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Math 7510, Fall 2009, Homework 4
due 2 October 2009

1. Let $p : E \longrightarrow B$ be a covering with E and B connected. Show that p is a homeomorphism iff $p_* : \pi_1(E) \longrightarrow \pi_1(B)$ is an isomorphism.

2. Let $p : S^1 \longrightarrow S^1$ be $p(z) = z^k$. What element in $\pi_1(S^1)$ is represented by p? Hint: what sort of map is p?

   The remaining problems are concerned with covers of $\mathbf{R}P^2 \vee \mathbf{R}P^2$. For $0 < r \in \mathbf{R}$ and $p \in \mathbf{R}^3$, let $C_r(p) = \{v \mid |v - p| = r\}$ be the 2-sphere of radius r centered at $p \in \mathbf{R}^3$. The space

   $$E = \bigcup_{n \in \mathbf{Z}} C_{1/2}(n, 0, 0)$$

   is the universal cover of $\mathbf{R}P^2 \vee \mathbf{R}P^2$. The covering map $p : E \longrightarrow \mathbf{R}P^2 \vee \mathbf{R}P^2$ is

   $$p(x, y, z) = \begin{cases} ([x - 2n, y, z], *) & N(x) = 2n \\ (*, [x - 2n - 1, y, z]) & N(x) = 2n + 1 \end{cases}$$

   where $N(x)$ is the nearest integer to x. Though N isn't well-defined, p is, if we choose $[1, 0, 0] = [1/2, 0, 0]$ as our basepoint, since

   $$([1/2, 0, 0], *) = (*, *) = (*, [1/2, 0, 0]).$$

   The group of covering transformations, $G = Cov(p) = \langle a, b \mid a^2 = b^2 = 1 \rangle$, where

   $$a(x, y, z) = (-x, -y, -z) \qquad \text{and} \qquad b(x, y, z) = (2 - x, -y, -z).$$

   There is another presentation of G which may be more useful:

   $$G = \langle a, c \mid a^2 = 1, aca = c^{-1} \rangle.$$

   The element $c = ba$ and generates an infinite cyclic subgroup with quotient $G/\langle c \rangle \cong C_2$. It acts on E by

   $$c(x, y, z) = (x + 2, y, z)$$

3. Show that G acts freely upon E.

4. Show that $p(v_1) = p(v_2)$ iff there exists $g \in G$ such that $g(v_1) = v_2$. Conclude that the elements of G are covering transformations and that p is the quotient map $E \longrightarrow E/G$.

   Since $\pi_1(E) = 0$, this shows that $G = \pi_1(\mathbf{R}P^2 \vee \mathbf{R}P^2)$.

   For each of the following subgroups $H < G$

   (a) Describe the covering $p' : E' \longrightarrow \mathbf{R}P^2 \vee \mathbf{R}P^2$ with $p'_* \pi_1(E') = H$. (A drawing will suffice.)

(b) Exhibit lifts of the generators a and b to E', both starting at your chosen basepoint of E'. (Again, a drawing will suffice.)

(c) Compute the group of covering transformations $Cov(p')$ and describe its effect on E'.

5. $\langle a \rangle$

6. $\langle (ab)^2 \rangle$

7. $\langle a, bab \rangle$

8. $\langle (ab)^4 \rangle$