HW 5

1. Exer 9.7 on p.48

Define $\partial^{\#} : S_0(X) \to R$ by $\partial^{\#}(\sum r_\sigma \sigma) = \sum r_\sigma$. Show

3 (i) $\partial^{\#}\partial : S_1(X) \to R$ is 0, so we may define $H_*^{\#} = \widetilde{H}_*$, the reduced homology.

(ii) X path conn. $\Rightarrow \widetilde{H}_0(X) = 0$

(iii) $\widetilde{H}_0(X) \cong R^{r-1}$ if X has r path components.

Proof: (i) $\partial^{\#}\partial(\sum r_\sigma \sigma) = \partial^{\#} \sum r_\sigma (\sigma(1) - \sigma(0)) = \sum r_\sigma (1-1) = 0$.

(ii) Follows from (iii) but directly: Suppose $z \in \widetilde{Z}_0(X)$. Then $z = \sum r_\sigma \sigma$ and $\sum r_\sigma = 0$. For each σ, let $\tau_\sigma : \Delta_1 \to X$ with $\tau_\sigma(0) = x_0$, a fixed chosen basepoint in X, and $\tau_\sigma(1) = \sigma(0)$. Then $\partial \sum r_\sigma \tau_\sigma = \sum r_\sigma \sigma - (\sum r_\sigma) x_0 = \sum r_\sigma \sigma = z$. Hence $\widetilde{Z}_0 = B_0$ and $\widetilde{H}_0 = \widetilde{Z}_0 / B_0 = 0$.

(iii) Suppose $X = X_1 \cup \cdots \cup X_r$ is the decomposition of X into path components. Choose $x_i \in X_i$ for each i. Suppose $z = \sum r_\sigma \sigma \in S_0(X)$. Let τ_σ, as in (ii), be a 1-simplex with $\tau_\sigma(1) = \sigma$ and $\tau_\sigma(0) = x_{i(\sigma)}$, where $\sigma \in X_{i(\sigma)}$. Then

$$\partial \sum r_\sigma \tau_\sigma = \sum r_\sigma \sigma - \sum r_\sigma x_{i(\sigma)} .$$

Hence $S_0(X) = B_0(X) + \bigoplus_{i=1}^{r} R x_i$. Since $\partial^{\#}(B_0(X)) = 0$, $\partial^{\#}$ factors as $S_0(X) \to \frac{S_0(X)}{B_0(X)}$ followed by $\bar{\partial} : \frac{S_0(X)}{B_0(X)} \to R$. Since $\bar{\partial}(\sum r_\sigma \sigma + B_0) = \partial^{\#}(\sum r_\sigma \sigma) = \sum r_\sigma$, $\operatorname{Ker}(\bar{\partial}) = \widetilde{Z}_0(X) / B_0(X) = \widetilde{H}_0(X)$. Finally if $\bar{\partial}(\sum r_i x_i + B_0) = \sum r_i = 0$ then $\sum_{i=1}^{r} r_i x_i = \sum_{i=1}^{r} r_i x_i - (\sum_{i=1}^{r} x_i) x_1 = \sum_{i=2}^{r} r_i (x_i - x_1)$.

Hence $\widetilde{Z}_0 / B_0 = \bigoplus_{i=2}^{r} R(x_i - x_1) \cong R^{n-1}$. //

(2.) Exer 9.12 on p.50

Let X be an affine space with $A, B \in X$, $A \neq B$. Let

$$\sigma = (A,B) \quad \text{and} \quad \tau = (B, A).$$

Then $-\sigma \neq \tau$ but $\sigma \sim -\tau$.

Proof: $S_1(X)$ is the free R-module on 1-simplices. The 1-simplices $\sigma : \Delta_1 \to X$ and $\tau : \Delta_1 \to X$ are distinct since $\sigma(0) = A \neq B = \tau(0)$.

Hence a linear combination $a\sigma + b\tau = 0$ iff $a = b = 0$. Thus $\sigma \neq -\tau$.

