Characteristic Classes in K-Theory
Connective K-theory of BG, G Compact Lie

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Outline

1. Notation
2. Tori
3. Special Unitary Groups
4. Unitary Groups
5. Symplectic Groups
6. Orthogonal Groups
7. Special Orthogonal Groups
For tori and ‘symplectic tori’ we have:

- $RU(T^n) = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$
- $RU(Sp(1)^n) = \mathbb{Z}[s_1, \ldots, s_n]$

If $\lambda_i$ is the $i^{\text{th}}$ exterior power of the defining representation, then:

- $RU(Sp(n)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_n]$
- $RU(SU(n)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_{n-1}]$
- $RU(U(n)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}]$
- $RU(O(n)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_n]/(\lambda_n^2 - 1, \lambda_i \lambda_n - \lambda_{n-i})$
- $RU(SO(2n+1)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_n]$ with $\lambda_{n+i} = \lambda_{n+1-i}$
- $RU(SO(2n)) = \mathbb{Z}[^n\lambda_1, \ldots, \lambda_{n-1}, ^{n+}\lambda_n, ^{-}\lambda_n]/R$
  with $\lambda_{n+i} = \lambda_{n-i}$, $\lambda_n = ^{n+}\lambda_n + ^{-}\lambda_n$ and $R = ((^n\lambda_n + \sum_i \lambda_{n-2i})(^{-}\lambda_n + \sum_i \lambda_{n-2i}) - (\sum \lambda_{n-1-2i})^2)$
\[ RU(T^n) = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \]

All simple representations but the trivial one are complex. The integral cohomology ring is

\[ H^* BT^n = \mathbb{Z}[y_1, \ldots, y_n] \]

with \( y_i = c_1(t_i) \).

**Theorem**

\[ ku^* BT^n = ku^* [[y_1, \ldots, y_n]] \]

with \( y_i = c_1^{ku}(t_i) \) and

\[ ku^* T^n = \text{MRees}(RU(T^n)) = \frac{ku^*[y_1, \bar{y}_1 \ldots, y_n, \bar{y}_n]}{(vy_i\bar{y}_i = y_i + \bar{y}_i)} \]

with \( \bar{y}_i = c_1^{ku}(t_i^{-1}) \mapsto -y_i/(1 - vy_i) \).
The calculation of $ku_T^n$ uses the pullback square

$$
\begin{array}{c}
ku_T^n \rightarrow KU_T^n \\
\downarrow \\
ku^*BT^n \rightarrow KU^*BT^n
\end{array}
$$

We have the relations $t_i = 1 - vy_i$ and $t_i^{-1} = 1 - v\bar{y}_i$, hence

$$KU_T^n = KU^*[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \cong KU^*[y_1, \bar{y}_1, \ldots, y_n, \bar{y}_n]/R,$$

where $R$ is the ideal generated by the relations $vy_i\bar{y}_i = y_i + \bar{y}_i$. Then

$$ku_T^n = ku^*[[y_1, \ldots, y_n]] \cap KU^*[y_1, \bar{y}_1, \ldots, y_n, \bar{y}_n]/R$$
Real connective K-theory

\( H^*BT^n \) is free over \( E[Sq^2] = H^*C\eta \), and \( ko \land C\eta = ku \).

Hence, the Adams spectral sequence for \( ko^*BT^n \) collapses and is concentrated in even degrees.

Hence \( \eta \) acts trivially and the \( \eta - c - R \) sequence is just a short exact sequence

\[
0 \longrightarrow ko^*BT^n \xrightarrow{c} ku^*BT^n \xrightarrow{R} ko^{*+2}BT^n \longrightarrow 0
\]

Complex conjugation acts by

- \( \tau(v) = -v \)
- \( \tau(y_i) = \tau \left( \frac{1 - t_i}{v} \right) = \frac{1 - t_i^{-1}}{-v} = -\overline{y}_i \)
The Bockstein differential $cR$ is then exact and $ko^*$-linear, hence $2v^2$ linear. But $ku^*BT^n$ is 2-torsion-free, so $cR$ is $v^2$-linear. Hence

$$v^2 cR(x) = cR(v^2 x) = cr(vx) = (1 + \tau)(vx) = vx - v\tau(x)$$

or

$$cR(x) = \frac{x - \tau(x)}{v}$$

**Theorem**

$$ko^*_T = (ku^*_T)^C_2 \text{ where } C_2 \text{ acts by conjugation.}$$

**Proof.**

$$ko^*_T = \text{im}(c) = \ker(R) = \ker(vcR) = \ker(1 - \tau).$$
\[ RU(SU(n)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_{n-1}] \]

We have \( \overline{\lambda_i} = \lambda_{n-i} \), so the \( \lambda_i \) are all complex unless \( n = 2m \), when \( \lambda_m \) is real if \( m \) is even and quaternionic if \( m \) is odd. The integral cohomology is

\[ H^* BSU(n) = \mathbb{Z}[c_2, \ldots, c_n] \]

with \( c_i = c_i(\lambda_1) \). The connective complex \( K \)-theory is easy to compute.

