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AN ASYMPTOTIC ANALYSIS ON THE FORM OF NAGHDI TYPE ARCH MODEL

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We consider a one-dimensional model of generally curved elastic arches whose cross-sections are rectangular. The model is of Naghdi's type which is a generalization of the Timoshenko beam model, which allows bending, membrane and transverse shearing deformations. Its form is basically determined in the literature, except for the value of a shear correction factor. With this factor being set to 1, we prove that the modelling error in the interior relative energy norm is proportional to the arch thickness. This result holds for the full range of arch shapes and very general loads. Lower modelling accuracy is proven to hold up to the arch ends. Any shear correction factor other than 1 makes the model diverge from the elasticity theory when a significant shear is involved in the deformation.

Keywords: Elastic arch; Timoshenko beam; shear correction factor.

AMS Subject Classification: 73K10, 73C02

1. Introduction

For a straight beam of length L and small rectangular cross section of dimension ϵ, the Timoshenko beam bending model is an approximation to the three-dimensional (3D) linear elasticity theory, which determines the transverse deflection w and normal fiber rotation θ, both being single variable functions, by minimizing the functional

$$\frac{1}{2}\int_0^L \left[\frac{1}{3}\epsilon^2 E \left(\frac{d\theta}{dx}\right)^2 + \kappa\mu\left(\frac{dw}{dx}+\theta\right)^2\right] dx - \langle \mathbf{f}, (\theta, w)\rangle \tag{1.1}$$

in a space of admissible functions. In the model, $\mathbf{f}$ is the resultant loading functional that is expressible in terms of the loading force densities, E is the Young's modulus of the elastic material, μ the shear modulus, and κ a dimensionless quantity called the *shear correction factor*. This model is widely used in engineering computations and has been extensively analyzed. It has the advantages of allowing transverse shear deformations and offering more accuracy than the fourth-order bending model that determines the transverse deflection only, especially when a significant shear is involved in the beam deformation.[10] The form of this model is generally agreed upon, except for the value of κ. This value was determined as 0.667 by Timoshenko,[21] 0.822 by Mindlin,[7] 5/6 by Roark,[18] $10(1+\nu)/(12+11\nu)$ by Cowper,[7] and many more. (Here, ν is the Poisson ratio of the elastic material.) These values were derived, for example, from comparing the model solutions with some known semi-analytic solutions of 3D elasticity for special problems, like the cantilever.[14] In the more recent literature of mathematical and numerical analysis of this model, κ is often mentioned as a shear correction factor without a specified value.[1,3,8,16] The shear correction factor also appears in the Reissner–Mindlin plate bending model, the Naghdi shell model, and a generalization of the Timoshenko beam model (1.1) to curved arches. This arch model, to which this paper is mainly devoted, is to Naghdi's shell model as Timoshenko beam model is to Reissner–Mindlin plate model. The value of κ is one of the unresolved issues in these models, of which the common feature is the transverse shear deformability. The value 5/6 is often viewed as the best.[5,6] However, there are also theories favoring other values.[13,25]

In this paper, we present an analysis for the arch model. For beams, our results reduce to the conclusion that the shear correction factor should be taken as $\kappa = 1$. This is based on an asymptotic error estimate between the model solution and the elasticity solution. The argument is as follows. We consider a sequence of beams of varying thickness ϵ, of fixed length L, subject to the same kind of boundary conditions, and made of the same elastic material that is homogeneous and isotropic. Under the usual assumption on the dependence of loading force densities on ϵ, the deformation determined by the Timoshenko model is either bending dominated or shear dominated when $\epsilon \to 0$. The latter is the case in which a significant shear arises for small ϵ. In the bending dominated case, the relative difference in the energy norm between Timoshenko solutions for different κ values is of the order $\mathcal{O}(\epsilon^2)$. This is also the order of difference between the Timoshenko solution and the solution of the fourth-order beam bending model in this case. It is the shear dominated case in which the Timoshenko solution is sensitive to the value of κ, in which case the fourth-order beam bending model is totally useless and the validity of the Timoshenko model requires $\kappa = 1$ in it. Shear dominance occurs when the classical fourth-order bending model yields a zero solution, in which case it is the surface couple (odd part of the tangential surface loads) that is responsible for the significant shear. In many of the classical works that concern the shear correction factor, the tangential surface force was often assumed to be zero, with which the effect of the shear correction factor is actually negligible.

Our analysis is for the more general curved arches that involve issues more than the shear correction factor. When the curvature of its middle curve is identically equal to zero, the arch reduces to a straight beam and the arch model decouples

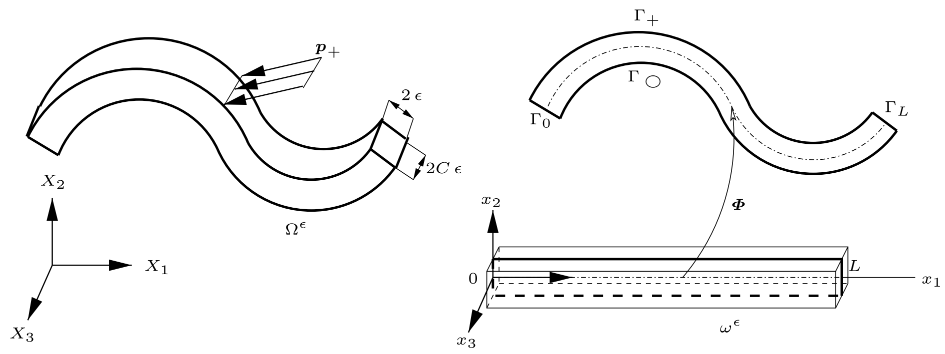

Fig. 1. An arch body and its curvilinear coordinates.

to the Timoshenko beam bending model (1.1) and a beam membrane model. In the remainder of this Introduction, we briefly summarize our results for arches. An arch is a curved thin body with rectangular cross section that occupies a domain Ω^ϵ in the 3D Euclidean space, in which the rectangular coordinate is denoted by (X_1, X_2, X_3), see Fig. 1. The body has a planar middle curve $\mathcal{S}$ that lies in the X_1X_2-plane whose length is L. We parametrize $\mathcal{S}$ by its arclength parameter x_1 through the mapping $\boldsymbol{\phi} : (0, L) \to \mathbb{R}^2$ such that $\mathcal{S} = \{(X_1, X_2, 0); (X_1, X_2) = \boldsymbol{\phi}(x_1), x_1 \in (0, L)\}$. The tangent vector $\mathbf{a}_1 = d\boldsymbol{\phi}/dx_1$ is a unit vector at any point on $\mathcal{S}$. At each point, we define the unit vector $\mathbf{a}_2$ that is in the X_1X_2-plane and orthogonal to $\mathbf{a}_1$ such that $\mathbf{a}_3 = \mathbf{a}_1 \times \mathbf{a}_2$ is the unit vector in the X_3 direction. The unit vectors $\mathbf{a}_i$ furnish the covariant basis on $\mathcal{S}$. (Following conventions, we denote the contravariant basis by $\mathbf{a}^i$ that, in this case, is identical to the covariant basis on $\mathcal{S}$.) We denote the curvature of the middle curve $\mathcal{S}$ at the point of coordinate x_1 by $b(x_1) = \mathbf{a}_2(x_1) \cdot d\mathbf{a}_1(x_1)/dx_1$. The domain Ω^ϵ occupied by the arch is then the image of the rectangular domain $\omega^\epsilon = (0, L) \times (-\epsilon, \epsilon) \times (-C\epsilon, C\epsilon)$ through the mapping $\mathbf{\Phi}(x_1, x_2, x_3) = \boldsymbol{\phi}(x_1) + x_2\mathbf{a}_2(x_1) + x_3\mathbf{a}_3$. I.e.

$$\Omega^\epsilon = \{(X_1, X_2, X_3) = \mathbf{\Phi}(x_1, x_2, x_3); (x_1, x_2, x_3) \in \omega^\epsilon\}.$$

The triple (x_1, x_2, x_3) furnishes the curvilinear coordinates on Ω^ϵ. The mapping $\mathbf{\Phi}$ continuously extends to $\overline{\omega}^\epsilon$ so that the curvilinear coordinates apply to the boundary of Ω^ϵ as well. A function defined on $\overline{\Omega}^\epsilon$ will be identified with a function on $\overline{\omega}^\epsilon$ through this mapping, and denoted by the same symbol. A function of fewer variables will be identified with a function of more variables that is constant in the additional variables. For example, $\mathbf{a}_1$ shall be viewed as a vector field defined on $\overline{\omega}^\epsilon$ such that $\mathbf{a}_1(x_1, x_2, x_3) = \mathbf{a}_1(x_1)$ for all $(x_1, x_2, x_3) \in \overline{\omega}^\epsilon$, etc. We shall often replace x_1 by x, and denote the derivative with respect to x by ∂, even when it is an ordinary derivative of a single variable function. Greek subscripts and superscripts, except ϵ that stands for the half thickness of the arch, always take their values in $\{1, 2\}$, and Latin scripts in $\{1, 2, 3\}$, except e, o, a and m, which are used to indicate even, odd, average, and moment, respectively. Summation convention with respect to repeated superscripts and subscripts will be used together with this rule. We denote second-order tensors by boldface Greek letters, and vectors by boldface Latin letters. Vectors and tensors will be given in terms of their covariant components, or contra-variant components, depending on convenience. The notation $P \lesssim Q$ means that there exists a constant C independent of ϵ, P and Q such that $P \leq CQ$. The notation $P \simeq Q$ means $P \lesssim Q$ and $Q \lesssim P$.

The boundary of Ω^ϵ is composed of the upper and lower faces $\Gamma_\pm$ on which $x_2 = \pm\epsilon$, the left and right end faces Γ_0 and Γ_L where $x_1 = 0$ and L, respectively, and the front and rear faces where $x_3 = \pm C\epsilon$. We assume that the arch is made of homogeneous and isotropic elastic material whose Lamé constants are λ and μ. To make the presentation more general, we let the arch be clamped (with which we mean that it is welded to a rigid surface) at Γ_0, free on the front and rear faces, loaded by surface forces on $\Gamma_\pm$ and Γ_L, and loaded by a body force. We assume that all the forces are parallel to the X_1X_2-plane and constant in x_3. Furthermore, we assume the body force density changes linearly in x_2. (This assumption seems reasonable when one considers a sequence of thinner and thinner arches hanging in a given force field, gravitational or electro-magnetic.) These force densities are denoted by $\mathbf{p}_\pm(x_1)$ on $\Gamma_\pm$, $\mathbf{d}(x_2)$ on Γ_L, and $\mathbf{q}(x_1, x_2)$ over Ω^ϵ, respectively. We define the following resultant loading functions. The average (q_a^α) and moment (q_m^α) of the body force density are defined by

$$q_a^\alpha = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} \mathbf{q}\cdot\mathbf{a}^\alpha dx_2, \quad q_m^\alpha = \frac{1}{2\epsilon^3}\int_{-\epsilon}^{\epsilon} x_2\mathbf{q}\cdot\mathbf{a}^\alpha dx_2. \tag{1.2}$$

The couple (that is the odd part p_o^α) and resultant (that is the even part p_e^α) of the surface forces on $\Gamma_\pm$ are defined by

$$p_o^\alpha = \frac{(\mathbf{p}_+ - \mathbf{p}_-)\cdot\mathbf{a}^\alpha}{2}, \quad p_e^\alpha = \frac{(\mathbf{p}_+ + \mathbf{p}_-)\cdot\mathbf{a}^\alpha}{2\epsilon}. \tag{1.3}$$

We define the components of the resultant d_a^α and moment d_m^α of the surface force on the right end Γ_L by

$$d_a^\alpha = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} \mathbf{d}\cdot\mathbf{a}^\alpha dx_2, \quad d_m^\alpha = \frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} x_2\mathbf{d}\cdot\mathbf{a}^\alpha dx_2. \tag{1.4}$$

The arch model determines the transverse deflection w, the normal fiber rotation θ, and the membrane displacement u, all are single variable functions, by minimizing the functional

$$\frac{1}{2}\int_0^L \left[\frac{1}{3}\epsilon^2 E\left(\frac{d\theta}{dx} + b\left[\frac{du}{dx} - bw\right]\right)^2 + E\left(\frac{du}{dx} - bw\right)^2 + \kappa\mu\left(\theta + \frac{dw}{dx} + bu\right)^2\right] dx \\ - \langle \mathbf{f}, (\theta, u, w)\rangle \tag{1.5}$$

in the space $H = [H^1_D(0, L)]^3$. Here $H^1_D(0, L)$ is the L^2 based first-order Sobolev space whose functions vanish at 0. The loading functional is defined as

$$\langle \mathbf{f}, (\theta, u, w)\rangle = d^1_m \theta(L) + [d^1_a - \nu p^2_o(L)]u(L) + d^2_a w(L) + \int_0^L [p^1_o \theta + (p^1_e + q^1_a - bp^1_o + \nu \partial p^2_o)u + (p^2_e + q^2_a + (\nu - 1)bp^2_o)w]dx. \tag{1.6}$$

This model involves the functions $\gamma(u, w) = \partial u - bw$, $\rho(\theta, u, w) = \partial\theta + b(\partial u - bw)$, and $\tau(\theta, u, w) = \theta + \partial w + bu$, which are the membrane strain, bending strain, and transverse shear strain engendered by (θ, u, w), respectively. In addition to the issue about the value of the shear correction factor κ, the definition of bending strain also has some uncertainty.[6] In analogue to Naghdi's shell model, one would have $\rho(\theta, u, w) = \partial\theta - b(\partial u - bw)$. Following Budianski–Sanders shell theory, one defines $\rho(\theta, u, w) = \partial\theta - \frac{1}{2}b(\partial u - bw)$. In terms of the mini-model, one simply defines the bending strain as $\partial\theta$. It can be seen from our analysis that the validity of the model is not affected by these variances in the bending strain. One way to derive the model is integrating, with some numerical quadrature, the 3D elasticity equation with respect to x_2. If the arch is circular, then a more accurate numerical integration scheme leads to our definition. Our definition also has the merit of more cancellations in the constitutive residual. However, the effect of such modification in the bending strain is insignificant as far as the rate of convergence of the model toward the 3D elasticity is concerned. When $b \equiv 0$, the arch becomes a straight beam and the model (1.5) decouples to the Timoshenko beam bending model (1.1) that determines (θ, w) and a beam membrane model that determines u. The resultant loading functional $\mathbf{f}$ decouples to a functional of (θ, w) and a functional of u, both of which are standard.

