THE SEMI-DIHEDRAL ALGEBRA IN ALGEBRAIC TOPOLOGY

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ABSTRACT. We explain how the 8 dimensional semi-dihedral algebra is of interest to algebraic topologists.

The semi-dihedral algebra of dimension 8 over $F_2$ arises when considering real connective K-theory. Since real K-theory is the $C_2$-fixed part of complex K-theory (either periodic or connective), the difference between real and complex is insignificant at primes other than 2. Hence we shall focus attention purely on the 2-primary situation.

We start by giving a little background on connective K-theory to explain the context, and then describe where the semi-dihedral algebra comes in.

Let $H = HF_2$ and $HZ$ be the spectra (in the sense of stable homotopy theory) representing ordinary mod 2 cohomology and ordinary integral cohomology, respectively. Let $ko$ and $ku$ be the spectra representing connective real and complex K-theory, respectively, and let $KO$ and $KU$ be the spectra representing periodic real and complex K-theory. Thus, $H$ and $HZ$ are determined by simplices, e.g., via singular chains and cochains of a space, while $KO$ and $KU$ are determined by real and complex vector bundles, e.g., on a space $X$, respectively. (The value of $KU^*(X)$ when $X$ is a spectrum, rather than a space, is a more homotopy theoretical object, not quite as directly related to vector bundles.)

The connective theories, $ko$ and $ku$, mix $KO$ and $KU$ with $HZ$. There are a couple of ways to look at this. First, we have cofiber sequences exposing the relation to cohomology:

$$
\xymatrix{\Sigma ko \ar[r]^-{\eta} & ko \ar[r]^-{c} & ku \ar[r]^-{R} & \Sigma^2 ko \\
\Sigma^2 ku \ar[r]^-{\nu} & ku \ar[r]^-{\nu} & HZ \ar[r]^-{\nu} & \Sigma^3 ku \\
HZ \ar[r]^-{2} & HZ \ar[r]^-{2} & HF_2 \ar[r]^-{2} & \Sigma HZ 
}
$$

The latter sequence is purely arithmetic, of course, but is included to show the analogy between $ko$'s relation to $HZ$ and $HZ$'s better known relation to $HF_2$. Here $\nu$ is the map representing complex Bott periodicity, $c$ is complexification, and $R$ is a lift of realification, in that $\Sigma^2 ku \ar[r]^-{\nu} & ku \ar[r]^-{R} & \Sigma^2 ko$ is the double suspension of realization. The map $\eta$ is related to the fact that $ko = (ku)^C_2$, the part of complex K-theory fixed by complex conjugation. Second, the periodic theories are obtained from the connective ones by inverting Bott periodicity:

$$
KU = ku[1/v] = \lim(ku \ar[r]^-{\nu} & \Sigma^{-2} ku \ar[r]^-{\nu} & \Sigma^{-4} ku \ar[r]^-{\nu} & \cdots)
$$

and similarly $KO = ko[1/v^4]$. This latter makes sense because the map of coefficients $ko \ar[r]^-{c} & ku$ hits the subalgebra generated by $v^4$ (and more). So, roughly speaking, $ku$ modulo $v$ is cohomology, $ku/v = HZ$, while $ku$ with $v$ inverted is bundle theory, $ku[1/v] = KU$.

On the level of coefficient rings this is especially clear. Recall that the coefficient ring of a ring spectrum is its homotopy, $E_i = \pi_i(E)$, with product induced by the ring structure $E \wedge E \ar[r] & E$. Here, we have

- $HF_{2,*} = F_2$
- $HZ_* = Z$
- $ku_* = Z[v]$
- $KU_* = Z[v, v^{-1}]$
- $ko_* = Z[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta \alpha, \alpha^2 - 4 \beta)$, and
- $KO_* = ko_*[\beta^{-1}]$

Thus $Spec(ku_*)$ is composed of the open set $\{v \neq 0\} = Spec(KU_*)$, and its complement, the closed set $\{v = 0\} = Spec(HZ_*)$. Similarly, $Spec(ku^*(X))$ is augmented over $Spec(ku_*)$ for any space $X$, and the parts sitting over $\{v = 0\}$ and $\{v \neq 0\}$ are given by the cohomology and the periodic K-theory of $X$ respectively.
The coefficients for real K-theory are more complex, and I will only note here that \( \eta \) induces the map \( \eta \) mentioned in the relation between \( ko \) and \( ku \), and that complexification induces the ring homomorphism \( c_\ast : ko_\ast \longrightarrow ku_\ast \) which satisfies \( c_\ast (\eta) = 0, c_\ast (\alpha) = 2v, \) and \( c_\ast (\beta) = v^4 \). From this and degree considerations, the relation \( \alpha^2 = 4\beta \) is obvious. The most algebraic derivation of these facts is by considering real and complex Clifford algebras, as in [2] and [1]. The difference between \( ko \) and \( ku \) is thus not visible on the level of varieties, since \( c_\ast \) is an F-isomorphism.