**Theorem**

\[ ku^* BSU(n) = ku^*[[c_2, \ldots, c_n]] \text{ and } ku^*_SU(n) = \text{MRees}(RU(SU(n))) = ku^*[c_2, \ldots, c_n]. \]
Proof.

Since $H^* \text{BSU}(n)$ is concentrated in even degrees, the Atiyah-Hirzebruch spectral sequence implies $ku^* \text{BSU}(n)$ must be the complete $ku^*$ algebra freely generated by $c_2, \ldots, c_n$.

In $KU^*_{\text{SU}(n)}$, we have

$$
\lambda_i = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} c_j^R = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} v^j c_j^{ku}.
$$

and $\lambda_n = 1$.

Hence, the $c_j = c_j^{ku}$ generate, and $c_1 - vc_2 + \cdots + (-v)^{n-1}c_n = 0$. Thus $KU^*_{\text{SU}(n)}$ is polynomial on any $n - 1$ of $c_1, \ldots, c_n$. In particular, $KU^*_{\text{SU}(n)} = KU^*[c_2, \ldots, c_n]$. 

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Proof.

(Cont.) The pullback square

\[
\begin{array}{ccc}
ku^*_{SU(n)} & \longrightarrow & KU^*[c_2, \ldots, c_n] \\
\downarrow & & \downarrow \\
k_u^*[\llbracket c_2, \ldots, c_n \rrbracket] & \longrightarrow & KU^*[\llbracket c_2, \ldots, c_n \rrbracket]
\end{array}
\]

shows that \( ku^*_{SU(n)} = ku^*[c_2, \ldots, c_n] \).
\[ RU(U(n)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}] \]

The integral cohomology is
\[ H^* BU(n) = \mathbb{Z}[c_1, \ldots, c_n] \]
where \( c_i = c_i(\lambda_1) \). Again, the complex connective \( K \)-theory follows immediately.

**Theorem**

\[ ku^* BU(n) = ku^*[c_1, \ldots, c_n] \text{ and } \]
\[ ku^*_U(n) = M\text{Rees}(RU(U(n))) = ku^*[c_1, \ldots, c_n, \Delta^{-1}] \]

where \( \Delta = \lambda_n = 1 - vc_1 + v^2 c_2 - \cdots + (-v)^n c_n \).

**Proof.**

The argument is nearly the same as for \( SU(n) \), except that \( KU^*_U(n) \) is not polynomial, but is instead \( KU^*[c_1, \ldots, c_n, \Delta^{-1}] \). \[ \square \]
In cohomology, restriction along the inclusion $SU(n) \rightarrow U(n)$ is the quotient which sends $c_1$ to 0. The proper way to think of this is that we are taking the quotient by the determinant of the defining representation. In $K$-theory, the map this induces is more interesting.

**Theorem**

The restriction homomorphism $ku^*_U(n) \rightarrow ku^*_SU(n)$ is the quotient $ku^*[c_1, \ldots, c_n, \Delta^{-1}] \rightarrow ku^*[c_2, \ldots, c_n]$ which sends $\Delta$ to 1 and $c_1$ to $vc_2 - v^2c_3 + \cdots - (-v)^{n-1}c_n$.

**Proof.**

$SU(n)$ is the kernel of the determinant $U(n) \rightarrow U(1)$. The determinant sends $y = (1 - \lambda_1)/v \in ku^2_U(1)$ to $(1 - \lambda_n)/v = c_1 - vc_2 + v^2c_3 - \cdots$, so this must go to zero in $ku^*_{SU(n)}$. After dividing by this, we have an isomorphism, by the calculation of $ku^*_{SU(n)}$. □
Consider the conjugate Chern classes $\overline{c}_i(V) = c_i(\overline{V})$.

Proposition

Restriction $ ku^*_U(n) \longrightarrow ku^*_T(n) $ sends $\overline{c}_i$ to $\sigma_i(\overline{y}_1, \ldots, \overline{y}_n)$. There is a relation

$$\Delta \overline{c}_i = \sum_{k=i}^{n} (-1)^k \binom{k}{i} v^{k-i} c_k$$

The conjugate $\overline{\Delta} = \overline{\lambda}_n = 1 - v\overline{c}_1 + v^2\overline{c}_2 - \cdots \pm v^n\overline{c}_n$ satisfies $\Delta \overline{\Delta} = 1$. Collecting terms we find

Proposition

In $ ku^*_U(n) $,

$$c_1 + \overline{c}_1 = - \sum_{k=2}^{2n} (-v)^{k-1} \sum_{i+j=k} c_i \overline{c}_j$$
\( RU(\text{Sp}(n)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_n] \)

The \( \lambda_{2i} \) are real and the \( \lambda_{2i+1} \) are quaternionic, hence all are self conjugate. 