The subspace $K = \{(\theta, u, w) \in H; \tau(\theta, u, w) = 0, \gamma(u, w) = 0\} \subset H$ is the subspace of pure bending deformations (without membrane or shear). It plays a pivotal role in the analysis. We shall assume that the above-defined loading functions are all independent of ϵ, which is consistent with assumptions in asymptotic analysis of plate in Refs. 5 and 9. Then, the loading functional $\mathbf{f}$ is independent of ϵ. We shall assume that $\mathbf{f} \neq 0$. (Indeed, there are very rare cases in which an arch could be loaded in such a way that its response is very small and in which $\mathbf{f} = 0$.) When $\epsilon \to 0$, the unique solution $(\theta^\epsilon, u^\epsilon, w^\epsilon)$ of (1.5) asymptotically behaves in two distinctively different ways, depending on whether $\mathbf{f}$ induces pure bending. I.e., whether $\mathbf{f}|_K \neq 0$. Let $B^\epsilon = \frac{1}{3}\epsilon^2 E \int_0^L \rho^2(\theta^\epsilon, u^\epsilon, w^\epsilon)dx$ be the bending strain energy, $S^\epsilon = \int_0^L [E\gamma^2(u^\epsilon, w^\epsilon) + \kappa\mu\tau^2(\theta^\epsilon, u^\epsilon, w^\epsilon)]dx$ be the membrane-shear strain energy, and $E^\epsilon = B^\epsilon + S^\epsilon$ the total energy. The key features of the asymptotic behavior are as follows.

- If $\mathbf{f}|_K \neq 0$, then $B^\epsilon/S^\epsilon \simeq \epsilon^{-2}$ and $E^\epsilon \simeq \epsilon^{-2}$.
- If $\mathbf{f}|_K = 0$, then $B^\epsilon/S^\epsilon \simeq \epsilon^2$ and $E^\epsilon \simeq 1$.

Thus the condition $\mathbf{f}|_K \neq 0$ characterizes the arch behavior as bending dominated, and $\mathbf{f}|_K = 0$ membrane-shear dominated. In response to applied forces of given magnitude, the strain energy E^ϵ arising in a membrane-shear dominated arch is lower in orders of magnitude than that in a bending dominated arch. So is the magnitude of displacement. This indicates that a membrane-shear dominated arch is much stiffer and exhibits much greater strength than a bending dominated arch. The condition $\mathbf{f}|_K = 0$ is a delicate balance between the shape of an arch and the loads on it. For example, if an arch is to support its own weight only then a catenary shape makes this condition, and if an arch is to support vertical and horizontally uniform surface loads, then a parabolic shape achieves such balance. The delicate balance implies that such a situation is highly unstable (a small perturbation of the loading and/or of the shape leads to large displacements). While arch structures are generally bending dominated, there are important arches designed to achieve greater strength resisting against certain, often major, load, by making the arch shape and such load together satisfy the membrane-shear domination condition. One such example is the Saint Louis Gateway Arch that is in the shape of catenary, and thus strongly resists against gravity. For a similar detailed discussion on shells, see Ref. 4. The two distinctive behaviors are peculiar to elastic arches and beams. This is not valid for Naghdi shell or Reissner–Mindlin plate, for which there are the so-called intermediate behaviors.[4,19,24,25]

Let $\mathbf{u}^{\epsilon *}$ be the arch displacement solution of the 3D elasticity. Based on the solution of the model (1.5), we define a displacement field $\mathbf{u}^\epsilon$, by explicit formulas, on the arch such that $\mathbf{u}^\epsilon|_{\mathcal{S}} = (u^\epsilon \mathbf{a}_1 + w^\epsilon \mathbf{a}_2)$, and it deforms a flat rectangular cross-section to a warped surface. We estimate the difference between $\mathbf{u}^{\epsilon *}$ and $\mathbf{u}^\epsilon$ in the interior energy norm. The energy norm of a stress tensor field $\boldsymbol{\sigma}$ on a subset $\Omega \subset \Omega^\epsilon$ is defined by $\|\boldsymbol{\sigma}\|_{E(\Omega)} = \left[(\mathcal{A}\boldsymbol{\sigma} : \boldsymbol{\sigma})_{L^2(\Omega)}\right]^{1/2}$, and for a strain tensor field $\boldsymbol{\epsilon}$, the energy norm is $\|\boldsymbol{\epsilon}\|_{E(\Omega)} = \left[(\mathcal{C}\boldsymbol{\epsilon} : \boldsymbol{\epsilon})_{L^2(\Omega)}\right]^{1/2}$. Here, $\mathcal{C}$ is the elasticity tensor of the arch, and $\mathcal{A}$ is the compliance tensor that is the inverse of $\mathcal{C}$. We prove that in the bending dominated case

$$\frac{\|\boldsymbol{\epsilon}(\mathbf{u}^{\epsilon *} - \mathbf{u}^\epsilon)\|_{E(\Omega_0^\epsilon)}}{\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)}} \lesssim \epsilon,$$

which is valid for κ being any number. Here Ω_0^ϵ is the interior portion of Ω^ϵ obtained by cutting off the two end portions of length $C\epsilon$. Changing the value of κ will only change $\mathbf{u}^\epsilon$ by $\mathcal{O}(\epsilon^2)$ in the relative energy norm. In the membrane-shear dominated case we prove the same estimate for $\kappa = 1$. In this latter case, if $\kappa \neq 1$ and the applied couple p_o^1 is not zero then

$$\frac{\|\boldsymbol{\epsilon}(\mathbf{u}^{\epsilon *} - \mathbf{u}^\epsilon)\|_{E(\Omega_0^\epsilon)}}{\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)}} \simeq |1 - \kappa|.$$

Thus $\kappa \neq 1$ makes the model diverge from the elasticity theory in this case.

Since the behavior of $\mathbf{u}^{\epsilon *}$ is very elusive when $\epsilon \to 0$, it is rather hard to obtain these estimates by directly comparing $\mathbf{u}^\epsilon$ with $\mathbf{u}^{\epsilon *}$. We shall use the Prager–Synge hyper-circle theorem[17] and Saint-Venant's principle[22] to establish these estimates. For this purpose, we need to construct a statically admissible stress field and a kinematically admissible displacement field. This is done in several steps. The most difficult part is the construction of a stress field $\boldsymbol{\sigma}^\epsilon$ that satisfies the equilibrium equation $\operatorname{div} \boldsymbol{\sigma}^\epsilon + \mathbf{q} = 0$ in Ω^ϵ and the surface force conditions $\boldsymbol{\sigma}^\epsilon \mathbf{n} = \mathbf{p}_\pm$ on $\Gamma_\pm$ and the free condition on the front and rear faces. The force condition on the right end is not precisely satisfied since we can only make $\boldsymbol{\sigma}^\epsilon \mathbf{n} = \bar{\mathbf{d}}$ on Γ_L. Here, $\bar{\mathbf{d}}$ bears the resultant and moment of $\mathbf{d}$, $\bar{\mathbf{d}} \cdot \mathbf{a}_1$ is linear in x_2, and $\bar{\mathbf{d}} \cdot \mathbf{a}_2$ is quadratic in x_2 such that the continuity condition $\bar{\mathbf{d}} \cdot \mathbf{a}_2(\pm\epsilon) = \pm \mathbf{p}_\pm(L) \cdot \mathbf{a}_1$ are satisfied. Thus the residual $\mathbf{d} - \bar{\mathbf{d}}$ has zero resultant and moment. The construction of $\boldsymbol{\sigma}^\epsilon$ and the definition of $\mathbf{u}^\epsilon$ ensure the smallness of the constitutive residual $\boldsymbol{\sigma}^\epsilon - \mathcal{C}\boldsymbol{\epsilon}\mathbf{u}^\epsilon$. However, neither $\mathbf{u}^\epsilon$ nor $\boldsymbol{\sigma}^\epsilon$ is admissible, and they fail to meet the requirements of the Prager–Synge theorem. They both need some modifications at the arch ends. We define $\boldsymbol{\sigma}_L^\epsilon$ and $\mathbf{u}_L^\epsilon$ as the stress and displacement solutions of the 3D elasticity theory on the arch such that $\mathbf{u}_L^\epsilon|_{\Gamma_0} = 0$ and $\boldsymbol{\sigma}_L^\epsilon \mathbf{n} = \mathbf{d} - \bar{\mathbf{d}}$ on Γ_L, and the arch is free on upper, lower, front and rear faces and free of body force. Saint-Venant's principle shows that such fields exponentially decay from the right end, and they are negligible at a distance $C\epsilon$ away from the right end. (Although the proof of the principle given by Toupin[22] assumes that the thin body is infinitely long, his argument actually applies to our case.) On the left end, due to a modification involved in $\mathbf{u}^\epsilon$, the welding displacement condition is not satisfied. We define $\mathbf{u}_0^\epsilon$ as the solution of the elasticity theory on the arch such that $\mathbf{u}_0^\epsilon = -\mathbf{u}^\epsilon$ on Γ_0, and the arch is free on all the other faces and subject to zero body force. The stress field $\boldsymbol{\sigma}_0^\epsilon = \mathcal{C}\boldsymbol{\epsilon}\mathbf{u}_0^\epsilon$ then exponentially decay from the left end. This is because the reacting force (the Lagrange multiplier) on Γ_0 has zero resultant and zero moment there, and Saint-Venant's principle applies. Our statically admissible stress field is then $\bar{\boldsymbol{\sigma}}^\epsilon = \boldsymbol{\sigma}^\epsilon + \boldsymbol{\sigma}_0^\epsilon + \boldsymbol{\sigma}_L^\epsilon$ and kinematically admissible displacement field is $\bar{\mathbf{u}}^\epsilon = \mathbf{u}^\epsilon + \mathbf{u}_0^\epsilon + \mathbf{u}_L^\epsilon$. And we have $\bar{\boldsymbol{\sigma}}^\epsilon - \mathcal{C}\boldsymbol{\epsilon}\bar{\mathbf{u}}^\epsilon = \boldsymbol{\sigma}^\epsilon - \mathcal{C}\boldsymbol{\epsilon}\mathbf{u}^\epsilon$. According the Prager–Synge hyper-circle theorem, we have

$$\|\boldsymbol{\epsilon}(\bar{\mathbf{u}}^\epsilon - \mathbf{u}^{\epsilon*})\|_{E(\Omega^\epsilon)}^2 + \|\boldsymbol{\sigma}^{\epsilon*} - \bar{\boldsymbol{\sigma}}^\epsilon\|_{E(\Omega^\epsilon)}^2 = \|\boldsymbol{\sigma}^\epsilon - \mathcal{C}\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)}^2.$$

Since $\mathbf{u}^\epsilon$ and $\bar{\mathbf{u}}^\epsilon$ are virtually equal on Ω_0^ϵ, it follows the estimate

$$\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon) - \boldsymbol{\epsilon}(\mathbf{u}^{\epsilon*})\|_{E(\Omega_0^\epsilon)} \le \|\boldsymbol{\sigma}^\epsilon - \mathcal{C}\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)},$$

of which the right side is amenable to a rigorous analysis.

Although $\bar{\mathbf{u}}^\epsilon$ well approximates $\mathbf{u}^{\epsilon*}$ up to the arch ends, it could be very complicated near the ends, and hard to compute. However, we prove that $\|\boldsymbol{\epsilon}(\mathbf{u}_0^\epsilon)\|_{E(\Omega^\epsilon)}/\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)} \lesssim \sqrt{\epsilon}$. Thus the relative error $\sqrt{\epsilon}$ between $\mathbf{u}^\epsilon$ and $\mathbf{u}^{\epsilon*}$ holds up to the clamped end. Since the arch model can only incorporate the resultant and moment applied on Γ_L and the missed residual $\mathbf{d} - \bar{\mathbf{d}}$ could cause arbitrarily large strain at the end, it is impossible for the arch model to provide valid approximation up to the right end, except if one assumes that $\mathbf{d} - \bar{\mathbf{d}}$ is small.

The paper is organized as follows. In Sec. 2 we analyze the asymptotic dependence on ϵ of the solution of the arch model (1.5). For this purpose, we introduce a functional equation and present an abstract analysis, from which the asymptotic estimates on the model solution follow. This abstract estimate will also be used to analyze the dependence on ϵ of the stress field $\boldsymbol{\sigma}^\epsilon$ that is constructed in Sec. 3. The construction of this stress field represents the main effort of this work. We introduce new rescaled components that differ from the physical components commonly used in the classical literature.[12,15,20] It seems that only by this rescaled components one could write the 3D equilibrium equation in a form that allows construction of stress fields that are essentially one-dimensional functions. It seems advantageous to use this method to deal with the more general surface loads that could reveal the necessity of defining a certain shear correction factor. Finally, in Sec. 4 we prove the modelling error estimates using the techniques outlined above. Some of the results of this paper can be extended, in weaker version, to general elastic shells.[23] Relevant results for plates can be found in Refs. 2 and 25.

