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Math 7510, Fall 2009, Homework 7
due 4 November 2009

1. Prove that if M is an R-module and $\{M_\alpha\}$ is the collection of finitely generated submodules of M ordered by inclusion, then $\{M_\alpha, \subseteq\}$ is directed and the natural map

$$\varinjlim M_\alpha \longrightarrow M$$

is an isomorphism. (This is usually abbreviated as 'any module is the direct limit of its finitely generated submodules'.)

2. For $n \geq 1$, we can view the unit sphere S^n as

$$S^n = \{(z, x) \in \mathbf{C} \times \mathbf{R}^{n-1} \mid z\bar{z} + |x|^2 = 1\}$$

and then define $f_k : S^n \longrightarrow S^n$ by $f_k(z, x) = (z^k, x)$. Show that f_k has degree k. (In other words, $f_{k*} : H_n(S^n) \longrightarrow H_n(S^n)$ is multiplication by k.) Because of this, we will often simply refer to 'the map' $k : S^n \longrightarrow S^n$ to mean any map homotopic to f_k.
Hints: Start an induction on n at $n = 1$ by using the homework problem on the effect of this map on the fundamental group and the homework problem on the Hurewicz map $\pi_1(X) \longrightarrow H_1(X)$.

3. The mod k Moore space $M^n(k)$ of dimension n is the cofiber of $k : S^n \longrightarrow S^n$, or, in other words, the cell complex with an n and an $n+1$ cell whose attaching map is k. Compute

   (a) $H_*(M^n(k); \mathbf{Z})$ and $H^*(M^n(k); \mathbf{Z})$

   (b) $H_*(M^n(k); \mathbf{Z}[1/k])$ and $H^*(M^n(k); \mathbf{Z}[1/k])$

   (c) $H_*(M^n(k); \mathbf{Z}/(k)$ and $H^*(M^n(k); \mathbf{Z}/(k))$

4. Construct cell complexes X, X_1 and X_2 such that

$$H^*(X_1; \mathbf{Z}[1/k]) \cong H_*(X; \mathbf{Z}[1/k])$$

and

$$H^*(X_2; \mathbf{Z}/(k)) \cong H_*(X; \mathbf{Z}/(k))$$

but $X_1 \not\simeq X \not\simeq X_2$. Hint: Use homology with appropriate coefficients to show $X \not\simeq X_i$. We may summarize this by saying that X is like X_1 *away from* k and like X_2 *at* k.

The End