Note that \( \lambda_1 = H^n = C^{2n} \), which is \( 2n \) dimensional, but its higher exterior powers \( \lambda_{n+1}, \ldots \lambda_{2n} \) can be expressed in terms of the first \( n \).

The integral cohomology is 

\[ H^* B\text{Sp}(n) = \mathbb{Z}[p_1, \ldots, p_n] \]

with \( |p_i| = 4i \).

Restriction along \( \text{Sp}(1)^n \rightarrow \text{Sp}(n) \) will play much the same role for \( \text{Sp}(n) \) as restriction along \( T^n = U(1)^n \rightarrow U(n) \) plays for \( U(n) \), so we start by considering \( \text{Sp}(1)^n \).
\[ RU(Sp(1)^n) = \mathbb{Z}[s_1, \ldots, s_n] \]

where
\[ s_i : Sp(1)^n \xrightarrow{p_i} Sp(1) \cong SU(2) \subset U(2) \]

Hence \( c_{1}^{ku}(s_i) = \nu c_{2}^{ku}(s_i) \) and
\[ \nu^2 c_{2}^{ku}(s_i) = \nu c_{1}^{ku}(s_i) = c_{2}^{R}(s_i) = c_{1}^{R}(s_i) = 2 - s_i. \]

Thus, we have classes \( z_i = c_{2}^{ku}(s_i) \in ku^4(BSp(1)^n) \) which satisfy
\[ \nu^2 z_i = 2 - s_i. \]

We will see that \( z_i \) comes from \( ko^* \). The integral cohomology ring is
\[ H^* BSp(1)^n = \mathbb{Z}[z_1, \ldots, z_n] \]

with \( z_i = p_1(s_i) \), the first Pontrjagin class of \( s_i \).
Theorem

There are compatible generators $z_i$ so that

- $ku^* \text{BSp}(1)^n = ku^*[z_1, \ldots, z_n]$
- $ko^* \text{BSp}(1)^n = ko^*[z_1, \ldots, z_n]$
- $ku_{Sp(1)^n}^* = ku^*[z_1, \ldots, z_n] = \text{MRees}(RU(Sp(1)^n))$
- $ko_{Sp(1)^n}^* = ko^*[z_1, \ldots, z_n]$

In particular, $z_i^{ku} \in ku_{Sp(1)^n}^4$ and $z_i^{ko} \in ko_{Sp(1)^n}^4$ satisfy

- $\nu^2 z_i^{ku} = 2 - s_i \in ku_{Sp(1)^n}^0 = RU(Sp(1)^n)$,
- $\alpha z_i^{ko} = 2(2 - s_i) \in ko_{Sp(1)^n}^0 = RO(Sp(1)^n)$ and
- $\beta z_i^{ko} = 2 - s_i \in ko_{Sp(1)^n}^{-4} = RSp(Sp(1)^n)$. 
Proof.

The Adams spectral sequence collapses at

$$E_{2,*,*}^* = H^*BSp(1)^n \otimes \text{Ext}_{\mathcal{A}(1)}^*(F_2, F_2) \implies ko^*BSP(1)^n$$

and similarly for $E(1)$ and $ku^*$.

The equivariant cases then follow by the defining pullback squares

$$
\begin{array}{ccc}
ku^*_{Sp(1)^n} & \rightarrow & KU^*[z_1, \ldots, z_n] \\
\downarrow & & \downarrow \\
kku^*[\llbracket z_1, \ldots, z_n \rrbracket] & \rightarrow & KU^*[\llbracket z_1, \ldots, z_n \rrbracket]
\end{array}
$$

The periodic groups are as claimed because we can change generators from the $s_i$ to the $z_i = (2 - s_i)/\nu^2$. This is $\text{MRees}(RU(Sp(1)^n))$: all irreducible representations are two dimensional, so $JU_{2n} = JU_{2n-1} = (JU_2)^n$. \qed
Pontrjagin classes

**Definition**

The $k^{\text{th}}$ representation theoretic Pontrjagin class of an $n$-dimensional symplectic representation $V : G \rightarrow Sp(n)$ is

$$p_k^R(V) = \sum_{j=0}^{k} (-1)^j 2^{k-j} \left( \begin{array}{c} n-j \\ n-k \end{array} \right) \wedge^j(V)$$

**Proposition**

The restriction $RU(Sp(n)) \rightarrow RU(Sp(1)^n)$ sends $p_k^R$ to $\sigma_k(2 - s_1, \ldots, 2 - s_n)$. The representation $p_k^R$ is real if $k$ is even, and quaternionic if $k$ is odd.