2. Asymptotic Estimates on the Arch Model Solution

2.1. *An abstract theory*

Notations in this subsection are independent of the rest of the paper. By properly defining spaces and operators, the beam model (1.1) and the arch model (1.5) can be written in the form of the functional equation (2.2) below. Let H, U and V be Hilbert spaces, $B : H \to U$ a bounded linear operator, and $S : H \to V$ a bounded linear operator with closed range. We assume that

$$\|B\mathfrak{u}\|_U + \|S\mathfrak{u}\|_V \simeq \|\mathfrak{u}\|_H \quad \forall\, \mathfrak{u} \in H. \tag{2.1}$$

Given an $f \in H^*$, the dual of H, we consider the variational problem of finding $\mathfrak{u}^\epsilon \in H$ such that

$$\epsilon^2(B\mathfrak{u}^\epsilon, B\mathfrak{v})_U + (S\mathfrak{u}^\epsilon, S\mathfrak{v})_V = \langle f, \mathfrak{v}\rangle \quad \forall\, \mathfrak{v} \in H. \tag{2.2}$$

The problem obviously has a unique solution $\mathfrak{u}^\epsilon \in H$ whose asymptotic behavior is drastically different depending on whether $f|_K = 0$ or not. Here $K \subset H$ is the kernel space of the operator S. The two lemmas below describe the distinctive behaviors. By the equivalence assumption (2.1), the bilinear form $(\mathfrak{u}, \mathfrak{v})_{\mathcal{H}} = (B\mathfrak{u}, B\mathfrak{v})_U + (S\mathfrak{u}, S\mathfrak{v})_V$ defines an inner product on H, which is equivalent to the original one. With this new inner product, the space H will be denoted by $\mathcal{H}$. Without loss of generality, we assume that the operator S maps H onto V. Otherwise, we just replace V by the range of S in it. The operator S is then an isomorphism between $K^\perp_{\mathcal{H}}$, the orthogonal complement of K with respect to the $\mathcal{H}$-norm, and V.

Lemma 2.1. *If* $f|_K = 0$, *by the closed range theorem, there exists a unique* $\xi \in V$ *such that* $(\xi, S\mathfrak{v})_V = \langle f, \mathfrak{v}\rangle$ *for all* $\mathfrak{v} \in H$ *and* $\|\xi\|_V \simeq \|f\|_{H^*}$. *We have the estimate*

$$\|\mathfrak{u}^\epsilon\|_H + \epsilon^{-2}\|S\mathfrak{u}^\epsilon - \xi\|_V \lesssim \|f\|_{H^*}. \tag{2.3}$$

Proof. First, it is easy to see that when $f|_K = 0$, the solution $\mathfrak{u}^\epsilon$ of (2.2) lies in the subspace $K^\perp_{\mathcal{H}}$. As S is an isomorphism between $K^\perp_{\mathcal{H}}$ and V, we have

$$\|\mathfrak{u}^\epsilon\|_H \simeq \|S\mathfrak{u}^\epsilon\|_V. \tag{2.4}$$

Since $\epsilon^2(B\mathfrak{u}^\epsilon, B\mathfrak{u}^\epsilon)_U + (S\mathfrak{u}^\epsilon, S\mathfrak{u}^\epsilon)_V = (\xi, S\mathfrak{u}^\epsilon)_V$, we have

$$(S\mathfrak{u}^\epsilon, S\mathfrak{u}^\epsilon)_V \le |(\xi, S\mathfrak{u}^\epsilon)_V| \le \|\xi\|_V\|S\mathfrak{u}^\epsilon\|_V.$$

Therefore, $\|S\mathfrak{u}^\epsilon\|_V \le \|\xi\|_V \simeq \|f\|_{H^*}$. Thus $\|\mathfrak{u}^\epsilon\|_H \lesssim \|f\|_{H^*}$. Let $\mathfrak{u}^0 \in K^\perp_{\mathcal{H}}$ be the unique element such that $S\mathfrak{u}^0 = \xi$. Note that

$$\epsilon^2(B\mathfrak{u}^\epsilon, B(\mathfrak{u}^\epsilon - \mathfrak{u}^0))_U + (S\mathfrak{u}^\epsilon, S(\mathfrak{u}^\epsilon - \mathfrak{u}^0))_V = (\xi, S(\mathfrak{u}^\epsilon - \mathfrak{u}^0))_V.$$

We see

$$\|S\mathfrak{u}^\epsilon - \xi\|^2_V = \epsilon^2|(B\mathfrak{u}^\epsilon, B(\mathfrak{u}^\epsilon - \mathfrak{u}^0))_U| \lesssim \epsilon^2\|\mathfrak{u}^\epsilon\|_H\|S\mathfrak{u}^\epsilon - \xi\|_V.$$

Therefore,

$$\|S\mathfrak{u}^\epsilon - \xi\|_V \lesssim \epsilon^2\|\mathfrak{u}^\epsilon\|_H \lesssim \epsilon^2\|f\|_{H^*}.$$

The desired result then follows. □

Lemma 2.2. *If* $f|_K \ne 0$, *then there exists a unique nonzero element* $\tilde{\mathfrak{u}}^0 \in K$ *such that*

$$(B\tilde{\mathfrak{u}}^0, B\mathfrak{v})_U = \langle f, \mathfrak{v}\rangle \quad \forall\ \mathfrak{v} \in K.$$

Obviously, $\|\tilde{\mathfrak{u}}^0\|_H \lesssim \|f\|_{H^*}$. *We have the estimate on* $\tilde{\mathfrak{u}}^\epsilon = \epsilon^2\mathfrak{u}^\epsilon$ *that*

$$\|\tilde{\mathfrak{u}}^\epsilon - \tilde{\mathfrak{u}}^0\|_H + \|S\tilde{\mathfrak{u}}^\epsilon\|_V \lesssim \epsilon^2\|f\|_{H^*}. \tag{2.5}$$

Proof. The key observation is that $\tilde{\mathfrak{u}}^\epsilon - \tilde{\mathfrak{u}}^0$ satisfies the equation

$$\epsilon^2(B(\tilde{\mathfrak{u}}^\epsilon - \tilde{\mathfrak{u}}^0), Bv)_U + (S(\tilde{\mathfrak{u}}^\epsilon - \tilde{\mathfrak{u}}^0), Sv)_V = \epsilon^2[\langle f, \mathfrak{v}\rangle - (B\tilde{\mathfrak{u}}^0, B\mathfrak{v})_U] \quad \forall\ \mathfrak{v} \in H.$$

The right-hand side of this equation is a functional that annihilates K. Therefore, the estimate (2.3) of Lemma 2.1 is applicable to estimating $\tilde{\mathfrak{u}}^\epsilon - \tilde{\mathfrak{u}}^0$. We have

$$\|\tilde{\mathfrak{u}}^\epsilon - \tilde{\mathfrak{u}}^0\|_H + \epsilon^{-2}\|S\tilde{\mathfrak{u}}^\epsilon - \epsilon^2\eta\|_V \lesssim \epsilon^2\|f\|_{H^*}.$$

Here $\eta \in V$ is the unique element such that $\langle f, \mathfrak{v}\rangle - (B\tilde{\mathfrak{u}}^0, B\mathfrak{v})_U = (\eta, S\mathfrak{v})_V\ \forall\ \mathfrak{v} \in H$. It is easy to see that $\|\eta\|_V \lesssim \|f\|_{H^*}$. Obviously, $\epsilon^{-2}\|S\tilde{\mathfrak{u}}^\epsilon\|_V \lesssim \|\eta\|_V + \epsilon^2\|f\|_{H^*}$. Therefore, $\|S\tilde{\mathfrak{u}}^\epsilon\|_V \lesssim \epsilon^2\|f\|_{H^*}$. The estimate (2.6) then follows. □

From these lemmas we see that if $f|_K = 0$ then $\mathfrak{u}^\epsilon$ converges to a nonzero limit in the H norm at the rate ϵ^2. The limit resides in the subspace $K^\perp_{\mathcal{H}}$. We have the S-domination in the sense that $\epsilon^2(B\mathfrak{u}^\epsilon, B\mathfrak{u}^\epsilon)_U/(S\mathfrak{u}^\epsilon, S\mathfrak{u}^\epsilon)_V \simeq \epsilon^2$ and the "total energy" tends to a nonzero constant since we have $\epsilon^2(B\mathfrak{u}^\epsilon, B\mathfrak{u}^\epsilon)_U + (S\mathfrak{u}^\epsilon, S\mathfrak{u}^\epsilon)_V \to (S\mathfrak{u}^0, S\mathfrak{u}^0)_V \neq 0$.

If $f|_K \neq 0$ then $\tilde{\mathfrak{u}}^\epsilon = \epsilon^2 \mathfrak{u}^\epsilon$ converges to a nonzero limit $\tilde{\mathfrak{u}}^0 \in K$ in the H norm at the rate ϵ^2. Therefore, $\mathfrak{u}^\epsilon$ has the magnitude of order ϵ^{-2}. In this case, we have the B-domination in the sense that $\epsilon^2(B\mathfrak{u}^\epsilon, B\mathfrak{u}^\epsilon)_U/(S\mathfrak{u}^\epsilon, S\mathfrak{u}^\epsilon)_V \simeq \epsilon^{-2}$, and the "total energy" blows up at the rate ϵ^{-2} since we have $\epsilon^2(B\mathfrak{u}^\epsilon, B\mathfrak{u}^\epsilon)_U + (S\mathfrak{u}^\epsilon, S\mathfrak{u}^\epsilon)_V \simeq \epsilon^{-2}$. In this case, if we assume that $f = \epsilon^2 F$ with F being independent of ϵ, then $\mathfrak{u}^\epsilon$ itself converges to a limit $\mathfrak{u}^0 \in K$ defined by

$$(B\mathfrak{u}^0, B\mathfrak{v})_U = \langle F, \mathfrak{v}\rangle \quad \forall\, \mathfrak{v} \in K.$$

And we have the estimate

$$\|\mathfrak{u}^\epsilon - \mathfrak{u}^0\|_H + \|S\mathfrak{u}^\epsilon\|_V \lesssim \epsilon^2 \|F\|_{H^*}. \tag{2.6}$$

The B-dominance remains, but the "total energy" will be reduced from the order ϵ^{-2} to ϵ^2. This result will be used in this way.

Remark 2.1. Both the estimates (2.3) and (2.5) are sharp, and so is (2.6). Actually equivalence holds.[24] The estimates of this subsection crucially hinge on the assumption that the operator S has closed range in V. This condition shall be verified for the Timoshenko beam bending model (1.1) and the arch model (1.5). But it is not met by Reissner–Mindlin plate model and Naghdi shell model, for which refined analysis is needed.[24]

2.2. *Asymptotic behavior of the model solution*

The arch model (1.5) fits in the abstract framework (2.2) in an obvious manner. We let $H = [H^1_D]^3$, $U = L^2$, and $V = [L^2]^2$. Here and henceforth, in default of the domain, a function space is a space of functions defined on $(0, L)$. The inner product in H is the usual one. The inner products in U and V need to be changed slightly but equivalently. For $\rho_1, \rho_2 \in U$, we define $(\rho_1, \rho_2)_U = E\frac{1}{3}(\rho_1, \rho_2)_{L^2}$, and for $[\gamma_1, \tau_1], [\gamma_2, \tau_2] \in V$, we define $([\gamma_1, \tau_1], [\gamma_2, \tau_2])_V = E(\gamma_1, \gamma_2)_{L^2} + \kappa\mu(\tau_1, \tau_2)_{L^2}$. We define the operators by $B(\theta, u, w) = \rho(\theta, u, w)$ and $S(\theta, u, w) = [\gamma(u, w), \tau(\theta, u, w)]$. Recall that

$$\gamma(u, w) = \partial u - bw,\ \ \rho(\theta, u, w) = \partial\theta + b(\partial u - bw),\ \ \tau(\theta, u, w) = \theta + \partial w + bu \tag{2.7}$$

are the membrane, bending, and transverse shear engendered by the displacement functions (θ, u, w), respectively. To apply the above lemmas, we need to verify the equivalence (2.1) and prove that the operator S has a closed range in V. These are addressed by the following lemmas.

Lemma 2.3. *The equivalence*

$$\|\rho(\theta, u, w)\|_{L^2} + \|\gamma(u, w)\|_{L^2} + \|\tau(\theta, u, w)\|_{L^2} \simeq \|(\theta, u, w)\|_H \quad \forall\ (\theta, u, w) \in H \tag{2.8}$$

holds.

