HW 6

1. Compute the homology of

(a)

$$\mathbb{Z}^2 \xleftarrow{\begin{pmatrix}1&-1\\-1&1\end{pmatrix}} \mathbb{Z}^2 \xleftarrow{\begin{pmatrix}1&1\\1&1\end{pmatrix}} \mathbb{Z}^2 \xleftarrow{\begin{pmatrix}1&-1\\-1&1\end{pmatrix}} \mathbb{Z}^2$$

Z_i	$\mathbb{Z}^2$	$\langle \binom{1}{1} \rangle$	$\langle \binom{1}{-1} \rangle$	$\langle \binom{1}{1} \rangle$
B_i	$\langle \binom{1}{-1} \rangle$	$\langle \binom{1}{1} \rangle$	$\langle \binom{1}{-1} \rangle$	0
H_i	$\mathbb{Z}$	0	0	$\mathbb{Z}$

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(b) For $R = \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}[\frac{1}{2}], \mathbb{Z}_{(2)}$ compute the homology of

$$R \xleftarrow{0} R \xleftarrow{2} R \xleftarrow{0} R \xleftarrow{2} \cdots$$

$R_2 = \{x \in R \mid 2x = 0\}$

general R: Z_i	R	R	R_2	R		H_{2i}	H_{2i+1}
B_i	0	(2)	0	(2)			
H_i	R	$R/(2)$	R_2	$R/(2)$	$\cdots$	R_2	$R/(2)$
$R = \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/(2)$	0	$\mathbb{Z}/(2)$	$\cdots$	0	$\mathbb{Z}/(2)$
$= \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\cdots$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$= \mathbb{Z}/3$	$\mathbb{Z}/3$	0	0	0	$\cdots$	0	0
$= \mathbb{Z}[\frac{1}{2}]$	$\mathbb{Z}[\frac{1}{2}]$	0	0	0	$\cdots$	0	0
$= \mathbb{Z}_{(2)}$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\cdots$	0	$\mathbb{Z}/(2)$

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2. 5-Lemma

(a)

$f_3 a_3 = 0$	hypothesis
$f_4 i_3 a_3 = 0$	$f_4 i_3 = j_3 f_3$
$i_3 a_3 = 0$	f_4 mono
$a_3 = i_2 a_2$	exactness
$j_2 f_2 a_2 = 0$	$j_2 f_2 = f_3 i_2$
$f_2 a_2 = j_1 b_1$	exactness
$f_2 a_2 = j_1 f_1 a_1 = f_2 i_1 a_1$	f_1 epi
$a_2 = i_1 a_1$	f_2 mono
$a_3 = i_2 i_1 a_1 = 0$	exactness //

4

(b)

$b_3 \in B_3 \Rightarrow j_3 b_3 = f_4 a_4$	f_4 epi
$f_5 i_4 a_4 = j_4 j_3 b_3 = 0$	exactness
$i_4 a_4 = 0$	f_5 mono
$a_4 = i_3 a_3$	exactness
$j_3 (f_3 a_3 - b_3) = 0$	$j_3 f_3 = f_4 i_3$
$b_3 = f_3 a_3 + j_2 b_2$	exactness
$b_3 = f_3 a_3 + f_3 i_2 a_2$	f_2 epi and $f_3 i_2 = j_2 f_2$
$b_3 = f_3 (a_3 + i_2 a_2)$ //	

Problem 3

Lemmas about $h : \pi_1(X, x_0) \to H_1(X)$

(1.) If α, β are paths $I \to X$ which can be concatenated ($\alpha(1) = \beta(0)$) then $\alpha * \beta : \Delta_1 = I \longrightarrow X$ is homologous to $\alpha + \beta$ (the sum of $\alpha : \Delta_1 \longrightarrow X$ and $\beta : \Delta_1 \longrightarrow X$)

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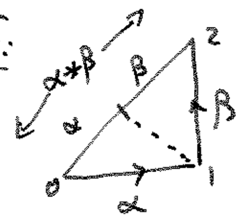
Pf:



The boundary of the 2-simplex $\Delta_2 \xrightarrow{\sigma} X$ obtained by orthogonal projection onto the edge (02) followed by the map $\alpha * \beta$ from that edge to X is

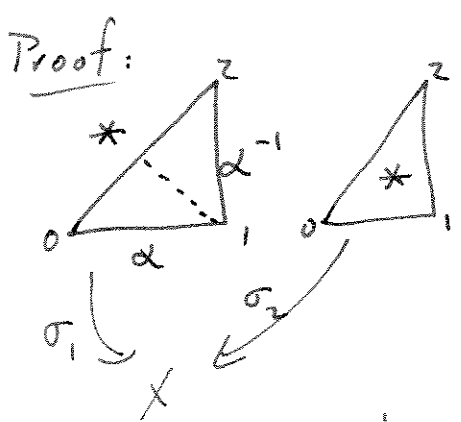
$$\partial \sigma = \beta - \alpha * \beta + \alpha . \quad \text{Hence } \alpha + \beta \simeq \alpha * \beta. //$$

(Note: this applies to any paths which can be concatenated, not just to loops.)

(2.) If $\alpha : \Delta_1 = I \longrightarrow X$ and $\alpha^{-1} : \Delta_1 \to X$ is its "inverse"

2 $\alpha^{-1}(t) = \alpha(1-t)$, then $\alpha + \alpha^{-1} \simeq 0$, i.e. $-\alpha \simeq \alpha^{-1}$.

Proof:



Let $\sigma_1 : \Delta_2 \to X$ be a nullhomotopy of $\alpha\alpha^{-1}$ as shown

α on (01)

α^{-1} on (12)

$*$ on (02) $\quad$ ($* = x_0$ of course)

Let $\sigma_2 : \Delta_2 \to X$ ~~be~~ the constant two simplex.

Then $\partial(\sigma_1 + \sigma_2) = \alpha^{-1} - * + \alpha + * - * + * = \alpha^{-1} + \alpha$.

Thus $\alpha^{-1} + \alpha \simeq 0$, or $\alpha^{-1} \simeq -\alpha$. //

c) $\boxed{h : \pi_1(X, x_0) \longrightarrow H_1(X) \quad \text{is surjective.}}$

Proof: Let $z \in Z_1(X, x_0)$. Then $z = \sum n_\sigma \sigma$ for some integers n_σ. Expanding, we may assume each $n_\sigma = \pm 1$, so we may write $z = \sigma_1 + \cdots + \sigma_m - \sigma_{m+1} - \cdots - \sigma_n$. By Lemma 2,

$$z \simeq \sigma_1 + \cdots + \sigma_m + \sigma_{m+1}^{-1} + \cdots + \sigma_n^{-1}$$

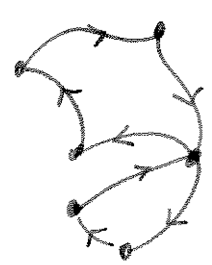
so we may simplify the notation and write $z = \sigma_1 + \cdots + \sigma_n$, with each $\sigma_i : \Delta_1 \to X$. Now $\partial z = 0$ since $z \in Z_1(X, x_0)$, so

$$\sum_{i=1}^{n} \sigma_i(1) = \sum_{i=1}^{n} \sigma_i(0)$$

That is, each 0-simplex $\sigma_i(0)$ or $\sigma_i(1)$ occurs as an "incoming vertex" $\sigma_i(1)$ exactly as often as it occurs as an "outgoing vertex" $\sigma_i(0)$.

So, we may arrange the σ_i in <u>cycles</u>

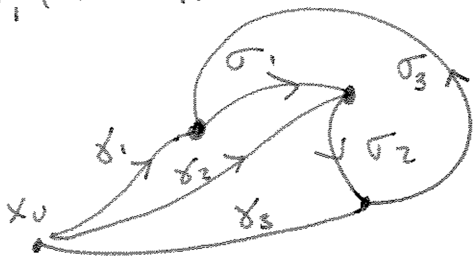
E.G.