Accordingly, we shall generally consider $p_{2i}^R$ as an element of $RO(G)$ and $p_{2i+1}^R$ as an element of $RSp(G)$. Note, however, that representations which are not irreducible can be both real and quaternionic.
**Theorem**

We have

- \( ku^* BSp(n) = ku^*[[p_1, \ldots, p_n]] \)
- \( ko^* BSp(n) = ko^*[[p_1, \ldots, p_n]] \)
- \( ku_{Sp(n)}^* = ku^*[p_1, \ldots, p_n] \)
- \( ko_{Sp(n)}^* = ko^*[p_1, \ldots, p_n] \).

In each case, \( p_k \) restricts to \( \sigma_k(z_1, \ldots, z_n) \).

In \( ku^* \), \( v^{2k} p_k^{ku} = p_k^R \in ku_{Sp(n)}^0 = RU(Sp(n)) \).

In \( ko^* \), \( \beta^k p_{2k}^{ko} = p_{2k}^R \in ko_{Sp(n)}^0 = RO(Sp(n)) \) and \( \beta^k p_{2k+1}^{ko} = p_{2k+1}^R \in ko_{Sp(n)}^4 = JSp(Sp(n)) \).
**Definition**

Let $V : G \rightarrow Sp(n)$ be a symplectic representation. For $E = RU$, $ko$, $KO$, $ku$, $KU$ or $H$, we define the *Pontrjagin class* $p^E_i(V) \in E^G_{4i}$ to be $V^*(p_i)$. It is convenient to collect these into the *total Pontrjagin class*

$$p^E_•(V) = 1 + p^E_1(V) + p^E_2(V) + \cdots + p^E_n(V)$$

and to let $p^E_i(V) = 0$ if $i > n$.

**Corollary**

$$p^E_•(V \oplus W) = p^E_•(V)p^E_•(W)$$
Lemma

The restriction \( ku_{Sp(1)^n}^* \to ku_{T(n)}^* \) maps \( z_i \) to \( y_i y_i \).

Write \( \overline{c}_i(V) = c_i(\overline{V}) \) for the Chern classes of the complex conjugate of a representation.

Theorem

The restriction maps \( ku_{U(2n)}^* \to ku_{Sp(n)}^* \to ku_{U(n)}^* \) obey

\[
\begin{align*}
c_k & \mapsto \sum_{0 \leq 2i \leq k} \binom{k-i}{i} v^{k-2i} p_{k-i} \\
& \mapsto \sum_{i+j=k} c_i \overline{c}_j.
\end{align*}
\]

Specializing to ordinary cohomology by setting \( v = 0 \) we obtain the usual relations (up to sign):

\[
q^*(p_n^H) = \sum_{i+j=2n} c_i^H \overline{c}_j^H = \sum_{i+j=2n} (-1)^i c_i^H c_j^H,
\]

\[
\overline{c}^*(c_{2i-1}) = 0, \quad \text{and} \quad \overline{c}^*(c_{2i}) = p_i.
\]
For $Sp(4)$, for example,

\[
\begin{align*}
    c_1 & \mapsto vp_1 \\
    c_2 & \mapsto p_1 + v^2 p_2 \\
    c_3 & \mapsto 2vp_2 + v^3 p_3 \\
    c_4 & \mapsto p_2 + 3v^2 p_3 + v^4 p_4 \\
    c_5 & \mapsto 3vp_3 + 4v^3 p_4 \\
    c_6 & \mapsto p_3 + 6v^2 p_4 \\
    c_7 & \mapsto 4vp_4 \\
    c_8 & \mapsto p_4
\end{align*}
\]

\[
\begin{align*}
    & \mapsto c_1 + \overline{c}_1 \\
    & \mapsto c_2 + c_1 \overline{c}_1 + \overline{c}_2 \\
    & \mapsto c_3 + c_2 \overline{c}_1 + c_1 \overline{c}_2 + \overline{c}_3 \\
    & \mapsto c_4 + c_3 \overline{c}_1 + c_2 \overline{c}_2 + c_1 \overline{c}_3 + \overline{c}_4 \\
    & \mapsto c_4 \overline{c}_1 + c_3 \overline{c}_2 + c_2 \overline{c}_3 + c_1 \overline{c}_4 \\
    & \mapsto c_4 \overline{c}_2 + c_3 \overline{c}_3 + c_2 \overline{c}_4 \\
    & \mapsto c_4 \overline{c}_3 + c_3 \overline{c}_4 \\
    & \mapsto c_4 \overline{c}_4
\end{align*}
\]
It is more difficult to get good expressions for the images of the individual $p_i$. However, for $i = 1$, using the fact that $v$ acts monomorphically on $ku^*_U(n)$ we have

$$p_1 \mapsto \frac{c_1 + \bar{c}_1}{v} = \sum_{k=2}^{n} (-v)^{k-2} \sum_{i+j=k} c_i \bar{c}_j$$

$$= c_2 + c_1 \bar{c}_1 + \bar{c}_2 - v \sum_{k=3}^{n} (-v)^{k-3} \sum_{i+j=k} c_i \bar{c}_j.$$

