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Math 7510, Fall 2009, Homework 6
due 21 October 2009

1. Compute the homology of the following complexes.

- (a) $C_n = \mathbf{Z} \oplus \mathbf{Z}$ for $0 \leq n \leq 3$ and $C_n = 0$ otherwise. $\partial_n(x, y) = (x - y, y - x)$ for $n = 1, 3$ and $\partial_2(x, y) = (x + y, x + y)$.
- (b) $C_n = R$ for $n \geq 0$. $\partial_{2i} = 2$ (i.e., multiplication by 2) and $\partial_{2i+1} = 0$, where R is \mathbf{Z} , $\mathbf{Z}/2\mathbf{Z}$, $\mathbf{Z}/3\mathbf{Z}$, $\mathbf{Z}[1/2]$ (rationals with denominator a power of 2), and $\mathbf{Z}_{(2)}$ (rationals with odd denominator). (So this is really 5 closely related problems.)

2. Prove the 5-Lemma, in the following form: given a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 A_1 & \xrightarrow{i_1} & A_2 & \xrightarrow{i_2} & A_3 & \xrightarrow{i_3} & A_4 & \xrightarrow{i_4} & A_5 \\
 \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\
 B_1 & \xrightarrow{j_1} & B_2 & \xrightarrow{j_2} & B_3 & \xrightarrow{j_3} & B_4 & \xrightarrow{j_4} & B_5
 \end{array}$$

- (a) If f_1 is epi and f_2 and f_4 are mono, then f_3 is mono.
- (b) If f_5 is mono and f_2 and f_4 are epi, then f_3 is epi.
3. Let $\iota : \Delta_1 \longrightarrow S^1$ be the singular 1-cycle $\iota(t) = e^{2\pi it}$, and let $\iota \in H_1(S^1)$ also denote its homology class. Define $h : \pi_1(X, x_0) \longrightarrow H_1(X)$ as follows. For $\alpha \in \pi_1(X, x_0)$, $h(\alpha) = \alpha_*(\iota)$. Show
- (a) h is well-defined, i.e., depends only upon the homotopy class α .
- (b) h is a homomorphism. (Hint: construct a 2-simplex whose boundary is $h(\alpha) + h(\beta) - h(\alpha\beta)$.)
- (c) h is onto, if X is path connected.
- (d) $\text{Ker } h$ is the commutator subgroup of $\pi_1(X)$