The relation to the semi-dihedral algebra arises when considering the relation between mod 2 cohomology and real connective K-theory. We shall simultaneously describe the relation of mod 2 cohomology to integral cohomology and complex K-theory, to provide context.

Let \( \mathcal{A} \) be the mod 2 Steenrod algebra, i.e., \( \text{End}(H^F\mathbb{F}_2) \), the algebra of natural transformations of mod 2 cohomology. The Steenrod operations \( Sq^n, i \geq 0 \), generate \( \mathcal{A} \) subject to the Adem relations, such as \( Sq^0 = 1, \) \( Sq^1 Sq^j = 0, \) \( Sq^2 Sq^2 = Sq^1 Sq^3 Sq^1, \) etc. The Steenrod algebra is a Hopf algebra with coproduct \( \psi(Sq^n) = \sum_{i+j=n} Sq^i \otimes Sq^j \). When we say algebra or subalgebra from now on, we shall mean Hopf algebra or subalgebra.

The subalgebra \( \mathcal{A}(n) \) is generated by \( Sq^i \) for \( i \leq 2^n \). Thus \( \mathcal{A}(0) \) is the exterior algebra generated by \( Sq^1 \), while \( \mathcal{A}(1) \), which is generated by \( Sq^1 \) and \( Sq^2 \) subject to the two relations mentioned above, is the semi-dihedral algebra which has dimension 8 over \( \mathbb{F}_2 \). In this topological setting, \( \mathcal{A} \) and \( \mathcal{A}(n) \) are graded by \( \text{deg}(Sq^n) = i \). Thus, for example, the socle of \( \mathcal{A}(1) \) is \( \Sigma^6 \mathbb{F}_2 = \mathbb{F}_2[6] \), i.e., \( \mathbb{F}_2 \) concentrated in degree 6, and is generated by \( Sq^2 Sq^3 Sq^2 = (Sq^1 Sq^2)^2 = (Sq^2 Sq^1)^2 \). The Hilbert series of \( \mathcal{A}(1) \) is \( (1+t)(1+t^2) \).

Following Milnor ([5]), we define \( Q_0 = Sq^1 \) and \( Q_n = [Q_{n-1}, Sq^2] \) for \( n > 0 \). Let \( E(n) \) be the subalgebra of \( \mathcal{A} \) generated by \( Q_0, \ldots, Q_n \), which Milnor shows is an exterior algebra. Clearly \( E(n) \) is a subalgebra of \( \mathcal{A}(n) \). In particular, \( E(0) = \mathcal{A}(1) \) while \( E(1) \) has index 2 in \( \mathcal{A}(1) \).

Now, it turns out that
\[
H^\ast (HZ) = A \otimes_{E(0)} \mathbb{F}_2, \\
H^\ast (ku) = A \otimes_{E(1)} \mathbb{F}_2,
\]
and
\[
H^\ast (ko) = A \otimes_{A(1)} \mathbb{F}_2.
\]

Therefore, by the Künneth Theorem and a standard lemma about extended modules over Hopf algebras, we have
\[
H^\ast (HZ \wedge X) = A \otimes_{E(0)} H^\ast X, \\
H^\ast (ku \wedge X) = A \otimes_{E(1)} H^\ast X,
\]
and
\[
H^\ast (ko \wedge X) = A \otimes_{A(1)} H^\ast X.
\]

It follows that the mod 2 cohomology of \( E \wedge X \) as a \( A \)-module, where \( E \) is \( HZ, ku, \) or \( ko \), is entirely determined by the mod 2 cohomology of \( X \) as a module over \( E(0), E(1), \) and \( A(1), \) respectively. This is significant because, by definition, \( E_\ast (X) = \pi_\ast (E \wedge X) \) and \( E^\ast (X) = \pi_\ast F(X, E) = \pi_\ast (F_E(E \wedge X, E)) \), where \( F(\cdot, \cdot) \) and \( F_E(\cdot, \cdot) \) are the function spectra of all maps and of \( E \)-module maps, respectively. Thus, information about \( E \wedge X \) is relevant to computing the \( E \)-homology and cohomology of \( X \).