In cohomology, where $v = 0$ and $\bar{c}_i = (-1)^i c_i$, we have

$$p_1 \mapsto c_2 + c_1 \bar{c}_1 + \bar{c}_2 = 2c_2 - c_1^2$$

with our normalization of the $p_i$. Thus, if $c_1 = 0$, then $c_2 = p_1/2$. 
Finally, we provide the following *symplectic splitting principle*.

**Theorem**

Let $\xi$ be an $Sp(n)$ bundle over $X$. Then there exists a map $f : Y \rightarrow X$ such that $f^*\xi$ is a sum of symplectic line bundles and $f^* : H^*X \rightarrow H^*Y$ is a monomorphism.

**Proof.**

Let $Y$ be the pullback

\[
\begin{array}{ccc}
Y & \longrightarrow & BSp(1)^n \\
\downarrow f & & \downarrow \\
X & \longrightarrow & BSp(n)
\end{array}
\]

along the classifying map of the bundle $\xi$. The universal bundle over $BSp(n)$ splits as a sum of line bundles over $BSp(1)^n$, so $f^*\xi$ also splits in this manner.

*The Serre spectral sequence gives the cohomology statement.*
$RU(O(n)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_n]/(\lambda_n^2 - 1, \lambda_i \lambda_n - \lambda_{n-i})$.

These representations are all real, so that complexification and quaternionification are isomorphisms

$$RO(O(n)) \xrightarrow{\sim} RU(O(n)) \xrightarrow{\sim} RSp(O(n)).$$

The integral cohomology is complicated. The best approach is to give the mod 2 cohomology, and if integral issues matter, the cohomology localized away from 2. We have

$$HF^*_2 BO(n) = F_2[w_1, \ldots, w_n]$$

where $w_i = w_i(\lambda_1)$. 
Rewrite the representation ring in terms of Chern classes, as usual: let
\[ c_i = c^K_i(\lambda_1) \in K^2_{O(n)}, \]
so that
\[ \lambda_i = \sum_{j=0}^{i} (-1)^j \binom{n-j}{n-i} v^j c_j. \]

Rather than replace \( \lambda_n \) by the top Chern class, \( c_n \), we use the first Chern class of the determinant representation, \( c = c^K_1(\lambda_n) \in K^2_{O(n)} \). This satisfies \( vc = 1 - \lambda_n \), which is much more convenient than
\[ v^n c_n = 1 - \lambda_1 + \cdots + (-1)^n \lambda_n. \]

**Proposition**

\[ KU^*_O(2n+1) = KU^*[c_1, \ldots, c_n, c]/(vc^2 - 2c) \]

and

\[ KU^*_O(2n) = KU^*[c_1, \ldots, c_n, c]/(vc^2 - 2c, c \sum_{i=0}^{n} \binom{2n-i}{n} (-v)^i c_i) \]
To compute $ku^* BO(n)$, we need to determine the $E(1)$-module structure of $HF_2^* BO(n)$. We start with its stable type. Let $\epsilon$ be 0 or 1.

First, the submodule

$$F_2[w_2^2, w_4^2, \ldots, w_{2n}^2] \longrightarrow H^* BO(2n + \epsilon)$$

is a trivial $E(1)$-submodule.

Second, the reduced homology of $BO(1)$ is the ideal $(w_1)$ in $F_2[w_1]$, and as an $E(1)$-submodule,

$$(w_1) \otimes F_2[w_2^2, w_4^2, \ldots, w_{2n-2}^2] \longrightarrow H^* BO(2n - \epsilon)$$

is a direct sums of suspensions of $(w_1)$.

The sum of these two submodules exhausts the ‘interesting’ part of $H^* BO(n)$, in the sense that the complementary summand is $E(1)$-free.
Theorem

The inclusions

\[ \mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n}^2] \oplus (w_1) \otimes \mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n-2}^2] \to H^*BO(2n) \]

and

\[ \mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n}^2] \oplus (w_1) \otimes \mathbf{F}_2[w_2^2, w_4^2, \ldots, w_{2n}^2] \to H^*BO(2n + 1) \]

induce isomorphisms in \( Q_0 \) and \( Q_1 \) homology.