$\sigma_1, \sigma_2, \ldots, \sigma_{K_1}$, $\quad \sigma_i(1) = \sigma_{i+1}(0)$ and $\sigma_{K}(1) = \sigma_1(0)$

$\sigma_{K_1+1}, \ldots \sigma_{K_2}$, etcetera.

If each $\sigma_1 + \cdots \sigma_K$ forming such a cycle is in $\mathrm{Im}(h)$, then h is onto, so WLOG $z = \sigma_1 + \sigma_2 + \cdots + \sigma_K$ with $\sigma_i(1) = \sigma_{i+1}(0)$ for each i, and $\sigma_K(1) = \sigma_1(0)$.



Choose paths $\gamma_i : x_0 \longrightarrow \sigma_i(0)$.

Here is where we need X to be path connected.

Let $\beta_i = \gamma_i * \sigma_i * \gamma_{i+1}^{-1}$ and $\beta_K = \gamma_K * \sigma_K * \gamma_1^{-1}$. Then each $\beta_i \in \pi_1(X, x_0)$ and

$$h(\beta_1\beta_2\ldots\beta_k) = \gamma_1 + \sigma_1 + \gamma_2^{-1} + \gamma_2 + \sigma_2 + \cdots + \sigma_k + \gamma_1^{-1}$$
$$\simeq \gamma_1 + \sigma_1 - \gamma_2 + \gamma_2 + \sigma_2 + \cdots + \sigma_k - \gamma_1$$
$$= \sigma_1 + \sigma_2 + \cdots + \sigma_k. //$$

(b) $\boxed{h \text{ is a homomorphism}}$

(2) This is Lemma 1. //

(a) Well defined $\alpha \simeq \beta \Rightarrow \alpha_* = \beta_*$

(2) $\Rightarrow h(\alpha) = \alpha_*(2) = \beta_*(2) = h(\beta)$. //

To show $\text{Ker}(h) = [\pi_1, \pi_1]$ we need a Lemma which will help us recognize elements of the commutator subgroup.

Lemma: In a group G, if $g_0 = g_1^{n_1} g_2^{n_2} \cdots g_k^{n_k}$ satisfies

$$\sum_{\substack{1 \le i \le k \\ g_i = g}} n_i = 0 \quad \text{for every } g \in G, \text{ then } g_0 \in [G,G].$$

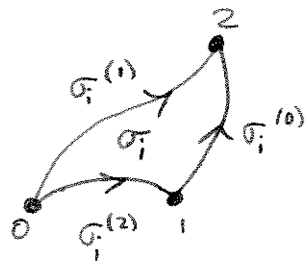
Proof: The homomorphism $G \longrightarrow G/[G,G]$ sends g_0 to $\sum n_i \bar{g}_i$, $\bar{g}_i =$ coset of g_i, and this is 0. Hence $g_0 \in [G,G]$. //

(d) Now $[\pi_1, \pi_1] \subseteq \text{Ker}\, h$, since $H_1(X)$ is abelian. It remains to

4 show: $\boxed{\text{Ker}\, h \subseteq [\pi_1, \pi_1]}$

Proof: If $h(\alpha) = 0$ in $H_1(X)$ then, in $S_1(X)$, $h(\alpha) = \partial z =$ $\partial\left(\sum_1^k n_i \sigma_i\right) = \sum_1^k n_i (\sigma_i^{(0)} - \sigma_i^{(1)} + \sigma_i^{(2)})$. Now, such an equation of 1-simplices means that the 1-simplex $h(\alpha)$ is one of the $\sigma_i^{(j)}$, while all the others cancel one another.

Using this, we will rewrite α as α times some trivial loops in such a way that the product is clearly a commutator (by the Lemma), and it will follow that α is a commutator.