This result establishes the equivalence (2.1), and also the well-posedness of the model (1.5) as long as the resultant loading functional belongs to the dual of H. To prove this result, we need Peetre's lemma[11]:

Lemma 2.4. *Let X, Y_1, Y_2 be Hilbert spaces, and let $A_1 : X \to Y_1$ and $A_2 : X \to Y_2$ be bounded linear operators with A_1 injective and A_2 compact. If there exists a constant $c > 0$ such that $\|x\|_X \le c(\|A_1 x\|_{Y_1} + \|A_2 x\|_{Y_2})\ \forall\ x \in X$, then there exists a constant $c' > 0$ such that $\|x\|_X \le c'\|A_1 x\|_{Y_1}\ \forall\ x \in X$.*

Proof. (of Lemma 2.3) It is obvious that the right-hand side is an upper bound of the left-hand side. We need to show that it is also a lower bound. We first see that the left-hand side is bounded from below by a constant multiple of $\|\partial\theta\|_{L^2} + \|\partial u - bw\|_{L^2} + \|\partial w + \theta + bu\|_{L^2}$. We consider the operators A_1 and A_2 from H to $[L^2]^3$ defined by $A_1(\theta, u, w) = (\partial\theta, \partial u - bw, \partial w + \theta + bu)$ and $A_2(\theta, u, w) = (0, bw, \theta + bu)$. The operator A_1 is injective, since if $(\theta, u, w) \in \ker A_1$, then $\partial\theta = 0$, $\partial u - bw = 0$ and $\partial w + \theta + bu = 0$. Since $\theta(0) = 0$, we have $\theta = 0$, and so $u\partial u + w\partial w = 0$. Therefore, $u^2 + w^2 = \text{const}$. Since $u(0) = w(0) = 0$, we must have $u = w = 0$. The operator A_2 is obviously compact. The statement then follows from Lemma 2.4. □

Lemma 2.5. *If the curvature of the middle curve $\mathcal{S}$ of the arch is not identically equal to zero, then the operator S maps the closed subspace $[H_0^1]^3$ of H onto V. Therefore, S maps H onto V.*

Proof. We consider the restriction of S on $[H_0^1]^3$, which is still denoted by S. We show that the dual operator S^* of S is injective and has closed range. The surjectivity of S then follows from the closed range theorem. The dual operator $S^* : [L^2]^2 \to [H^{-1}]^3$ is defined by

$$S^*(\zeta, \eta) = (\eta, b\eta - \partial\zeta, -\partial\eta - b\zeta) \quad \forall\ (\zeta, \eta) \in [L^2]^2.$$

We first show that S^* is injective. If $(\zeta, \eta) \in \ker S^*$, then $\|\eta\|_{-1} = 0$, $\|b\eta - \partial\zeta\|_{-1} = 0$, and $\|\partial\eta + b\zeta\|_{-1} = 0$, so we have $\eta = 0$, $\|\partial\zeta\|_{-1} = 0$, and $\|b\zeta\|_{-1} = 0$. Since the curvature b is not identically equal to zero, we must have $\zeta = 0$. We then show that S^* has closed range. By viewing S^* as the operator A_1 in Lemma 2.4, and considering the compact operator $A_2 : [L^2]^2 \to [H^{-1}]^3$ defined by $A_2(\eta, \zeta) = (0, b\eta, b\zeta)\ \forall\ (\eta, \zeta) \in [L^2]^2$, the fact that S^* has closed range will follow from Lemma 2.4. □

Remark 2.2. If the curvature of the middle curve $\mathcal{S}$ is identically equal to zero, i.e. the arch is a straight beam, the operator S maps $[H_0^1]^3$ onto the closed subspace $[L^2/\mathbb{R}] \times L^2$ of V. However, S still maps H onto V.

According to Lemmas 2.1 and 2.2, the behavior of the model solution $(\theta^\epsilon, u^\epsilon, w^\epsilon)$ is dramatically different for whether $\mathbf{f}|_K = 0$ or $\mathbf{f}|_K \neq 0$. Here K is the kernel space of S. The former means S-domination. For our arch problem, this is the membrane-shear dominated case. The latter means B-domination that is the bending dominated case. For brevity, in the following we denote $\rho^\epsilon = \rho(\theta^\epsilon, u^\epsilon, w^\epsilon)$, $\gamma^\epsilon = \gamma(u^\epsilon, w^\epsilon)$, and $\tau^\epsilon = (\theta^\epsilon, u^\epsilon, w^\epsilon)$. We will need the following sufficient condition for the problem to be bending dominated. We recall that

$$\langle \mathbf{f}, (\phi, y, z)\rangle = d_m^1\phi(L) + [d_a^1 - \nu p_o^2(L)]y(L) + d_a^2 z(L) \\ + \int_0^L \left[p_o^1\phi + (p_e^1 + q_a^1 - bp_o^1 + \nu\partial p_o^2)y + (p_e^2 + q_a^2 + (\nu-1)bp_o^2)z\right] dx. \tag{2.9}$$

This can be rewritten as

$$\langle \mathbf{f}, (\phi, y, z)\rangle = \int_0^L p_o^1\tau(\phi, y, z)dx - \int_0^L \nu p_o^2\gamma(y, z)dx + \int_0^L [(p_e^1 + q_a^1 - 2bp_o^1)y \\ + (p_e^2 + q_a^2 - bp_o^2 + \partial p_o^1)z]dx + d_a^1 y(L) + [d_a^2 - p_o^1(L)]z(L) + d_m^1\phi(L). \tag{2.10}$$

Lemma 2.6. *If d_m^1, the bending moment applied on Γ_L, is not zero, then the deformation determined by the arch model is bending dominated. I.e.* $\mathbf{f}|_K \neq 0$.

Proof. If $\mathbf{f}|_K = 0$, then by the closed range theorem and Lemma 2.5, there would exist two functions γ_0 and τ_0 in L^2 such that

$$\langle \mathbf{f}, (\phi, y, z)\rangle = E\int_0^L \gamma_0\gamma(y, z)dx + \kappa\mu\int_0^L \tau_0\tau(\phi, y, z)dx \quad \forall\ (\phi, y, z) \in H. \tag{2.11}$$

We choose a sequence $\{(\phi_n, 0, 0)\} \subset H$ such that $\|\phi_n\|_{L^2}$ is bounded but $\phi_n(L) \to \infty$ when $n \to \infty$. We see that $\langle \mathbf{f}, (\phi_n, 0, 0)\rangle = \kappa\mu\int_0^L \tau_0\phi_n dx$ would be bounded. On the other hand, from the formula (2.9), we see that if $d_m^1 \neq 0$ then $\langle \mathbf{f}, (\phi_n, 0, 0)\rangle = d_m^1\phi_n(L) + \int_0^L p_o^1\phi_n dx \to \infty$ when $n \to \infty$. □

We shall need to estimate the derivatives of γ^ϵ and ρ^ϵ in the modelling error estimate. For this purpose, we write the arch model (1.5) in differential form to obtain the following two equations.

$$\begin{aligned} \frac{1}{3}\epsilon^2 E\partial\rho^\epsilon &= \kappa\mu\tau^\epsilon - p_o^1, \\ E\partial\gamma^\epsilon &= -\frac{1}{3}\epsilon^2 E\rho^\epsilon\partial b - (p_e^1 + q_a^1 + \nu\partial p_o^2). \end{aligned} \tag{2.12}$$

We first consider the case of $\mathbf{f}|_K = 0$. In this case, we have $d_m^1 = 0$ in (2.10). Comparing it with (2.11), we see that $\tau^0 = \frac{1}{\kappa\mu}p_o^1$ in (2.11). Therefore, $\xi \in V$ defined in Lemma 2.1 has the expression $\xi = [\gamma^0, \frac{1}{\kappa\mu}p_o^1]$. The following result follows from the estimate (2.3).

Lemma 2.7. *If the resultant loading functional* $\mathbf{f}$ *does not induce pure bending deformation, i.e.* $\mathbf{f}|_K = 0$*, then the arch model solution is membrane-shear dominated. We have the estimate*

$$\|\rho^\epsilon\|_{L^2} + \epsilon^{-2}\|\gamma^\epsilon - \gamma^0\|_{L^2} + \epsilon^{-2}\|\kappa\mu\tau^\epsilon - p_o^1\|_{L^2} \lesssim \|\mathbf{f}\|_{H^*}. \tag{2.13}$$

From this estimate, we see that if $\mathbf{f}|_K = 0$ and the applied surface couple $p_o^1 \neq 0$, then the shear strain τ^ϵ converges to a finite limit $\frac{1}{\kappa\mu}p_o^1$. This is the case in which a significant shear arises in the arch deformation, and the shear strain sensitively depends on the shear correction factor κ. In this case, it follows from (2.12) and (2.13) that there exists a constant C independent of ϵ such that

$$\|\partial\rho^\epsilon\|_{L^2} \le C, \quad \|\partial\gamma^\epsilon\|_{L^2} \le C. \tag{2.14}$$

If $\mathbf{f}|_K \neq 0$, the model solution blows up at the rate of ϵ^{-2}. To ease the analysis, we scale loading force densities by ϵ^2. This is to say that we assume

$$p^\alpha_{o,e} = \epsilon^2 P^\alpha_{o,e}, \quad q^\alpha_{a,m} = \epsilon^2 Q^\alpha_{a,m}, \quad d^\alpha_{a,m} = \epsilon^2 D^\alpha_{a,m}, \tag{2.15}$$

in which the quantities denoted by P, Q and D are independent of ϵ. This makes $\mathbf{f} = \epsilon^2\mathbf{F}$ with $\mathbf{F}$ being independent of ϵ. Since we will estimate the relative error of the model solution, this assumption is not a restriction on the loading forces. The expressions for $\mathbf{F}$ is the the same as (2.9), had p, q and d been replaced by P, Q and D, respectively. The following result follows from Lemma 2.2 and the estimate (2.6).

Lemma 2.8. *In the bending dominated case, i.e.* $\mathbf{f}|_K \neq 0$*, we assume that* $\mathbf{f} = \epsilon^2\mathbf{F}$ *with* $\mathbf{F}$ *being independent of* ϵ*. Then we have*

$$\|\rho^\epsilon - \rho^0\|_{L^2} + \|\gamma^\epsilon\|_{L^2} + \|\tau^\epsilon\|_{L^2} \lesssim \epsilon^2\|\mathbf{F}\|_{H^*}. \tag{2.16}$$

Here, $\rho^0 = \rho(\theta^0, u^0, w^0)$ and $(\theta^0, u^0, w^0) \in K$ is the solution of the limiting bending model

$$\frac{1}{3}E\int_0^L \rho(\theta^0, u^0, w^0)\rho(\phi, y, z)dx = \langle \mathbf{F}, (\phi, y, z)\rangle \quad \forall\ (\phi, y, z) \in K. \tag{2.17}$$

Since $\mathbf{F}|_K \neq 0$, we have $\rho^0 \neq 0$.

It follows from this estimate and Lemma 2.3 that in the bending dominated case, we have $\|\theta^\epsilon - \theta^0\|_{H^1} \lesssim \epsilon^2$, $\|u^\epsilon - u^0\|_{H^1} \lesssim \epsilon^2$, and $\|w^\epsilon - w^0\|_{H^1} \lesssim \epsilon^2$. So the arch model solution is very close to that of the limiting bending model. It is also obvious that a change in the value of κ shall only lead to a small change in the arch model solution by a magnitude of $\mathcal{O}(\epsilon^2)$ in the H-norm. This is the insensitivity to the shear correction factor in the bending dominated case, which we mentioned in the Introduction. In this case, and under the loading scaling (2.15), there exists a constant C such that

$$\|\partial\rho^\epsilon\|_{L^2} \le C, \quad \|\partial\gamma^\epsilon\|_{L^2} \le C\epsilon^2. \tag{2.18}$$

3. The Admissible Stress Field on the Arch

3.1. *Construction of the stress field* $\boldsymbol{\sigma}^\epsilon$

The mapping $\mathbf{\Phi}$ from the slender rectangular domain ω^ϵ to the arch body Ω^ϵ furnishes curvilinear coordinates x_i on the latter, in terms of which the linear elasticity theory of the arch can be written as a set of equations on ω^ϵ and its boundary. Pertaining to this curvilinear coordinates, the covariant basis vectors $\mathbf{g}_i = \partial\mathbf{\Phi}/\partial x_i$ are $\mathbf{g}_1 = (1 - bx_2)\mathbf{a}_1$, $\mathbf{g}_2 = \mathbf{a}_2$, and $\mathbf{g}_3 = \mathbf{a}_3$. The contravariant basis vectors are $\mathbf{g}^1 = 1/(1 - bx_2)\mathbf{a}^1$, $\mathbf{g}^2 = \mathbf{a}^2$, and $\mathbf{g}^3 = \mathbf{a}^3$. Recall that $\mathbf{a}_i$ and $\mathbf{a}^i$ are the covariant and contravariant basis on $\mathcal{S}$, respectively, and b is the curvature, at the point of coordinate x_1. The covariant components $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ of the metric tensor are $g_{11} = (1 - bx_2)^2$, $g_{22} = g_{33} = 1$, and all the others are zero. The contravariant components of the metric tensor $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$ are $g^{11} = 1/(1 - bx_2)^2$, $g^{22} = g^{33} = 1$, and all the others are zero. Many of the Christoffel symbols $\Gamma^i_{jk} = \mathbf{g}^i \cdot \partial_k \mathbf{g}_j$ are zero. The nonzero ones are $\Gamma^1_{11} = -x_2\partial b/(1 - bx_2)$, $\Gamma^1_{12} = \Gamma^1_{21} = -b/(1 - bx_2)$, $\Gamma^2_{11} = b(1 - bx_2)$. A vector $\mathbf{v}$ can be given by its covariant components v_i or contravariant components v^i such that $\mathbf{v} = v_i\mathbf{g}^i = v^i\mathbf{g}_i$. A tensor $\boldsymbol{\tau}$ can be given in terms of its covariant τ_{ij} or contravariant τ^{ij} such that $\boldsymbol{\tau} = \tau_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \tau^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$, etc. The components of the body force density $\mathbf{q}$, of the surface force densities $\mathbf{p}_\pm$ on $\Gamma_\pm$, of the end force density $\mathbf{d}$ on Γ_L, and of the normal $\mathbf{n}$ to the boundary of Ω^ϵ, can all be calculated.