Corollary

As an \( E[Q_0, Q_1] \)-module, \( H^*BO(n) \) is the sum of trivial modules, suspensions of \( H^*BO(1) \), and free modules.

Proof.

The corollary follows by the result of Adams and Margolis, that \( Q_0 \) and \( Q_1 \) homology detects the stable isomorphism type of the module.
In principle, this describes $H^* BO(n)$ as an $E(1)$-module but finding a good parametrization of the complementary $E(1)$-free submodule is non-trivial. The $A(1)$-module structure is not as simple, as $Sq^2$ does not annihilate all squares. The $w_{2i}^2$ detect Pontrjagin classes $p_i$ of the defining representation and $w_1^2$ in the ($w_1$) summand detects the first Chern class of the determinant representation.

**Corollary**

*The Adams spectral sequence converging to $ku^* BO(n)$ collapses at $E_2$, and the natural homomorphism*

$$ku^* BO(n) \longrightarrow H^* BO(n) \oplus KU^* BO(n)$$

*is a monomorphism.*
Comments on the proof

The $H(-, Q_0)$ isomorphism is straightforward, but the $H(-, Q_1)$ isomorphism requires a careful choice of generators.

Once the correct generators are identified, it turns out that the general case is just a regraded version of $H^* BO(4)$ tensored with an $E(1)$-trivial subalgebra.

See the book with Greenlees for details.
Recall that $ku_{O(1)}^* = ku^*[c]/(vc^2 - 2c)$ by the pullback square

$$
ku_{O(1)}^* \rightarrow KU_{O(1)}^* = KU^*[c]/(vc^2 - 2c)
$$

$$
ku^* BO(1) = ku^*[c]/(vc^2 - 2c) \rightarrow KU^* BO(1) = KU^*[c]/(vc^2 - 2c)
$$

The Bockstein spectral sequence then gives

**Theorem**

There are unique elements $p_0 \in ko_{O(1)}^0$ and $p_1 \in ko_{O(1)}^4$ with complexifications $c(p_0) = vc$ and $c(p_1) = c^2$. The ring

$$
ko_{O(1)}^* = \frac{ko^*[p_0, p_1]}{(\eta p_1, \alpha p_1 - 4p_0, \beta p_1 - \alpha p_0, p_0 p_1 - 2p_1, p_0^2 - 2p_0)}
$$
In terms of representation theory, this can be written as follows.

**Corollary**

*O(1)-equivariant connective real K-theory has coefficient ring*

\[
ko^i_{O(1)} = \begin{cases} 
RSp & i = -8k - 4 \leq 0 \\
RO/2 & i = -8k - 2 \leq 0 \\
RO/2 & i = -8k - 1 \leq 0 \\
RO & i = -8k \leq 0 \\
JSp_k = JSp^k & i = 4k > 0 \\
0 & \text{otherwise}
\end{cases}
\]

To justify the notation \( p_i \):

**Theorem**

*The restriction* \( ku^*_{Sp(1)} \to ku^*_{O(1)} *\) *is:*

\[
z = p_1(\lambda_1) \mapsto p_1, \quad \alpha z \mapsto 4p_0, \quad \text{and} \quad \beta z \mapsto \alpha p_0,
\]
Proof.

That $z$ maps to $p_1$ is evident by comparison with $ku^*$. The rest follows by the relations in $ko^*_{O(1)}$.

Thus, $p_1$ really is the first Pontrjagin class of the quaternionic representation induced up from the defining representation of $O(1)$, while $p_0$ is a genuinely real class. We call it $p_0$ because of the relations which tie it to $p_1$. 
**O(2)**

**Corollary**

\[
KU^*_O(2) = KU^*[c, c_1]/(vc^2 - 2c, c(2 - vc_1)) = KU^*[c, c_2]/(vc^2 - 2c, cc_2)
\]

**Proof.**

The calculation

\[
v^2 c_2 = 1 - \lambda_1 + \lambda_2 = 1 - (2 - vc_1) + (1 - vc) = v(c_1 - c)
\]

shows that \(c_1 = c + vc_2\). Then the relation \(0 = c(2 - vc_1)\) becomes \(v^2 cc_2 = 0\) since \(c(2 - vc) = 0\).  

[\(\square\)]
The connective $K$-theory is similar but somewhat larger.

