, 1) = (u - \alpha) q_N(u)
\]

where \( q_N \) is a polynomial of degree at most \( N \).

**Theorem 10.** If \( y = \alpha x + b \) is asymptotic to a solution of the differential equation (7) then

\[
(p_N(\alpha) + f_N(1, \alpha)) b + (g_N - 1 - 1) f_N - 1 - 1 \alpha f_N - 1 - 1 = 0.
\]

and if \( x = \alpha y + b \) is asymptotic to a solution of (7) then

\[
(q_N(\alpha) + g_N(\alpha, 1)) b + (f_N - 1 - 1 \alpha - \alpha g_N - 1 - 1) = 0.
\]

**Proof:** We may rewrite the flow (11) in the \((w, z)\)-plane as

\[
\begin{align*}
w' &= (w - \alpha) p_N(w) + \sum_{i=0}^{N-1} z^{N-i-1} (g_i(1, w) - w f_i(1, w)) \\
z' &= -\sum_{i=0}^{n} z^{N-i+1} f_i(1, w).
\end{align*}
\]
The induced flow in the \((r, s)\)-plane is then
\[
\begin{align*}
    r' &= r\left(p_N(\alpha + rs) + f_N(1, \alpha + rs)\right) + \sum_{i=0}^{N-1} s^{N-i-1}(g_i(1, \alpha + rs) - \alpha f_i(1, \alpha + rs)) \\
    s' &= -\sum_{i=0}^{n} s^{N-i+1} f_i(1, \alpha + rs).
\end{align*}
\]
Along \(s = 0\), the blow up of the point \((\alpha, 0)\), we have
\[(15)\]
\[
\begin{align*}
    r' &= r\left(p_N(\alpha) + f_N(1, \alpha)\right) + (g_{N-1}(1, \alpha) - \alpha f_{N-1}(1, \alpha)) \\
    s' &= 0.
\end{align*}
\]
If \(r'\) is not identically 0 on the \(r\)-axis, then trajectories starting in \(s \neq 0\) can only approach \((b, 0)\) if \(r' = 0\) there. This is what we had to show. The analysis in the \((u, v)\)-plane is the same.

If the linear polynomial in Theorem 10 is not constant, then there is a unique asymptotic line at slope \(\alpha\). If the polynomial is constant and nonzero, then trajectories with asymptotic slope \(\alpha\) will have infinite intercept, as described in Section 4. If it is the zero polynomial, then the flow is stationary along the entire \(r\)-axis and we cannot make any conclusions about the way in which the flow off the \(r\)-axis approaches it. Thus, in this case the theorem gives us no information. When this happens, the flow in the \((r, s)\)-plane can be rescaled by dividing by an appropriate power of \(s\), and analysis of the resulting flow will tell us the asymptotic intercepts.

For example, if we blow up \((u, v) = (0, 0)\) for the Riccati equation Example 7, we get
\[
\begin{align*}
    r' &= s \\
    s' &= rs^3 - s,
\end{align*}
\]
which is stationary along the \(r\)-axis. Upon removing the common factor \(s\), the flow becomes
\[
\begin{align*}
    r' &= 1 \\
    s' &= rs^2 - 1,
\end{align*}
\]
which has trajectories passing through every point of the \(r\) axis, in agreement with our earlier conclusion that every vertical asymptote occurs.

**Example 11.** As a final example of the use of blowing up, consider
\[
\begin{align*}
    x' &= y^2 - x^2 \\
    y' &= xy
\end{align*}
\]
(see Figure 4). The Jacobian at the origin is zero, so yields no information about trajectories there. Examining the nullclines suggests that there are trajectories approaching the origin in each quadrant at slopes greater than 1 in absolute value, in addition to the two trajectories in the \(x\)-axis. Blowing up the origin by \(x = s\) and \(y = rs\) yields the flow
\[
\begin{align*}
    s' &= s(r^2 - 1) \\
    r' &= r(2 - r^2)
\end{align*}
\]
after dividing out a common factor of $s$ (see Figure 5). Along $s = 0$ we have $s' = 0$ and $r' = r(2 - r^2)$, so the trajectories approaching $(x, y) = (0, 0)$ have slopes 0 and $\pm \sqrt{2}$. The Jacobians at $(s, r) = (0, 0)$ and $(s, r) = (0, \pm \sqrt{2})$ have one positive and one negative eigenvalue, so there are exactly two trajectories approaching each of these three points from $s \neq 0$. Hence, there are exactly two trajectories of each of these three slopes approaching the origin in the original flow.

Now that we know the slopes, we observe that if $y \pm \sqrt{2}x = 0$ then $(y \pm \sqrt{2}x)' = 0$ Thus, each line $y = \pm \sqrt{2}x$ contains two trajectories, one approaching $(0, 0)$ as $t$ increases and the other approaching $(0, 0)$ as $t$ decreases.

Applying Theorem 3, we see that the asymptotic slopes are also 0 and $\pm \sqrt{2}$, being the roots of $g_2(1, w) - w^2 = w(2 - w^2)$. Finally Theorem refsecondtheorem tells us that the asymptotic intercepts are all 0, so that all trajectories are asymptotic to the 6 straight line trajectories of slopes 0 and $\pm \sqrt{2}$. We therefore have a flow with 6 hyperbolic sectors bounded by the straight lines $y = 0$ and $y = \pm \sqrt{2}x$.

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