We assume that all the applied forces are in the X_1X_2-plane and constant in x_3. Therefore, the third components of these force vectors are all zero, (for example, $p^3_+ = 0$) and the nonzero components depend on x_1 and x_2 only. We also assume that the arch is free on the front and rear faces, and clamped on the left face Γ_0. Further, we assume that $\mathbf{q}$ is linear in x_2, and we shall replace $\mathbf{d}$ by $\bar{\mathbf{d}}$ such that (a) $\bar{\mathbf{d}} \cdot \mathbf{a}_1$ is linear in x_2 and $\bar{\mathbf{d}} \cdot \mathbf{a}_2$ quadratic in x_2, (b) $\mathbf{p}_+(L) \cdot \mathbf{a}_1 = \bar{\mathbf{d}} \cdot \mathbf{a}_2$ at $x_2 = \epsilon$, and $\mathbf{p}_-(L) \cdot \mathbf{a}_1 = -\bar{\mathbf{d}} \cdot \mathbf{a}_2$ at $x_2 = -\epsilon$, and (c) $\bar{\mathbf{d}}$ bears the resultant and moment of $\mathbf{d}$. Subject to these assumptions, we seek for a statically admissible stress field $\boldsymbol{\sigma}$ such that $\sigma^{3j} = 0$ and $\sigma^{\alpha\beta}$ depend on x_1 and x_2 only. Such field obviously satisfies the free condition $\boldsymbol{\sigma}\mathbf{n} = 0$ on the front and rear faces. For $\boldsymbol{\sigma}$ to be statically admissible, it must satisfy the equilibrium equation $\operatorname{div}\boldsymbol{\sigma} + \mathbf{q} = 0$ in Ω^ϵ, the surface force condition $\boldsymbol{\sigma}\mathbf{n} = \mathbf{p}_\pm$ on $\Gamma_\pm$, and $\boldsymbol{\sigma}\mathbf{n} = \bar{\mathbf{d}}$ on Γ_L. In terms of components, the equilibrium equation is $\sigma^{ij}\|_j = \partial_j\sigma^{ij} + \Gamma^i_{jk}\sigma^{jk} + \Gamma^j_{jk}\sigma^{ik} = -q^i$, of which the third one $\sigma^{3j}\|_j = -q^3$ is already satisfied, and the first two require that

$$\begin{aligned}\sigma^{1j}\|_j &= \partial_1\sigma^{11} + \partial_2\sigma^{12} - 2\frac{x_2\partial b}{1 - bx_2}\sigma^{11} - 3\frac{b}{1 - bx_2}\sigma^{12} = -q^1,\\ \sigma^{2j}\|_j &= \partial_1\sigma^{12} + \partial_2\sigma^{22} + b(1 - bx_2)\sigma^{11} - \frac{x_2\partial b}{1 - bx_2}\sigma^{12} - \frac{b}{1 - bx_2}\sigma^{22} = -q^2.\end{aligned} \tag{3.1}$$

The force conditions $\sigma^{ij}n_j = p^i_\pm$ on $\Gamma_\pm$ and $\sigma^{ij}n_j = \bar{d}^i$ on Γ_L are as follows. Note that the covariant components of the unit normal are $n_1 = 0$ and $n_2 = 1$ on

Γ_+, $n_1 = 0$ and $n_2 = -1$ on Γ_-, and $n_1 = 1 - bx_2$ and $n_2 = 0$ on Γ_L.

$$\begin{aligned} \sigma^{12}(x_1, \pm\epsilon) &= \pm p^1_\pm(x_1), \quad \sigma^{22}(x_1, \pm\epsilon) = \pm p^2_\pm(x_1), \quad x_1 \in (0, L), \\ \sigma^{11}(L, x_2)(1 - bx_2) &= \bar{d}^1(x_2), \quad \sigma^{12}(L, x_2)(1 - bx_2) = \bar{d}^2(x_2), \quad x_2 \in (-\epsilon, \epsilon). \end{aligned} \tag{3.2}$$

It seems rather difficult to find a $\sigma^{\alpha\beta}$ that exactly satisfies (3.1) and (3.2). We introduce the following rescaled components indicated by tilde, which reveal the possibility for these conditions to be satisfied. We define

$$\tilde{\sigma}^{11} = (1 - bx_2)^2\sigma^{11}, \quad \tilde{\sigma}^{12} = \tilde{\sigma}^{21} = (1 - bx_2)\sigma^{12}, \quad \tilde{\sigma}^{22} = (1 - bx_2)\sigma^{22}. \tag{3.3}$$

In terms of the rescaled components, the row divergence in (3.1) becomes

$$\begin{aligned} \sigma^{1j}\|_j &= \frac{1}{(1 - bx_2)^2}[\partial_1\tilde{\sigma}^{11} + (1 - bx_2)\partial_2\tilde{\sigma}^{12} - 2b\tilde{\sigma}^{12}], \\ \sigma^{2j}\|_j &= \frac{1}{1 - bx_2}[\partial_1\tilde{\sigma}^{12} + \partial_2\tilde{\sigma}^{22} + b\tilde{\sigma}^{11}], \end{aligned}$$

which is noticeably simpler. We define the rescaled force components as $\tilde{q}^1 = (1 - bt)^2 q^1$,$\tilde{q}^2 = (1 - bt)q^2$, $\tilde{p}^\alpha_\pm = (1 \mp b\epsilon)p^\alpha_\pm$, $\tilde{d}^1 = (1 - bx_2)\bar{d}^1$, and $\tilde{d}^2 = \bar{d}^2$. In terms of the rescaled components the equilibrium equation (3.1) and the surface force condition (3.2) becomes

$$\begin{aligned} \partial_1\tilde{\sigma}^{11} + (1 - bx_2)\partial_2\tilde{\sigma}^{12} - 2b\tilde{\sigma}^{12} &= -\tilde{q}^1, \\ \partial_1\tilde{\sigma}^{12} + \partial_2\tilde{\sigma}^{22} + b\tilde{\sigma}^{11} &= -\tilde{q}^2 \end{aligned} \tag{3.4}$$

and

$$\tilde{\sigma}^{\alpha 2}(\,\cdot\,, \pm\epsilon) = \pm\tilde{p}^\alpha_\pm, \quad \tilde{\sigma}^{1\alpha}(L, \,\cdot\,) = \tilde{d}^\alpha. \tag{3.5}$$

In terms of the resultant loading functions (1.2)–(1.4), we can express the rescaled force components as

$$\begin{gathered} \tilde{p}^1_\pm = p^1_o \pm \epsilon p^1_e, \quad \tilde{p}^2_\pm = p^2_o - \epsilon^2 b p^2_e \pm \epsilon(p^2_e - bp^2_o), \\ \tilde{d}^1 = d^1_a + 3\epsilon^{-2}d^1_m x_2, \quad \tilde{d}^2 = p^1_o(L) + p^1_e(L)x_2 + \frac{3}{2}[d^2_a - p^1_o(L)]\left(1 - \frac{x_2^2}{\epsilon^2}\right), \\ \tilde{q}^\alpha = q^\alpha_a + (3q^\alpha_m - bq^\alpha_a)x_2 - 3bq^\alpha_m x_2^2. \end{gathered}$$

The last one used the assumption that $\mathbf{q}$ is linear in x_2. A close inspection of (3.4) suggests that it is possible to exactly satisfy the equilibrium equation by choosing $\tilde{\sigma}^{11}$ and $\tilde{\sigma}^{12}$ as quadratic polynomials in x_2, and $\tilde{\sigma}^{22}$ cubic polynomial in x_2. We define three polynomials

$$r(x_2) = \frac{x_2^2}{\epsilon^2} - \frac{1}{3}, \quad q(x_2) = 1 - \frac{x_2^2}{\epsilon^2}, \quad s(x_2) = \frac{x_2}{\epsilon}\left(1 - \frac{x_2^2}{\epsilon^2}\right).$$

Taking the surface force condition (3.5) into consideration, we choose the following forms for the re-scaled stress components in which σ_0^{11}, σ_1^{11}, σ_2^{11}, σ_2^{12}, σ_2^{22}, σ_3^{22} are six single variable functions of x_1.

$$\begin{aligned}\tilde{\sigma}^{11} &= \sigma_0^{11} + \sigma_1^{11} x_2 + \sigma_2^{11} r(x_2),\\ \tilde{\sigma}^{12} = \tilde{\sigma}^{21} &= p_o^1 + p_e^1 x_2 + \sigma_2^{12} q(x_2),\\ \tilde{\sigma}^{22} &= (p_o^2 - \epsilon^2 b p_e^2) + (p_e^2 - b p_o^2) x_2 + \sigma_2^{22} q(x_2) + \sigma_3^{22} s(x_2).\end{aligned} \tag{3.6}$$

The surface force conditions on $\Gamma_\pm$ are automatically satisfied. To satisfy the force condition on Γ_L, we require that

$$\sigma_0^{11}(L) = d_a^1, \quad \sigma_1^{11}(L) = 3\epsilon^{-2} d_m^1, \quad \sigma_2^{12}(L) = \frac{3}{2}[d_a^2 - p_o^1(L)], \quad \sigma_2^{11}(L) = 0. \tag{3.7}$$

Both sides of the equilibrium equation (3.4) are quadratic polynomials in x_2. By equating coefficients of the polynomials, the equilibrium equation can be equivalently written in terms of the following system of ordinary differential equations.

$$\begin{aligned}\partial\sigma_0^{11} - \frac{1}{3}\partial\sigma_2^{11} - 2b\sigma_2^{12} + p_e^1 - 2bp_o^1 &= -q_a^1,\\ \partial\sigma_1^{11} - \frac{2}{\epsilon^2}\sigma_2^{12} - 3bp_e^1 &= bq_a^1 - 3q_m^1,\\ \partial\sigma_2^{11} + 4b\sigma_2^{12} &= 3\epsilon^2 b q_m^1,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}\partial\sigma_2^{12} + b\sigma_0^{11} - \frac{1}{3}b\sigma_2^{11} + \frac{1}{\epsilon}\sigma_3^{22} + \partial p_o^1 + p_e^2 - bp_o^2 &= -q_a^2,\\ b\sigma_1^{11} - \frac{2}{\epsilon^2}\sigma_2^{22} + \partial p_e^1 &= bq_a^2 - 3q_m^2,\\ b\sigma_2^{11} - \partial\sigma_2^{12} - \frac{3}{\epsilon}\sigma_3^{22} &= 3\epsilon^2 b q_m^2.\end{aligned} \tag{3.9}$$

The set of equations (3.8) is equivalent to the first equation in (3.4) and (3.9) is equivalent to the second one. We number the above six equations by the Roman numerals *I–VI*, and take the linear combinations

$$I + \frac{1}{3}III, \quad IV + \frac{1}{3}VI, \quad \frac{1}{3}\epsilon^2 II, \quad III, \quad \frac{1}{2}\epsilon^2 V, \quad \frac{1}{2}\epsilon(IV + VI),$$

to form a system of six equations, which is equivalent to (3.8) and (3.9):

$$\begin{aligned}\partial\sigma_0^{11} - \frac{2}{3}b\sigma_2^{12} &= 2bp_o^1 - p_e^1 - q_a^1 + \epsilon^2 b q_m^1,\\ b\sigma_0^{11} + \frac{2}{3}\partial\sigma_2^{12} &= bp_o^2 - p_e^2 - \partial p_o^1 - q_a^2 + \epsilon^2 b q_m^2,\\ \frac{1}{3}\epsilon^2\partial\sigma_1^{11} - \frac{2}{3}\sigma_2^{12} &= \epsilon^2 b p_e^1 + \frac{1}{3}\epsilon^2 b q_a^1 - \epsilon^2 q_m^1,\end{aligned} \tag{3.10}$$

with the boundary condition

$$\sigma_0^{11}(L) = d_a^1, \quad \sigma_1^{11}(L) = 3\epsilon^{-2} d_m^1, \quad \sigma_2^{12}(L) = \frac{3}{2}[d_a^2 - p_o^1(L)]. \tag{3.11}$$

And

$$\begin{aligned}\partial\sigma_2^{11} &= -4b\sigma_2^{12} + 3\epsilon^2 q_m^1,\\ \sigma_2^{22} &= \frac{1}{2}\epsilon^2(b\sigma_1^{11} + \partial p_e^1 - bq_a^2 - 3q_m^2),\\ \sigma_3^{22} &= \frac{1}{2}\epsilon\left(\frac{2}{3}b\sigma_2^{11} + b\sigma_0^{11} + p_e^2 - bp_o^2 + \partial p_o^1 + q_a^2 - 3\epsilon^2 bq_m^2\right),\end{aligned} \tag{3.12}$$

with the boundary condition $\sigma_2^{11}(L) = 0$. Equations (3.10) form a closed system for the three functions σ_0^{11}, σ_1^{11} and σ_2^{12}. Equations (3.12) determine σ_2^{11}, σ_2^{22} and σ_3^{22} in terms of σ_0^{11}, σ_1^{11}, σ_2^{12} and the loading functions. The problem of determining statically admissible stress fields with rescaled components of the form (3.6) then is reduced to solving the system of three ordinary differential equations (3.10) with the boundary condition (3.11) for the three single variable functions σ_0^{11}, σ_1^{11} and σ_2^{12}. In a sense, the functions σ_0^{11}, σ_1^{11}, and σ_2^{12} play a principal role in building the stress fields. We thus call them principal stress functions. The remaining functions σ_2^{11}, σ_2^{22}, and σ_3^{22} are supplementary.