Now $$\partial\big((A,B,A) + (A,A,A)\big) = (B,A) - (A,A) + (A,B) + (A,A) - (A,A) + (A,A)$$
$$= \tau + \sigma$$

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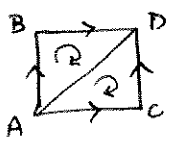
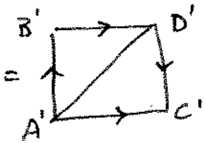
and therefore $\sigma \sim -\tau$. //

(3.) Exer 9.13 on p.15

$T =$ 

$K =$ 

$I^2 \xrightarrow{p} T$

$I^2 \xrightarrow{p'} K$

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$\sigma_1 = p \circ (A,B,D)$
$\sigma_2 = p \circ (A,C,D)$

$P = p(A) = p(B) = p(C) = p(D)$

$\tau_1 = p' \circ (A', B', D')$
$\tau_2 = p' \circ (A', D', C')$

$\omega = p' \circ (A', B')$
$= p' \circ (D', C')$

$Q = p'(A') = p'(B') = p'(C') = p'(D')$

Show $\partial(\sigma_1 - \sigma_2) = 0$
$\partial(\tau_1 + \tau_2) = 2\omega$, $\partial\omega = 0$.

Thus if $R = \mathbb{Z}$, $H_1(K) \supset \mathbb{Z}/2$
if $R = \mathbb{Z}/2$, $H_2(K) \supset \mathbb{Z}/2$, $H_1(K) \supseteq \mathbb{Z}/2$.

Proof: $$\partial(\sigma_1 - \sigma_2) = p\big(\partial(A,B,D) - \partial(A,C,D)\big)$$
$$= p\big((B,D) - (A,D) + (A,B) - (C,D) + (A,D) - (A,C)\big)$$

$$= p\big((B,D) - (A,C)\big) + p\big((A,B) - (C,D)\big) = 0 + 0 = 0.$$

$$\partial(\tau_1 + \tau_2) = p'\big(\partial(A',B',D') + \partial(A',D',C')\big)$$
$$= p'\big((B',D') - (A',D') + (A',B') + (D',C') - (A',C') + (A',D')\big)$$
$$= p'\big((B',D') - (A',C')\big) + p'\big((A',B') + (D',C')\big) = 0 + 2\omega = 2\omega$$

$\partial\omega = p'\big((B') - (A')\big) = 0$

$R = \mathbb{Z}/2$: $\langle[\omega]\rangle \cong \mathbb{Z}/2 \subseteq H_1(K; \mathbb{Z}/2)$
$\langle[\tau_1 + \tau_2]\rangle \cong \mathbb{Z}/2 \subseteq H_2(K; \mathbb{Z}/2)$

$R = \mathbb{Z}$: $\langle[\omega]\rangle \cong \mathbb{Z}/2 \subseteq H_1(K)$;

4. Exer 10.14 on p.57

Let $Q \in X$ and $c: X \to X$ be constant: $c(x) = Q$ $\forall x \in X$.

(i) Find $S_*(c) = c_* : S_*(X) \longrightarrow S_*(X)$

Let $c' : S_*(X) \longrightarrow S_*(X)$ be $c'(\sigma) = \begin{cases} 0 & |\sigma|>0 \\ \sigma_0 & |\sigma| = 0 \end{cases}$ where $\sigma_n : \Delta_n \to X$ is constant: $\sigma_n(t) = Q$ $\forall t \in \Delta_n$.

(ii) Show c' is a chain map

(iii) show $c' \simeq c_*$.

Proof. (i) If $(\sigma : \Delta_n \to X) \in S_n(X)$ then $c_*(\sigma) = \left(\Delta_n \xrightarrow{\sigma} X \xrightarrow{c} X\right) = \sigma_n$. Hence

$c_*\left(\sum r_\sigma \sigma\right) = \left(\sum r_\sigma\right) \sigma_n$. Note $\partial \sigma_n = \partial(Q, Q, \ldots, Q) = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases}$.

(ii)

$$\begin{tikzcd} S_0(X) \arrow[d, "c'"'] & S_1(X) \arrow[l, "\partial"'] \arrow[d, "c'"] \\ S_0(X) & S_1(X) \arrow[l, "\partial"] \end{tikzcd} \quad \text{is} \quad \begin{tikzcd} \sigma(1)-\sigma(0) \arrow[d, mapsto] & \sigma \arrow[l, mapsto] \arrow[d, mapsto] \\ \sigma_0 - \sigma_0 = 0 & 0 \arrow[l, mapsto] \end{tikzcd} \quad \text{okay.}$$

$$\begin{tikzcd} S_n(X) \arrow[d, "c'"'] & S_{n+1}(X) \arrow[l, "\partial"'] \arrow[d, "c'"] \\ S_n(X) & S_{n+1}(X) \arrow[l, "\partial"] \end{tikzcd} \quad \text{is } 0 = 0 \text{ for } n>0.$$

