HW 6  §7 #1a,c,e, 5
§8 #1,2

(a) \( V(y) = (x) \)

\[ x' = x \Rightarrow x = x_0 e^t \]
\[ y' = y \Rightarrow y = y_0 e^t \]

Then \( \frac{y}{x} = \frac{y_0}{x_0} \) is constant.

(\( \theta \) is an unstable node)

(c) \( V(y) = (2y, -x) \)

\[ x'' = (2y)' = -2x \]

So \( p(x) = \int -2x \, dx = x^2 \)

is our "potential energy".

Then \( \frac{1}{2} (x')^2 + x^2 = \frac{1}{2} (2y)^2 + x^2 = 2y^2 + x^2 \)

is constant.

(\( \theta \) is a center (stable))

(e) \( V(y) = (x) \)

\[ x' = x \Rightarrow x = x_0 e^t \]
\[ y' = -y \Rightarrow y = y_0 e^{-t} \]

So \( xy = x_0 y_0 \) is constant

Saddle (unstable)
For Figure 7.4 in the book

\[ C = \#ABC - \#ACB = 3 - 3 = 0 \]

\[ I_1 = \#AB - \#BA \text{ outside} = 2 - 1 = 1 \]

\[ I_2 = \#AB - \#BA \text{ inside} = 1 - 0 = 1 \]

So \( C = I_1 - I_2 \) for the annulus.

In general, let \( c_1 \) and \( c_2 \) be edges or sequences of edges which connect the inner and outer circles, as below:

These split the annulus into two cells

Then the Index Lemma applies to each cell, so we get

\[
\text{Content (D}_1\text{)} = I'_1 + I(c_1) - I'_2 + I(c_2)
\]

\[
\text{Content (D}_2\text{)} = I''_1 - I(c_1) - I''_2 - I(c_2)
\]

where \( I(c_1), I(c_2) \) is the index along \( c_1 \) and \( c_2 \).

So

\[
\text{Content (Annulus)} = \text{Content (D}_1\text{)} + \text{Content (D}_2\text{)}
\]

\[
= I'_1 - I'_2 + I''_1 - I''_2
\]

\[
= I_1 - I_2
\]

Since \( I(c_1) \) and \( I(c_2) \) cancel.
\( V(x) = (1-x^2) \) points due north when \( y=0 \) and \( 1-x^2>0 \).

(a) On \( x^2+y^2=2x \) this requires \( x=0 \) or \( 2 \) and \( 1-x^2>0 \), i.e. \( x=y=0 \).

Before (9), \( y>0 \), so \( V \) points in region A, and after (9), \( y<0 \), so \( V \) points into region B.

Hence \( W = +1 \).

(b) On \( x^2+y^2=-2x \), again (9) is the only place \( V \) points north.

Before (9), \( y<0 \), and after (9), \( y>0 \), so we have a BA transition. \( W = -1 \).

(c) On \( x^2+y^2=2y \), \( y=0 \) gives \( x=0 \) so (9) is again the only north pointing point. Before (9), and after (9), \( y>0 \), so we have an AA non-transition. \( W = 0 \).

(d) On \( x^2+y^2=-2y \), similarly, (9) is the only point where \( V \) points north, and \( y>0 \) both before and after this, so \( W = 0 \).
2. \( V(y) = \left( \frac{y(x^2-1)}{x(y^2-1)} \right) \) points due north when

\[(y = 0 \text{ or } x = \pm 1) \text{ and } x(y^2-1) > 0 \] (shaded below)

\[
\begin{array}{c|c|c}
\text{x-coord} & + & - & + \\
\hline
y=0 & + & - & y=1 \\
\hline
y=1 & - & + & y=0 \text{ (pos where shaded)} \\
\hline
y=-1 & - & + & y=1 \\
\hline
x=1 & - & + & x=0 \\
\hline
x=-1 & + & - & x=1 \\
\hline
\end{array}
\]

(a) \[ x^2 + y^2 - 2x - 2y + 1 = 0 \]
\[ (x-1)^2 + (y-1)^2 = 1 \]

One due north point \((1, \frac{1}{2})\). \(AB \) so \(W = 1\)

(b) \[ x^2 + y^2 + x + y = \frac{1}{2} \]
\[ (x + \frac{1}{2})^2 + (y + \frac{1}{2})^2 = 1 \]

North pointing points:
\[
\begin{pmatrix}
-1 \\
(\sqrt{3} - \frac{1}{2})
\end{pmatrix} \quad \text{B to A}
\]
\[
\begin{pmatrix}
-\sqrt{3} - 1/2 \\
0
\end{pmatrix} \quad \text{A to B}
\]

W = 0

(c) \((-1,0)\) only \(W = -1\)

(d) \(\pm \frac{1}{\sqrt{3}}\) and \((-2,0)\) \(W = 3\)