Here is a way to understand the relation between the theories \( HZ, ku \) and \( ko \) and the subalgebras of \( A \) to which they correspond. The cofiber sequence \( \Sigma ko \longrightarrow ko \longrightarrow ku \longrightarrow \Sigma HZ \) gives rise to an exact couple for computing \( HZ \wedge ko \wedge ku \) from \( H^\ast ku \) whose first differential is \( H^\ast \Sigma \longrightarrow \Sigma^2 \Sigma^3 \longrightarrow \Sigma HZ \), and this turns out to be \( Sq^1 \), which generates \( E(0) \). The cofiber sequence \( \Sigma^2 ku \longrightarrow ku \longrightarrow HZ \longrightarrow \Sigma^3 ko \) gives rise to an exact couple for computing \( HZ \wedge ku \) from \( HZ \wedge ku \) whose first differential is \( HZ \longrightarrow \Sigma^3 ku \longrightarrow \Sigma^4 HZ \), and this turns out to be a lift to integral cohomology of \( Q_1 \), which, together with \( E(0) \), generates \( E(1) \). Finally, the cofiber sequence \( \Sigma ko \longrightarrow ko \longrightarrow ku \longrightarrow \Sigma^2 ko \) gives rise to an exact couple for computing \( ko \wedge ku \) from \( ku \wedge ku \) whose first differential is \( ku \longrightarrow \Sigma^2 ko \longrightarrow \Sigma^2 ku \), and this turns out to be a lift to ku-theory of \( Sq^2 \), which, together with \( E(1), \) generates \( A(1) \).

This takes on additional force in connection with the Adams spectral sequence
\[
\text{Ext}_A(H^\ast Y, H^\ast X) \Longrightarrow [X, Y]_2
\]
converging to the 2-completion of the module of homotopy classes of maps from $X$ to $Y$. With $E = H\mathbb{Z}$, $ku$, or $ko$ and $B = E(0)$, $E(1)$, or $A(1)$ respectively, we get

$$\text{Ext}_A(H^*E, H^*X) \implies [X, E]_2^\wedge = (E^*X)_2^\wedge$$

and

$$\text{Ext}_A(H^*(E \wedge X), F_2) \implies [S, E \wedge X]_2^\wedge = (E_*X)_2^\wedge$$

By the isomorphisms above, these can be written

$$\text{Ext}_A(A \otimes_B F_2, H^*X) \implies [X, E]_2^\wedge = (E^*X)_2^\wedge$$

and

$$\text{Ext}_A(A \otimes_B H^*X, F_2) \implies [S, E \wedge X]_2^\wedge = (E_*X)_2^\wedge$$

and by standard change of rings isomorphisms, this is

$$\text{Ext}_B(F_2, H^*X) \implies [X, E]_2^\wedge = (E^*X)_2^\wedge$$

and

$$\text{Ext}_B(H^*X, F_2) \implies [S, E \wedge X]_2^\wedge = (E_*X)_2^\wedge$$

In particular, the structure of $H^*X$ as a module over the semi-dihedral algebra $A(1)$ determines the $E_2$-term of the Adams spectral sequences converging to $ko^*X$ and to $ko_*X$.

As usual, $E_2$ is not the end of the story, and it turns out that differentials in these spectral sequences are affected by the rest of the $A$-module structure of $H^*X$. This should be understood as telling us about the relation between the rest of the Steenrod algebra and $\text{End}(E)$.

I should admit that we have not used the structure theory of Crawley-Boevey. In fact, we are working over $F_2$ so would have had to do the Galois descent from $F_4$ to $F_2$ to do so. But more significantly, the small number of modules we have seen in algebraic topology can be dealt with on an ad hoc basis as they occur.

Note also, that the relation between $ko$ and $ku$, and its reflection in the relation between $E(1)$ and $A(1)$, suggests that we should be able to classify modules over $A(1)$ by some sort of Galois theory together with the extremely simple classification of modules over the exterior algebra on two generators, $E(1)$.

References

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4. Crawley-Boevey, Functorial Filtrations I, II, and III.
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