For each vertex x of a σ_i, choose a path γ_x depending only on x, and if $x = x_0$, choose γ_{x_0} to be the constant path. Then if $h(\alpha) = \sigma_i^{(j)}$ we have $\alpha \simeq \gamma_x \sigma_i^{(j)} \gamma_x^{-1}$. Further, if $\sigma_i^{(j)}$ and $\sigma_m^{(n)}$ cancel (i.e. are equal and have opposite coefficients) then so do their composites with the paths to the start and end vertices. Precisely, let

$$\left.\begin{aligned} \beta_{i0} &= \gamma_{\sigma_i(1)} * \sigma_i^{(0)} * \gamma_{\sigma_i(2)}^{-1} \\ \beta_{i1} &= \gamma_{\sigma_i(0)} * \sigma_i^{(1)} * \gamma_{\sigma_i(2)}^{-1} \\ \beta_{i2} &= \gamma_{\sigma_i(0)} * \sigma_i^{(2)} * \gamma_{\sigma_i(1)}^{-1} \end{aligned}\right\} \quad \text{and} \quad \begin{aligned} \beta_i &= \beta_{i0} \beta_{i1}^{-1} \beta_{i2} \\ &\simeq \gamma_{\sigma_i(1)} * \sigma_i^{(0)} * (\sigma_i^{(1)})^{-1} * \sigma_i^{(2)} * \gamma_{\sigma_i(1)}^{-1} \\ &\simeq \gamma_{\sigma_i(1)} * \gamma_{\sigma_i(1)}^{-1} \simeq * \end{aligned}$$

using the homotopy through the simplex σ_i to cancel $\sigma_i^{(0)} * \sigma_i^{(1)^{-1}} * \sigma_i^{(2)}$. Therefore

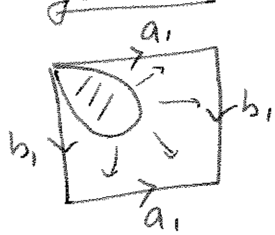
$$\alpha \simeq \alpha \beta_1^{-n_1} \beta_2^{-n_2} \cdots \beta_k^{-n_k}$$

This latter product lies in the commutator, by the Lemma, since α cancels one of the $\sigma_i^{(j)}$, which is therefore equal (i.e. homotopic to) β_{ij}, and the remaining β_{ij} must all cancel since the remaining $\sigma_i^{(j)}$ do. //

Final Note:

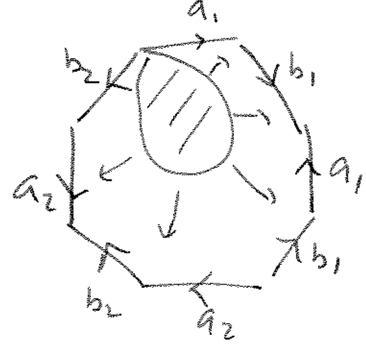
A closed disk in a compact orientable surface has boundary a commutator:

genus 1



$\partial = a_1 b_1 a_1^{-1} b_1^{-1}$

genus 2

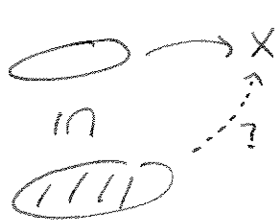


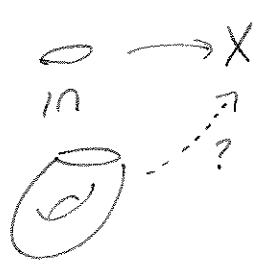
$\partial = (a_1 b_1 a_1^{-1} b_1^{-1})(a_2 b_2 a_2^{-1} b_2^{-1})$

Etc.

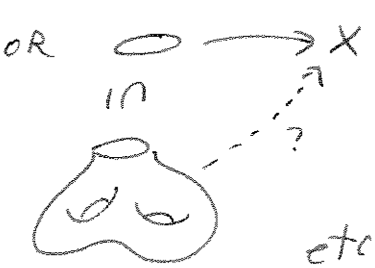
We may therefore compare $\pi_1(X)$ to $H_1(X)$ as follows. A loop $S^1 \xrightarrow{\alpha} X$ is trivial in $\pi_1(X)$ iff it extends over a disk D^2 (genus 0, $S^2 - D^2 = D^2$). A homology class $\Delta_1 \xrightarrow{\alpha} X$ is trivial in $H_1(X)$ iff it extends over a compact orientable surface with $\partial = S^1$ of any genus.

$\boxed{\pi_1}$ vs $\boxed{H_1}$





OR



etc.