Remark 3.1. Under our assumption that the arch is clamped on Γ_0 and subject to forces on Γ_L, the system (3.10) and (3.12) uniquely determine the stress field of the assumed variance in x_2. If the arch is clamped at both ends, then we have a set of such stress fields. If the arch is subject to forces on both ends, the stress field is unique. In this case, the existence is not obvious. It follows from an compatibility assumption on the overall loads on the arch.

We now connect the principal stress functions to a displacement field by minimizing the constitutive residual. We choose displacement fields $\hat{\mathbf{v}}$ with covariant components

$$\hat{v}_1 = (1 - bx_2)(\hat{u} + x_2\hat{\theta}), \quad \hat{v}_2 = \hat{w}, \quad \hat{v}_3 = 0. \tag{3.13}$$

Here, $(\hat{\theta}, \hat{u}, \hat{w})$ are three single variable functions satisfying $\hat{\theta}(0) = \hat{u}(0) = \hat{w}(0) = 0$. We construct a stress field $\hat{\boldsymbol{\sigma}}$ whose rescaled components are of the form (3.6) with the supplementary functions being zero. Thus

$$\begin{aligned}(1 - bx_2)^2\hat{\sigma}^{11} &= \sigma_0^{11} + x_2\sigma_1^{11},\\ (1 - bx_2)\hat{\sigma}^{12} &= p_o^1 + tp_e^1 + q(x_2)\sigma_0^{12},\\ (1 - bx_2)\hat{\sigma}^{22} &= [p_o^2 - \epsilon^2 bp_e^2] + x_2[p_e^2 - bp_o^2],\end{aligned} \tag{3.14}$$

in which the undetermined functions σ_0^{11}, σ_1^{11} and σ_2^{12} satisfy the boundary condition (3.11). For any $\hat{\mathbf{v}}$, we determine $\hat{\boldsymbol{\sigma}}$ such that the constitutive residual $\|\hat{\boldsymbol{\sigma}} - \mathcal{C}\boldsymbol{\epsilon}(\hat{\mathbf{v}})\|_{E(\Omega^\epsilon)}$ is minimized. This leads to the equations

$$\begin{aligned}\sigma_1^{11} &= E\hat{\rho} + \nu(1 - \epsilon^2 b)p_e^2,\\ \sigma_0^{11} &= \frac{1}{3}b\epsilon^2\sigma_1^{11} + E\hat{\gamma} + \nu(p_o^2 - \epsilon^2 bp_e^2),\\ \sigma_2^{12} &= \frac{5}{4}\mu\hat{\tau} - \frac{5}{4}p_o^1 + \frac{1}{4}b\epsilon^2 p_e^1.\end{aligned} \tag{3.15}$$

Here $\hat{\gamma} = \gamma(\hat{u}, \hat{w})$, $\hat{\rho} = \rho(\hat{\theta}, \hat{u}, \hat{w})$, and $\hat{\tau} = \tau(\hat{\theta}, \hat{u}, \hat{w})$, cf. (2.7). Through these expressions, Eq. (3.10) and the boundary condition (3.11) are enforced on the displacement functions $\hat{\theta}$, $\hat{u}$ and $\hat{w}$. We write the upshot in a weak formulation which better serves our purpose of modelling error estimates. Adding to and subtracting from σ_0^{11} the term $\frac{1}{3}\epsilon^2 b\sigma_1^{11}$ in each of the first two equations of (3.10), we write the system as

$$\begin{gathered}\partial\left(\sigma_0^{11} - \frac{1}{3}\epsilon^2 b\sigma_1^{11}\right) + \frac{1}{3}\epsilon^2\partial(b\sigma_1^{11}) - \frac{2}{3}b\sigma_2^{12} = 2bp_o^1 - p_e^1 - q_a^1 + \epsilon^2 bq_m^1, \\ b\left(\sigma_0^{11} - \frac{1}{3}\epsilon^2 b\sigma_1^{11}\right) + \frac{1}{3}\epsilon^2 b^2\sigma_1^{11} + \frac{2}{3}\partial\sigma_2^{12} = bp_o^2 - p_e^2 - \partial p_o^1 - q_a^2 + \epsilon^2 bq_m^2, \\ \frac{1}{3}\epsilon^2\partial\sigma_1^{11} - \frac{2}{3}\sigma_2^{12} = \epsilon^2 bp_e^1 + \frac{1}{3}\epsilon^2 bq_a^1 - \epsilon^2 q_m^1.\end{gathered} \quad (3.16)$$

Multiplying these three equations, respectively, by smooth single variable functions y, z, ϕ which vanish at 0, integrating by parts on the interval $(0, L)$, incorporating the boundary condition (3.11), adding, and invoking the definition (2.7), we get

$$\begin{aligned}&\frac{1}{3}\epsilon^2\int_0^L \sigma_1^{11}\rho(\phi, y, z)dx + \int_0^L\left(\sigma_0^{11} - \frac{1}{3}\epsilon^2 b\sigma_1^{11}\right)\gamma(y, x)dx + \frac{2}{3}\int_0^L \sigma_2^{12}\tau(\phi, y, z)dx \\ &\quad = \int_0^L \left[(p_e^1 + q_a^1 - 2bp_o^1 - \epsilon^2 bq_m^1)y + (p_e^2 - bp_o^2 + q_a^2 + \partial p_o^1 - \epsilon^2 bq_m^2)z\right]dx \\ &\qquad - \frac{1}{3}\epsilon^2\int_0^L (bq_a^1 + 3bp_e^1 - 3q_m^1)\phi dx + d_a^1 y(L) + [d_a^2 - p_o^1(L)]z(L) + d_m^1\phi(L).\end{aligned}$$

This is the weak formulation of the equilibrium equation (3.10) together with its boundary condition (3.11). Substituting (3.15) into this equation, we obtain the variational equation

$$\begin{aligned}&\frac{1}{3}\epsilon^2 E\int_0^L \rho(\hat{\theta}, \hat{u}, \hat{w})\rho(\phi, y, z)dx + E\int_0^L \gamma(\hat{u}, \hat{w})\gamma(y, z)dx \\ &\quad + \frac{5}{6}\mu\int_0^L \tau(\hat{\theta}, \hat{u}, \hat{w})\tau(\phi, y, z)dx = \langle\hat{\mathbf{f}}, (\phi, y, z)\rangle.\end{aligned} \quad (3.17)$$

Here $\hat{\mathbf{f}} = \hat{\mathbf{f}}_0 + \epsilon^2\hat{\mathbf{f}}_2 + \epsilon^4\hat{\mathbf{f}}_4$ and

$$\begin{aligned}\langle\hat{\mathbf{f}}_0, (\phi, y, z)\rangle = &\frac{5}{6}\int_0^L p_o^1\tau(\phi, y, z)dx - \nu\int_0^L p_o^2\gamma(y, z)dx \\ &+ \int_0^L [(p_e^1 + q_a^1 - 2bp_o^1)y + (p_e^2 - bp_o^2 + q_a^2 + \partial p_o^1)z]dx \\ &+ d_a^1 y(L) + [d_a^2 - p_o^1(L)]z(L) + d_m^1\phi(L),\end{aligned} \quad (3.18)$$

$$\langle \hat{\mathbf{f}}_2, (\phi, y, z) \rangle = -\frac{1}{3}\nu \int_0^L p_e^2 \rho(\phi, y, z) dx + \nu \int_0^L b p_e^2 \gamma(y, z) dx - \frac{1}{6} \int_0^L b p_e^1 \tau(\phi, y, z) dx$$
$$- \int_0^L \left[b q_m^1 y + b q_m^2 z + \frac{1}{3}(b q_a^1 + 3 b p_e^1 - 3 q_m^1)\phi \right] dx,$$
$$\langle \hat{\mathbf{f}}_4, (\phi, y, z) \rangle = \frac{1}{3}\nu \int_0^L b p_e^2 \rho(\phi, y, z) dx.$$

It follows from Lemma 2.3 that Eq. (3.17) has a unique solution $(\hat{\theta}^\epsilon, \hat{u}^\epsilon, \hat{w}^\epsilon) \in H$. These functions then define the principal stress functions by the formulas (3.15), which exactly satisfy Eq. (3.10) and the boundary condition (3.11). The supplementary functions are determined by the formulas (3.12). We thus obtained the rescaled components (3.6) of a stress field $\boldsymbol{\sigma}^\epsilon$ that satisfies the equilibrium equation (3.1) and surface force condition (3.2) exactly.

3.2. *Asymptotic estimates on* $\boldsymbol{\sigma}^\epsilon$

The stress field constructed above varies with ϵ. The key ingredients in this stress field are the functions $(\hat{\theta}^\epsilon, \hat{u}^\epsilon, \hat{w}^\epsilon) \in H$ determined by Eq. (3.17). We can split the solution as $(\hat{\theta}^\epsilon, \hat{u}^\epsilon, \hat{w}^\epsilon) = (\hat{\theta}_0^\epsilon, \hat{u}_0^\epsilon, \hat{w}_0^\epsilon) + \epsilon^2(\hat{\theta}_2^\epsilon, \hat{u}_2^\epsilon, \hat{w}_2^\epsilon) + \epsilon^4(\hat{\theta}_4^\epsilon, \hat{u}_4^\epsilon, \hat{w}_4^\epsilon)$, with $(\hat{\theta}_{2i}^\epsilon, \hat{u}_{2i}^\epsilon, \hat{w}_{2i}^\epsilon)$ being the solution of (3.17) in which the loading functional is the ϵ-independent $\hat{\mathbf{f}}_{2i}$ for $i = 0, 1, 2$, respectively. Each of these equations now fits in the abstract framework of Sec. 2.1. And we can estimate the three set of functions as we did for the model solution in Sec. 2. We then get the estimates on $(\hat{\theta}^\epsilon, \hat{u}^\epsilon, \hat{w}^\epsilon)$ by superposition.

The only difference between (3.18) and the model loading functional, cf. (2.10), is the factor 5/6 in the former. From this observation, we see that $\mathbf{f}|_K = \hat{\mathbf{f}}_0|_K$. So the behavior of $(\hat{\theta}_0^\epsilon, \hat{u}_0^\epsilon, \hat{w}_0^\epsilon)$ is similar to that of the model solution as described in Lemmas 2.7 and 2.8. The functions $(\hat{\theta}_2^\epsilon, \hat{u}_2^\epsilon, \hat{w}_2^\epsilon)$ and $(\hat{\theta}_4^\epsilon, \hat{u}_4^\epsilon, \hat{w}_4^\epsilon)$ affect the behavior of $(\hat{\theta}^\epsilon, \hat{u}^\epsilon, \hat{w}^\epsilon)$ only slightly since they are scaled by the factors ϵ^2 and ϵ^4, respectively. We have the following estimates, in which $\hat{\rho}^\epsilon = \rho(\hat{\theta}^\epsilon, \hat{u}^\epsilon, \hat{w}^\epsilon)$, $\hat{\gamma}^\epsilon = \gamma(\hat{u}^\epsilon, \hat{w}^\epsilon)$, $\hat{\tau}^\epsilon = \tau(\hat{\theta}^\epsilon, \hat{u}^\epsilon, \hat{w}^\epsilon)$, and C is a constant independent of ϵ.

If $\mathbf{f}|_K = 0$, then

$$\|\hat{\rho}^\epsilon\|_{L^2} + \epsilon^{-2}\|\hat{\gamma}^\epsilon - \gamma^0\|_{L^2} + \epsilon^{-2}\|\mu\hat{\tau}^\epsilon - p_o^1\|_{L^2} \le C. \tag{3.19}$$

It is important to note that the γ^0 is the same as that appeared in (2.13).

If $\mathbf{f}|_K \neq 0$, we scale the loading functions by ϵ^2, cf. (2.15). Then we have

$$\|\hat{\rho}^\epsilon - \rho^0\|_{L^2} + \|\hat{\gamma}^\epsilon\|_{L^2} + \|\hat{\tau}^\epsilon\|_{L^2} \le \epsilon^2 C. \tag{3.20}$$

Here, $\rho^0 = \rho(\theta^0, u^0, w^0) \neq 0$, and $(\theta^0, u^0, w^0) \in K$ is defined by (2.17), which also appeared in (2.16). Estimates on the principal and supplementary stress functions involved in $\boldsymbol{\sigma}^\epsilon$, cf. (3.15) and (3.12), then follow from these estimates.