(iii)

$$(c' - c_*)(\sigma) = \begin{cases} \sigma_0 - \sigma_0 & |\sigma| = 0 \\ 0 - \sigma_n & |\sigma| = n > 0 \end{cases} = \begin{cases} 0 & |\sigma| = 0 \\ -\sigma_n & |\sigma| = n > 0 \end{cases}.$$

Let $T : S_{2i}(X) \longrightarrow S_{2i+1}(X)$ be 0 and $T : S_{2i-1}(X) \longrightarrow S_{2i}(X)$ be $T(\sigma) = -\sigma_{2i}$.

Then $\partial T + T \partial = c' - c_*$:

[More simply: $T : S_i(X) \to S_{i+1}(X)$ by $T(\sigma) = -\sigma_{i+1}$ also works.]

$$\begin{tikzcd} S_0(X) \arrow[d, "c'-c_*"'] \arrow[dr, "T"] & \\ S_0(X) & S_1(X) \arrow[l, "\partial"] \end{tikzcd} \qquad \begin{tikzcd} \sigma \arrow[d, mapsto] \arrow[dr, mapsto] & \\ 0 & 0 \arrow[l, mapsto] \end{tikzcd} \quad \text{commutes}$$

[AND, there are others]

$$\begin{tikzcd} S_{2i}(X) \arrow[dr, "T"'] & S_{2i+1}(X) \arrow[l, "\partial"'] \arrow[d, "c'-c_*"] \arrow[dr, "T"] & \\ & S_{2i+1}(X) & S_{2i+2}(X) \arrow[l, "\partial"] \end{tikzcd}$$

$(\partial T + T\partial)(\sigma) = -\partial \sigma_{2i+2} + 0 = -\sigma_{2i+1}$

$(c' - c_*)(\sigma) = -\sigma_{2i+1}$ okay.

$$\begin{tikzcd} S_{2i-1}(X) \arrow[dr, "T"'] & S_{2i}(X) \arrow[l, "\partial"'] \arrow[d, "c'-c_*"] \arrow[dr, "T"] & \\ & S_{2i}(X) & S_{2i+1}(X) \arrow[l, "\partial"] \end{tikzcd}$$

$(\partial T + T\partial)(\sigma) = 0 + T(\partial \sigma) = -\sigma_{2i}$

$(c' - c_*)(\sigma) = -\sigma_{2i}$

okay.

So $T : c' \simeq c_*$. //

(5.) Exer 10.15 p. 57

Show that chain homotopy is an equivalence relation.

Proof: Reflexive: $f \simeq f$ since $\partial \cdot 0 + 0 \cdot \partial = f - f$.

Symmetric: $f \simeq g \Rightarrow \exists \tau$ s.t. $\partial \tau + \tau \partial = f - g$

$\Rightarrow \partial(-\tau) + (-\tau)\partial = g - f \Rightarrow g \simeq f$.

Transitive: $f \simeq g$ and $g \simeq h \Rightarrow \exists \tau, \sigma$ s.t. $\partial \tau + \tau \partial = f - g$

and $\partial \sigma + \sigma \partial = g - h$

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$\Rightarrow \partial(\tau + \sigma) + (\tau + \sigma)\partial = f - g + g - h = f - h \Rightarrow f \simeq h$. //

(6.) Exer 10.16 p. 58

Show that if $f_1 \simeq f_2$ and $g_1 \simeq g_2$ then $f_1 g_1 \simeq f_2 g_2$.

Proof: Suppose $\partial \tau + \tau \partial = f_1 - f_2$ and $\partial \sigma + \sigma \partial = g_1 - g_2$.

Then $\partial(g_1 \tau + \sigma f_2) + (g_1 \tau + \sigma f_2)\partial$

$= g_1(\partial \tau + \tau \partial) + (\partial \sigma + \sigma \partial) f_2$

$= g_1(f_1 - f_2) + (g_1 - g_2) f_2$

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$= g_1 f_1 - g_1 f_2 + g_1 f_2 - g_2 f_2$

$= g_1 f_1 - g_2 f_2$. //