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Math 7510, Fall 2009, Homework 10
due 11 December 2009

These are taken from Hatcher, pp. 228-9.

1. (a) Use the product $H^*(X, A) \otimes H^*(X, B) \longrightarrow H^*(X, A \cup B)$ to show that if X is the union of two contractible open subsets A and B, then all products of positive dimensional classes in $H^*(X)$ are 0. (This applies, for example, to any suspension.)

   (b) Show that if X is the union of n contractible open subsets, then all n-fold products of positive dimensional classes are 0.

2. (a) Use the product structure of $H^*(\mathbf{RP}^n; \mathbf{Z}/2)$ to show that if $n > m$ then there is no map $\mathbf{RP}^n \longrightarrow \mathbf{RP}^m$ which induces a nontrivial homomorphism on $H^1(\ ; \mathbf{Z}/2)$. State and prove the analogous statement for complex projective spaces.

   (b) Use the first part to show that for every map $g : S^n \longrightarrow \mathbf{R}^n$ there is an $x \in S^n$ such that $g(x) = g(-x)$, as follows. Suppose not. Then

   $$f(x) = \frac{(g(x) - g(-x))}{|g(x) - g(-x)|}$$

   defines a map $f : S^n \longrightarrow S^{n-1}$ satisfying $f(-x) = -f(x)$. Then ....

3. Let $f : \mathbf{CP}^n \longrightarrow \mathbf{CP}^n$ be induced by $(z_0, z_1, \ldots, z_n) \mapsto (z_0^d, z_1^d, \ldots, z_n^d)$ for an integer $d > 0$. Compute $f^* : H^*\mathbf{CP}^n \longrightarrow H^*\mathbf{CP}^n$. Hint: start with $n = 1$.

4. Show that if $j, k > 0$ then every map $S^{k+j} \longrightarrow S^k \times S^j$ induces the trivial homomorphism in *homology*, $H_{k+j}(S^{k+j}) \longrightarrow H_{k+j}(S^k \times S^j)$.

5. Let $q : \mathbf{RP}^\infty \longrightarrow \mathbf{CP}^\infty$ be the natural quotient map induced by regarding both projective spaces as quotients of S^∞, the unit sphere in $\mathbf{C}^\infty$. Show that $q^* : H^*(\mathbf{CP}^\infty, \mathbf{Z}) \longrightarrow H^*(\mathbf{RP}^\infty, \mathbf{Z})$ is the evident quotient map $\mathbf{Z}[y] \longrightarrow \mathbf{Z}[y]/(2y)$. Hint: it is sufficient to study the restriction of q to $\mathbf{RP}^2$, $q : \mathbf{RP}^2 \longrightarrow \mathbf{CP}^1 = S^2$, and then use the product structure.