4. Modelling Error Estimate

Based on the model solution $(\theta^\epsilon, u^\epsilon, w^\epsilon)$, we define a displacement field $\mathbf{u}^\epsilon$ with the covariant components

$$u_1^\epsilon = (1 - bx_2)(u^\epsilon + \theta^\epsilon x_2),$$
$$u_2^\epsilon = w^\epsilon + w_1^\epsilon x_2 + \frac{1}{2} w_2^\epsilon x_2^2 - \frac{1}{2} z_2^\epsilon x_3^2, \tag{4.1}$$
$$u_3^\epsilon = z_1^\epsilon x_3 + z_2^\epsilon x_2 x_3.$$

Here w_1^ϵ, w_2^ϵ, z_1^ϵ and z_2^ϵ are single variable functions defined by

$$\begin{aligned} w_1^\epsilon &= \frac{1-\nu^2}{E} p_o^2 - \nu\gamma^\epsilon, \quad w_2^\epsilon = \frac{1}{E}[(1-\nu^2)p_e^2 - bp_o^2] - \nu\rho^\epsilon, \\ z_1^\epsilon &= -\frac{\nu(1+\nu)}{E} p_o^2 - \nu\gamma^\epsilon, \quad z_2^\epsilon = -\frac{\nu}{E}[(1+\nu)p_e^2 - bp_o^2] - \nu\rho^\epsilon. \end{aligned} \tag{4.2}$$

We prove that the displacement field $\mathbf{u}^\epsilon$ is close to the displacement of the arch determined by the 3D elasticity theory in the interior energy norm. A major step to achieve this is to estimate the constitutive residual $\mathcal{A}(\boldsymbol{\sigma}^\epsilon) - \boldsymbol{\epsilon}(\mathbf{u}^\epsilon)$ in which $\boldsymbol{\sigma}^\epsilon$ is the stress field built in the previous section. The following formulas for the covariant components of the strain tensor $\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)$ are results of direct calculations. Note that $\epsilon_{ij}(\mathbf{u}^\epsilon) = (u^\epsilon_{i\|j} + u^\epsilon_{j\|i})/2$, where $u^\epsilon_{i\|j} = \partial_j u_i^\epsilon - \Gamma^k_{ij} u_k^\epsilon$ is the covariant derivative.

$$\begin{aligned} \epsilon_{11}(\mathbf{u}^\epsilon) &= (1-2bx_2)\gamma^\epsilon + x_2\rho^\epsilon - b(1-bx_2)\left(w_1^\epsilon x_2 + \frac{1}{2}w_2^\epsilon x_2^2 - \frac{1}{2} z_2^\epsilon x_3^2\right) - bx_2^2\partial\theta^\epsilon, \\ \epsilon_{12}(\mathbf{u}^\epsilon) &= \frac{1}{2}\tau^\epsilon + \frac{1}{2}\left(x_2\partial w_1^\epsilon + \frac{1}{2}x_2^2 \partial w_2^\epsilon - \frac{1}{2}x_3^2\partial z_2^\epsilon\right), \quad \epsilon_{22}(\mathbf{u}^\epsilon) = w_1^\epsilon + x_2 w_2^\epsilon, \\ \epsilon_{13}(\mathbf{u}^\epsilon) &= \frac{1}{2}(\partial z_1^\epsilon x_3 + \partial z_2^\epsilon x_2 x_3), \quad \epsilon_{23}(\mathbf{u}^\epsilon) = 0, \quad \epsilon_{33}(\mathbf{u}^\epsilon) = z_1^\epsilon + z_2^\epsilon x_2. \end{aligned} \tag{4.3}$$

The compliance tensor $\mathcal{A}$ acting on a stress tensor $\boldsymbol{\sigma}$ with contravariant components σ^{ij} yields a tensor with covariant components given by

$$\mathcal{A}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(g_{ik}g_{jl}\sigma^{kl} - \frac{\lambda}{2\mu + 3\lambda} g_{ij} g_{kl}\sigma^{kl}\right).$$

Using the formulas (3.15), (3.6) and (3.3), we obtain the following expressions for the covariant components of $\mathcal{A}\boldsymbol{\sigma}^\epsilon$ denoted by $\hat{\epsilon}_{ij}$:

$$\begin{aligned} \hat{\epsilon}_{11} &= (1-2bx_2)\hat{\gamma}^\epsilon + x_2\hat{\rho}^\epsilon + r_{11}, \quad \hat{\epsilon}_{12} = \frac{1}{2\mu}\left(p_o^1 + \frac{5}{4}(\mu\hat{\tau}^\epsilon - p_o^1)q(x_2)\right) + r_{12}, \\ \hat{\epsilon}_{22} &= \frac{1-\nu^2}{E}p_o^2 - \nu\hat{\gamma}^\epsilon + \left(\frac{1}{E}[(1-\nu^2)p_e^2 - bp_o^2] - \nu\hat{\rho}^\epsilon\right)x_2 + r_{22}, \\ \hat{\epsilon}_{33} &= -\frac{\nu(1+\nu)}{E}p_o^2 - \nu\hat{\gamma}^\epsilon - \left(\frac{\nu}{E}[(1+\nu)p_e^2 - bp_o^2] + \nu\hat{\rho}^\epsilon\right)x_2 + r_{33}, \end{aligned} \tag{4.4}$$

and $\hat\epsilon_{13} = \hat\epsilon_{23} = 0$. Here

$$
\begin{aligned}
r_{11} &= \frac{1}{E}(1-2bx_2)r(x_2)\sigma_2^{11} + \frac{1}{E}b^2x_2^2[\sigma_0^{11} + x_2\sigma_1^{11} + r(x_2)\sigma_2^{11}] \\
&\quad + \frac{1}{E}\left[\frac{1}{3}b\epsilon^2(1-2bx_2) - 2bx_2^2\right]\sigma_1^{11} \\
&\quad - \frac{\nu}{E}\left\{(1-bx_2)[q(x_2)\sigma_0^{22} + s(x_2)\sigma_1^{22}] - bx_2^2(p_e^2 - bp_o^2)\right\}, \\
r_{22} &= \frac{bx_2}{E(1-bx_2)}\tilde\sigma^{22} + \frac{1}{E}(\nu^2 - 1 + \nu^2 x_2)bp_e^2\epsilon^2 - \frac{\nu}{3E}b\sigma_1^{11}\epsilon^2 \\
&\quad + \frac{1}{E}[\sigma_2^{22}q(x_2) + \sigma_3^{22}s(x_2) - \nu\sigma_2^{11}r(x_2)], \\
r_{12} &= \frac{1}{2\mu}\left[x_2 + \frac{1}{4}q(x_2)b\epsilon^2\right]p_e^1 - \frac{1}{2\mu}bx_2[p_o^1 + x_2p_e^1 + q(x_2)\sigma_2^{12}], \\
r_{33} &= -\frac{\nu bx_2}{E(1-bx_2)}\tilde\sigma^{22} + \frac{\nu}{E}(\nu + 1 + \nu x_2)bp_e^2\epsilon^2 - \frac{\nu}{3E}b\sigma_1^{11}\epsilon^2 \\
&\quad - \frac{\nu}{E}[\sigma_2^{22}q(x_2) + \sigma_3^{22}s(x_2) + \sigma_2^{11}r(x_2)].
\end{aligned}
$$

The constitutive residual $\boldsymbol{\varrho}^\epsilon = \mathcal{A}\boldsymbol{\sigma}^\epsilon - \boldsymbol{\epsilon}(\mathbf{u}^\epsilon)$ then has the expression:

$$
\begin{aligned}
\varrho_{11}^\epsilon &= (1-2bx_2)(\hat\gamma^\epsilon - \gamma^\epsilon) + x_2(\hat\rho^\epsilon - \rho^\epsilon) \\
&\quad + b(1-bx_2)\left(w_1^\epsilon x_2 + \frac{1}{2}w_2^\epsilon x_2^2 - \frac{1}{2}z_2^\epsilon x_3^2\right) + bx_2^2\partial\theta^\epsilon + r_{11},
\end{aligned}
$$

$$
\begin{aligned}
\varrho_{12}^\epsilon &= \frac{1}{2\mu}\left(p_o^1 - \mu\tau^\epsilon + \frac{5}{4}(\mu\hat\tau^\epsilon - p_o^1)q(x_2)\right) \\
&\quad - \frac{1}{2}\left(x_2\partial w_1^\epsilon + \frac{1}{2}x_2^2\partial w_2^\epsilon - \frac{1}{2}x_3^2\partial z_2^\epsilon\right) + r_{12},
\end{aligned}
$$

$$
\varrho_{13}^\epsilon = -\frac{1}{2}(\partial z_1^\epsilon x_3 + \partial z_2^\epsilon x_2 x_3), \quad \varrho_{22}^\epsilon = \nu(\gamma^\epsilon - \hat\gamma^\epsilon) + \nu(\rho^\epsilon - \hat\rho^\epsilon)x_2 + r_{22},
$$
$$
\varrho_{23}^\epsilon = 0, \quad \varrho_{33}^\epsilon = \nu(\gamma^\epsilon - \hat\gamma^\epsilon) + \nu(\rho^\epsilon - \hat\rho^\epsilon)x_2 + r_{33}.
$$

Since $|x_2| \le \epsilon$ and $|x_3| \le C\epsilon$, we have

$$
\begin{gathered}
|r_{11}| \lesssim |\sigma_2^{11}| + |\sigma_2^{22}| + |\sigma_3^{22}| + \epsilon^2\left(|\sigma_0^{11}| + |\sigma_1^{11}| + |p_e^2| + |p_o^2|\right), \\
|r_{22}|,\ |r_{33}| \lesssim |\sigma_2^{11}| + |\sigma_2^{22}| + |\sigma_3^{22}| + \epsilon\left(|p_e^2| + |p_o^2|\right) + \epsilon^2|\sigma_1^{11}|, \\
|r_{12}| \lesssim \epsilon\left(|p_e^1| + |p_o^1| + |\sigma_2^{12}|\right).
\end{gathered} \tag{4.5}
$$

$$
\begin{gathered}
|\varrho_{11}^\epsilon| \lesssim |\hat\gamma^\epsilon - \gamma^\epsilon| + \epsilon|\hat\rho^\epsilon - \rho^\epsilon| + \epsilon|w_1^\epsilon| + \epsilon^2|w_2^\epsilon| + \epsilon^2|z_2^\epsilon| + \epsilon^2|\partial\theta^\epsilon| + |r_{11}|, \\
|\varrho_{12}^\epsilon| \lesssim |p_o^1 - \mu\tau^\epsilon| + |\mu\hat\tau^\epsilon - p_o^1| + \epsilon|\partial w_1^\epsilon| + \epsilon^2|\partial w_2^\epsilon| + \epsilon^2|\partial z_2^\epsilon| + |r_{12}|, \\
|\varrho_{13}^\epsilon| \lesssim \epsilon|\partial z_1^\epsilon| + \epsilon^2|\partial z_2^\epsilon|, \quad |\varrho_{22}^\epsilon| \lesssim |\gamma^\epsilon - \hat\gamma^\epsilon| + \epsilon|\rho^\epsilon - \hat\rho^\epsilon| + |r_{22}|, \\
|\varrho_{23}^\epsilon| = 0, \quad |\varrho_{33}^\epsilon| \lesssim |\gamma^\epsilon - \hat\gamma^\epsilon| + \epsilon|\rho^\epsilon - \hat\rho^\epsilon| + |r_{33}|.
\end{gathered} \tag{4.6}
$$

At this point, the analysis must proceed separately for the cases of $\mathbf{f}|_K = 0$ and $\mathbf{f}|_K \neq 0$.

Lemma 4.1. *If* $\mathbf{f}|_K = 0$ *and the shear correction factor* $\kappa = 1$ *in the arch model* (1.5), *then there exists a constant* C *independent of* ϵ *such that*

$$\frac{\|\boldsymbol{\varrho}^\epsilon\|_{E(\Omega^\epsilon)}}{\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)}} \le C\epsilon. \tag{4.7}$$

Proof. In (3.19), we proved

$$\|\hat{\rho}^\epsilon\|_{L^2} \le C, \quad \|\hat{\gamma}^\epsilon - \gamma^0\|_{L^2} \le C\epsilon^2, \quad \|\mu\hat{\tau}^\epsilon - p_o^1\|_{L^2} \le C\epsilon^2. \tag{4.8}$$

Here L^2 is $L^2(0, L)$. Insert these estimates into the expressions (3.15), we get

$$\|\sigma_1^{11}\|_{L^2} \le C, \quad \|\sigma_0^{11}\|_{L^2} \le C, \quad \|\sigma_2^{12}\|_{L^2} \le C\epsilon^2.$$

From (3.12) and the boundary condition $\sigma_2^{11}(L) = 0$, we see

$$\|\sigma_2^{22}\|_{L^2} \le C\epsilon^2, \quad \|\sigma_3^{22}\|_{L^2} \le C\epsilon, \quad \|\sigma_2^{11}\|_{H^1} \le C\epsilon^2.$$

Using the estimates (4.5), we thus have

$$\|r_{11}\|^2_{L^2(\omega^\epsilon)} + \|r_{12}\|^2_{L^2(\omega^\epsilon)} + \|r_{22}\|^2_{L^2(\omega^\epsilon)} + \|r_{33}\|^2_{L^2(\omega^\epsilon)} \le C\epsilon^4.$$

Recall (2.13), we have

$$\|\rho^\epsilon\|_{L^2} \le C, \quad \|\gamma^\epsilon - \gamma^0\|_{L^2} \le C\epsilon^2, \quad \|\kappa\mu\tau^\epsilon - p_o^1\|_{L^2} \le C\epsilon^2. \tag{4.9}$$

This and (2.8) shows that $\|\theta^\epsilon\|_{H^1} \le C$. Also, from (2.14) we have $\|\partial\rho^\epsilon\|_{L^2} \le C$ and $\|\partial\gamma^\epsilon\|_{L^2} \le C$. We thus have $\|w_1^\epsilon\|_{H^1} \le C$, $\|w_2^\epsilon\|_{H^1} \le C$, $\|z_1^\epsilon\|_{H^1} \le C$, $\|z_2^\epsilon\|_{H^1} \le C$. From (4.8) and (4.9), we see $\|\gamma^\epsilon - \hat{\gamma}^\epsilon\|_{L^2} \le C\epsilon^2$. All these together with the estimates (4.6) establish the estimate $\|\varrho_{ij}^\epsilon\|^2_{L^2(\omega^\epsilon)} \le C\epsilon^4$. Since the elasticity tensor and the compliance tensor are uniformly positive definite and bounded, we have $\|\boldsymbol{\varrho}^\epsilon\|_{E(\Omega^\epsilon)} \le C\epsilon^2$.