**Theorem**

$$ku^*_{O(2)} = ku^*[c, c_2]/(vc^2 - 2c, 2cc_2, vcc_2)$$

**Proof.**

Decomposing $H^*BO(2)$ as an $E[Q_0, Q_1]$-module shows that $c$ and $c_2$ are algebra generators for $ku^*BO(2)$. The monomorphism into $H^*BO(2) \oplus KU^*BO(2)$ then shows the relations are complete. The pullback square then gives us $ku^*_{O(2)}$.  

\[ \square \]
The Bockstein spectral sequence then gives

**Theorem**

\[ KO^*_O(2) = KO^*[p_0, r_0]/(p_0^2 - 2p_0, p_0r_0) \text{ and } \]

\[ ko^*_O(2) = ko^*[p_0, p_1, p_2, r_0, r_1, s]/I \]

where \( I \) is the ideal generated by the relations

\[
\begin{array}{ccc}
\eta p_1 = 0 & \alpha p_1 = 4p_0 & \beta p_1 = \alpha p_0 \\
\eta r_1 = 0 & \alpha r_1 = 4r_0 & \beta r_1 = \alpha r_0 \\
\eta s = 0 & \alpha s = 0 & \beta s = \eta^2 r_0 \\
p_0p_2 = 0 & p_1p_2 = s^2 \\
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
p_0 & p_1 & r_0 & r_1 & s \\
\hline
p_0 & 2p_0 & 2p_1 & 0 & 0 & 0 \\
p_1 & 2p_1 & p_1^2 & 0 & 0 & p_1s \\
r_0 & 0 & 0 & \beta p_2 & \alpha p_2 & \eta^2 p_2 \\
r_1 & 0 & 0 & \alpha p_2 & 4p_2 & 0 \\
\hline
\end{array}
\]
$p_0$ and $r_0$ are $1 - \det$ and the Euler class of the defining representation, respectively.

$p_1$ and $r_1$ are their images in $JSp = ko^4$. This explains the similarity of the action of $ko^*$ on them.

The class $p_2$ refines the square of the Euler class in the sense that $r_1^2 = 4p_2$, $r_0r_1 = \alpha p_2$ and $r_0^2 = \beta p_2$.

The class $s$ is a square root of the product $p_1p_2 = s^2$.

The relation $\beta s = \eta^2 r_0$ is hidden in the Bockstein spectral sequence. Representation theory (i.e., the map into $KO^*_{O(2)}$) and the Adams spectral sequence each work to recover it.
Corollary

\[ KU_{O(3)}^* = KU^*[c, c_1]/(vc^2 - 2c) \]

Proposition

\[ KO_{O(3)}^* = KO^*[p_0, q_0]/(p_0^2 - 2p_0) \]
where \( p_0 \) and \( q_0 \) have complexifications \( vc \) and \( vc_1 \) respectively. The restriction \( KO_{O(3)}^* \rightarrow KO_{O(2)}^* \) sends \( p_0 \) to \( p_0 \) and \( q_0 \) to \( p_0 + r_0 \).

The Chern classes no longer suffice to generate \( ku_{O(n)}^* \) for \( n > 2 \). Let \( \overline{Q}_0 : H \rightarrow \Sigma HZ \) and \( \overline{Q}_1 : HZ \rightarrow \Sigma^3 ku \) be the boundary maps in the cofiber sequences for \( 2 : HZ \rightarrow HZ \) and \( v : \Sigma^2 ku \rightarrow ku \). They are lifts of the Milnor primitives \( Q_0 \) and \( Q_1 \).
Definition

Let \( q_2 = \overline{Q}_1 Q_0(w_2) \in ku^6 BO(3) \) and \( q_3 = \overline{Q}_1 Q_0(w_3) \in ku^7 BO(3) \).

Proposition

The classes \( q_2 \) and \( q_3 \) are nonzero classes annihilated by \((2, v)\). The class \( q_3 \) is independent of \( c, c_2, \) and \( c_3 \), while \( q_2 = cc_2 - 3c_3 \). These are the only nonzero 2 or \( v \)-torsion classes in \( ku^6 BO(3) \) and \( ku^7 BO(3) \).

Theorem

\( ku^* BO(3) = ku^* [c, c_2, c_3, q_3]/R \), where \( R \) is an ideal containing \((vc^2 - 2c, 2(cc_2 - 3c_3), v(cc_2 - 3c_3), 2q_3, vq_3, vcc_3 - 2c_3)\).
**O(n) for larger n**

The free summands in $H^* BO(n)$ begin to get more complicated at $n = 4$. Let us write $w_S$ for the product $\prod_{i \in S} w_i$.