On the other hand, we see from (4.3)

$$|\epsilon_{11}(\mathbf{u}^\epsilon)| \gtrsim |\gamma^0| - |\gamma^0 - \gamma^\epsilon| - \epsilon(|\gamma^\epsilon| + |\rho^\epsilon| + |w_1^\epsilon| + \epsilon|w_2^\epsilon| + \epsilon|\partial\theta^\epsilon|),$$
$$|\epsilon_{12}(\mathbf{u}^\epsilon)| \gtrsim |\tau^0| - |\tau^0 - \tau^\epsilon| - \epsilon(|\partial w_1^\epsilon| + \epsilon|\partial w_2^\epsilon|).$$

Therefore, when ϵ is sufficiently small, we have $\|\epsilon_{11}(\mathbf{u}^\epsilon)\|^2_{L^2(\omega^\epsilon)} + \|\epsilon_{12}(\mathbf{u}^\epsilon)\|^2_{L^2(\omega^\epsilon)} \gtrsim \epsilon^2(\|\gamma^0\|^2_{L^2} + \|\tau^0\|^2_{L^2})$. Since $\|\gamma^0\|^2_{L^2} + \|\tau^0\|^2_{L^2} \simeq \|\mathbf{f}\|^2_{H^*} \neq 0$, we thus get the lower bound $\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)} \ge C\epsilon$. □

Remark 4.1. The bound $\|\varrho_{12}^\epsilon\|^2_{L^2(\omega^\epsilon)} \le C\epsilon^4$ crucially hinges on $\kappa = 1$ in (4.9). If $\kappa \neq 1$, then we only have $\|\varrho_{12}^\epsilon\|^2_{L^2(\omega^\epsilon)} \le C\epsilon^2$. If we use $\mathbf{u}_\kappa^\epsilon$ to denote the displacement field on the arch obtained from the solution of the model (1.5) with the shear correction factor κ, and use $\mathbf{u}_1^\epsilon$ when $\kappa = 1$, then we see from the formulas (4.3) that

$$\frac{\|\boldsymbol{\epsilon}(\mathbf{u}_1^\epsilon - \mathbf{u}_\kappa^\epsilon)\|_{E(\Omega^\epsilon)}}{\|\boldsymbol{\epsilon}(\mathbf{u}_1^\epsilon)\|_{E(\Omega^\epsilon)}} \simeq |1-\kappa| \|p_o^1\|_{L^2}. \tag{4.10}$$

Therefore, there is a finite relative difference between $\mathbf{u}_1^\epsilon$ and $\mathbf{u}_\kappa^\epsilon$ when $\kappa \neq 1$ and $p_o^1 \neq 0$.

Lemma 4.2. *If* $\mathbf{f}|_K \neq 0$, *then there exists a constant independent of* ϵ *such that*

$$\frac{\|\boldsymbol{\varrho}^\epsilon\|_{E(\Omega^\epsilon)}}{\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)}} \leq C\epsilon. \tag{4.11}$$

This estimate holds for κ *being any positive constant.*

Proof. In this case, we need to scale the loading force densities by ϵ^2, cf. (2.15), such that the model solution converges to a finite limit. We have already proved, cf. (3.20), that $\|\hat\rho^\epsilon - \rho^0\|_{L^2} \leq C\epsilon^2$, $\|\hat\gamma^\epsilon\|_{L^2} \leq C\epsilon^2$, and $\|\hat\tau^\epsilon\|_{L^2} \leq C\epsilon^2$. In (3.15), p_e^2 and other loading functions now are replaced by $\epsilon^2 P_e^2$ etc., from which we see

$$\|\sigma_1^{11}\|_{L^2} \leq C, \quad \|\sigma_0^{11}\|_{L^2} \leq C\epsilon^2, \quad \|\sigma_2^{12}\|_{L^2} \leq C\epsilon^2.$$

From (3.12), we have

$$\|\sigma_2^{11}\|_{H^1} \leq C\epsilon^2, \quad \|\sigma_2^{22}\|_{L^2} \leq C\epsilon^2, \quad \|\sigma_3^{22}\|_{L^2} \leq C\epsilon^3.$$

These bounds lead to

$$\|r_{11}\|^2_{L^2(\omega^\epsilon)} + \|r_{22}\|^2_{L^2(\omega^\epsilon)} + \|r_{33}\|^2_{L^2(\omega^\epsilon)} \leq C\epsilon^6, \quad \|r_{12}\|^2_{L^2(\omega^\epsilon)} \leq C\epsilon^8.$$

Recall the asymptotic estimates (2.16) and (2.18), we have $\|\rho^\epsilon - \rho^0\|_{L^2} \leq C\epsilon^2$, $\|\gamma^\epsilon\|_{L^2} \leq C\epsilon^2$, $\|\tau^\epsilon\|_{L^2} \leq C\epsilon^2$, $\|\partial\rho^\epsilon\|_{L^2} \leq C$, and $\|\partial\gamma^\epsilon\|_{L^2} \leq C\epsilon^2$. From these, we get $\|w_1^\epsilon\|_{H^1} \leq C\epsilon^2$, $\|w_2^\epsilon\|_{H^1} \leq C$, $\|z_1^\epsilon\|_{H^1} \leq C\epsilon^2$, $\|z_2^\epsilon\|_{H^1} \leq C$. All these together with the estimates (4.6) establish the estimate $\|\varrho_{ij}^\epsilon\|^2_{L^2(\omega^\epsilon)} \leq C\epsilon^6$. Thus $\|\boldsymbol{\varrho}^\epsilon\|_{E(\Omega^\epsilon)} \leq C\epsilon^3$.

We see that in the expression of $\epsilon_{11}(\mathbf{u}^\epsilon)$, cf. the first equation of (4.3), the second term $x_2\rho^\epsilon$ has a dominating magnitude. When ϵ is sufficiently small, we have $\|\epsilon_{11}(\mathbf{u}^\epsilon)\|^2_{L^2(\omega^\epsilon)} \gtrsim \|x_2\rho^0\|^2_{L^2(\omega^\epsilon)} \simeq \epsilon^4\|\rho^0\|^2_{L^2}$. Since $\rho^0 \neq 0$, cf. Lemma 2.8, we see the lower bound $\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)} \geq C\epsilon^2$. □

We need some modifications on both $\boldsymbol{\sigma}^\epsilon$ and $\mathbf{u}^\epsilon$ on the ends of the arch, since neither of these fields is admissible, and they fail to meet the requirements of the Prager–Synge theorem. We defined $\boldsymbol{\sigma}_L^\epsilon$ and $\mathbf{u}_L^\epsilon$ as the stress and displacement solutions of the 3D elasticity theory on the arch such that $\mathbf{u}_L^\epsilon|_{\Gamma_0} = 0$ and $\boldsymbol{\sigma}_L^\epsilon\mathbf{n} = \mathbf{d} - \bar{\mathbf{d}}$ on Γ_L and the arch is free of traction on upper, lower, front, and rear faces and free of body force. It follows from Saint-Venant's principle that such fields exponentially decay from the right end, and they are negligible at a distance $C\epsilon$ away from the right end. On the left end, the welding displacement condition is not satisfied by $\mathbf{u}^\epsilon$ due to corrections (4.2). We define $\mathbf{u}_0^\epsilon$ as the solution of the elasticity theory on the arch such that $\mathbf{u}_0^\epsilon = -\mathbf{u}^\epsilon$ on Γ_0, and the arch is free of traction on all the other faces and subject to zero body force. The stress field $\boldsymbol{\sigma}_0^\epsilon = \mathcal{C}\boldsymbol{\epsilon}(\mathbf{u}_0^\epsilon)$ then exponentially decay from the left end. This is because the reacting force on Γ_0 has zero resultant and zero moment, and Saint-Venant's principle applies. Our statically admissible stress field then is $\bar{\boldsymbol{\sigma}}^\epsilon = \boldsymbol{\sigma}^\epsilon + \boldsymbol{\sigma}_0^\epsilon + \boldsymbol{\sigma}_L^\epsilon$ and kinematically admissible displacement field is $\bar{\mathbf{u}}^\epsilon = \mathbf{u}^\epsilon + \mathbf{u}_0^\epsilon + \mathbf{u}_L^\epsilon$. And we have $\bar{\boldsymbol{\sigma}}^\epsilon - \mathcal{C}\boldsymbol{\epsilon}(\bar{\mathbf{u}}^\epsilon) = \boldsymbol{\varrho}^\epsilon$. According to the Prager–Synge hyper-circle theorem, we have

$$\|\boldsymbol{\epsilon}(\bar{\mathbf{u}}^\epsilon - \mathbf{u}^{\epsilon *})\|_{E(\Omega^\epsilon)}^2 + \|\boldsymbol{\sigma}^{\epsilon *} - \bar{\boldsymbol{\sigma}}^\epsilon\|_{E(\Omega^\epsilon)}^2 = \|\boldsymbol{\varrho}^\epsilon\|_{E(\Omega^\epsilon)}^2.$$

Since $\mathbf{u}^\epsilon$ and $\bar{\mathbf{u}}^\epsilon$ are virtually equal on the interior portion Ω_0^ϵ of Ω^ϵ, it follows the estimate

$$\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon) - \boldsymbol{\epsilon}(\mathbf{u}^{\epsilon *})\|_{E(\Omega_0^\epsilon)} \le \|\boldsymbol{\varrho}^\epsilon\|_{E(\Omega^\epsilon)}.$$

We, therefore, proved the following theorem.

Theorem 4.1. *Let $\mathbf{u}^{\epsilon *}$ be the displacement solution of the 3D elasticity theory for the arch. Let $\mathbf{u}^\epsilon$ be the displacement field defined on the arch by modifying the solution of the arch model* (1.5) *using the formulas* (4.1). *The arch deformation is either bending dominated or membrane-shear dominated, depending on the arch shape and the loading. If the arch deformation is bending dominated, then we have the error estimate*

$$\frac{\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon) - \boldsymbol{\epsilon}(\mathbf{u}^{\epsilon *})\|_{E(\Omega_0^\epsilon)}}{\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)}} \le C\epsilon, \tag{4.12}$$

which is valid for the shear correction factor κ in the model being an arbitrary positive number. Here Ω_0^ϵ is an interior portion of Ω obtained by cutting off the two end portions of length $C\epsilon$ and C is a constant independent of ϵ.

If the arch deformation is membrane-shear dominated, and if we set $\kappa = 1$ in the model, then the same estimate (4.12) *holds. In this latter case, if $\kappa \neq 1$, the model fails when there is a significant shear in the arch deformation, which occurs when there is a nonzero surface force couple applied on the upper and lower faces of the arch. In this case, we have*

$$\frac{\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon) - \boldsymbol{\epsilon}(\mathbf{u}^{\epsilon *})\|_{E(\Omega_0^\epsilon)}}{\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)}} \simeq |\kappa - 1|.$$

Finally, we prove the lower order accuracy of $\mathbf{u}^\epsilon$ up to the clamped end Γ_0 by showing that $\|\boldsymbol{\epsilon}(\mathbf{u}_0^\epsilon)\|_{E(\Omega^\epsilon)}/\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)} \lesssim \sqrt{\epsilon}$. From this we see that

$$\frac{\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon) - \boldsymbol{\epsilon}(\mathbf{u}^{\epsilon *})\|_{E(\Omega_1^\epsilon)}}{\|\boldsymbol{\epsilon}(\mathbf{u}^\epsilon)\|_{E(\Omega^\epsilon)}} \le C\sqrt{\epsilon}.$$

Here Ω_1^ϵ is a subset of Ω^ϵ obtained by cutting-off the right end a portion of length $C\epsilon$. We define a displacement field $\tilde{\mathbf{u}}^\epsilon$ by giving its covariant components $\tilde{u}_1^\epsilon = 0$, $\tilde{u}_2^\epsilon = \tilde{w}_1^\epsilon x_2 + \frac{1}{2}\tilde{w}_2^\epsilon x_2^2 - \frac{1}{2}\tilde{z}_2^\epsilon x_3^2$, $u_3^\epsilon = \tilde{z}_1^\epsilon x_3 + \tilde{z}_2^\epsilon x_2 x_3$. Where, $\tilde{w}_1^\epsilon = -w_1^\epsilon(0)e^{-x/\epsilon}$, $\tilde{w}_2^\epsilon = -w_2^\epsilon(0)e^{-x/\epsilon}$, $\tilde{z}_1^\epsilon = -z_1^\epsilon(0)e^{-x/\epsilon}$, $\tilde{z}_2^\epsilon = -z_2^\epsilon(0)e^{-x/\epsilon}$. This makes $\tilde{\mathbf{u}}^\epsilon = \mathbf{u}_0^\epsilon$ on Γ_0.

By the minimum potential energy principle, we have $\|\boldsymbol{\epsilon}(\mathbf{u}_0^\epsilon)\|_{E(\Omega^\epsilon)} \leq \|\boldsymbol{\epsilon}(\tilde{\mathbf{u}}^\epsilon)\|_{E(\Omega^\epsilon)}$. It is straightforward to show that $\|\boldsymbol{\epsilon}(\tilde{\mathbf{u}}^\epsilon)\|_{E(\Omega^\epsilon)}$ is bounded by $C\epsilon^{1.5}$ under the condition of Lemma 4.1, and bounded by $C\epsilon^{2.5}$ in the case of Lemma 4.2. As for the right end Γ_L, we remark that the estimate

$$\|\boldsymbol{\epsilon}(\mathbf{u}_L^\epsilon)\|_{E(\Omega^\epsilon)} \leq C\sqrt{\epsilon}\|\mathbf{d} - \bar{\mathbf{d}}\|_{H^1(-\epsilon,\epsilon)}$$

holds.[22] Based on this, one can impose a condition on variance of the end force $\mathbf{d}$ in x_2 so that the relative modelling error of the order $\sqrt{\epsilon}$ holds up to both ends.

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