**Proposition**

Maximal $E(1)$-free summands of $H^* BO(n)$ are:

- **$n = 4$**
  
  $$F_2[w_1^2, w_2^2, w_3^2, w_4^2] \langle w_2, w_3, w_4, w_234 \rangle$$
  
  $$\oplus F_2[w_1^2, w_2^2, w_4^2] \langle w_24 \rangle$$

- **$n = 5$**
  
  $$F_2[w_1^2, w_2^2, w_3^2, w_4^2, w_5^2] \langle w_2, w_3, w_4, w_5, w_234, w_{235}, w_{245}, w_{345} \rangle$$
  
  $$\oplus F_2[w_1^2, w_2^2, w_4^2, w_5^2] \langle w_24, w_{34} \rangle$$

- **$n = 6$**
  
  $$F_2[w_1^2, w_2^2, w_3^2, w_4^2, w_5^2, w_6^2] \langle w_2, w_3, w_4, w_5, w_6, w_{234}, w_{235}, w_{236}, w_{245}, w_{246}, w_{256}, w_{345}, w_{346}, w_{356}, w_{456}, w_{3456} \rangle$$
  
  $$\oplus F_2[w_1^2, w_2^2, w_3^2, w_4^2, w_5^2, w_6^2] \langle w_{236}, w_{246}, w_{256}, w_{346}, w_{456}, w_{23456} \rangle$$
  
  $$\oplus F_2[w_1^2, w_2^2, w_4^2, w_5^2, w_6^2] \langle w_{24}, w_{34}, w_{2456} \rangle$$
  
  $$\oplus F_2[w_1^2, w_2^2, w_4^2, w_5^2, w_6^2] \langle w_{46} \rangle$$
Remark

Each $w_S$ generating a free $E(1)$ will give rise to a $(2, ν)$-annihilated class $Q_1Q_0(w_S) \in ku^*BO(n)$. 
\[ RU(SO(2n + 1)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_n] \]

with \( \lambda_{n+i} = \lambda_{n+1-i} \) and

\[ RU(SO(2n)) = \mathbb{Z}[\lambda_1, \ldots, \lambda_{n-1}, \lambda_n^+, \lambda_n^-]/R \]

with \( \lambda_{n+i} = \lambda_{n-i} \) and \( \lambda_n = \lambda_n^+ + \lambda_n^- \). The ideal \( R \) is generated by one relation

\[
(\lambda_n^+ + \sum_i \lambda_{n-2i})(\lambda_n^- + \sum_i \lambda_{n-2i}) = (\sum \lambda_{n-1-2i})^2
\]

All the \( \lambda_i \) are real. \( RU(SO(2n)) \) is free over \( RU(SO(2n + 1)) \) on \( \{1, \lambda_n^+\} \).
$H^* BSO(n) = F_2[w_2, \ldots, w_n]$ where $w_i = w_i(\lambda_1)$.

We have already examined $SO(2) = T(1)$ and found (writing $c_1$ rather than $y_1$ here)

$$ku^*_{SO(2)} = ku^*[c_1, \bar{c}_1]/(vc_1\bar{c}_1 = c_1 + \bar{c}_1)$$

and

$$ku^* BSO(2) = ku^*[[c_1]].$$

The maps induced in $ku^*$ by the fibre sequence $SO(2) \xrightarrow{i} O(2) \xrightarrow{\text{det}} O(1)$ are

**Proposition**

$\det^*(c) = c$, while $i^*(c) = 0$, $i^*(c_2) = c_1\bar{c}_1$ and $i^*(c_1) = i^*(c + vc_2) = c_1 + \bar{c}_1$. 
**SO(3)**

\[ RU(SO(3)) = \mathbb{Z}[\lambda_1] \text{ with } \lambda_2 = \lambda_1 \text{ and } \lambda_3 = 1. \]

**Proposition**

\[ KU^*_{SO(3)} = KU^*[c_2] \]

**Proof.**

The Chern classes of the defining representation of \( SO(3) \) satisfy \( c_1 = vc_2, \ c_3 = 0 \) and \( v^2c_2 = vc_1 = 3 - \lambda_1 \).

**Theorem**

\[ ku^*_{SO(3)} = ku^*[c_2, c_3]/(2c_3, vc_3). \]

The first Chern class, \( c_1 = vc_2 \). The restriction \( ku^*_{O(3)} \to ku^*_{SO(3)} \) sends \( c \) and \( q_3 \) to 0, and sends each \( c_i \) to \( c_i \).
Proof.

The Adams spectral sequence again collapses and gives us a monomorphism into the sum of mod 2 cohomology and periodic K-theory. This makes it easy to show $\text{ku}^* \text{BSO}(3) = \text{ku}^*[c_2, c_3]/(2c_3, \nu c_3)$. The pullback square then gives $\text{ku}^*_\text{SO}(3) = \text{ku}^*[c_2, c_3]/(2c_3, \nu c_3)$.

In general, we expect $c_1 = \nu c_2 - \nu^2 c_3$, since this is true in $SU(n)$, but here $\nu^2 c_3 = 0$.

The restriction from $O(3)$ is computed by using the monomorphism to periodic $K$-theory plus mod 2 cohomology. Note that $q_2 = cc_2 - 3c_3$ restricts to $c_3$. 
Thank you