SHARP CONVERGENCE RATE OF DOMAIN EMBEDDING METHODS FOR VARIOUS BOUNDARY CONDITIONS

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Abstract. We prove sharp convergence rates for a class of domain embedding methods for elliptic boundary value problems. The theory is established for Dirichlet, Neumann, and Robin boundary conditions, and is unified in a new formulation for mixed boundary condition. On the continuous level, the methods for these boundary value problems perform equally well.

Key words. Domain embedding, fictitious domain, mixed boundary condition.
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1. Introduction

Domain embedding, or fictitious domain, methods have been extensively studied since the 1960's [22, 16]. General partial differential equation solvers based on this methodology have been developed and free softwares are available [9]. The methods (approximately) replace a problem defined on a domain by a problem defined on a larger regular (rectangular, e.g.) domain on which fast numerical methods could be applied. Uniform rectangular grids with no or little adjustment around the original domain boundary can be employed to produce accurate numerical solutions. This kind of methods is especially useful for problems defined on domains with complex, moving, or unknown boundaries [9].

When a domain is embedded in a larger domain, the complementary part of the original domain is said to be *fictitious*, and the boundary of the larger domain *artificial*. Many domain embedding methods have been proposed, tested, and analyzed. There is a class of methods that achieve the goal by extending an elliptic differential equation to the fictitious domain, where the ellipticity is retained but the coefficients in the equation are chosen to be very small or very large, depending on the boundary conditions in the original boundary value problem [22, 13, 16]. As these coefficients tend to zero/infinity, restriction of the solution of the extended equation on the original domain converges to the solution of the original problem. Or the solution of the extended equation converges to an extension of the original solution. Domain embedding methods have been extended beyond the realm of elliptic equations, and find their applications in fluid mechanics, electro-magnetics, etc [3, 7, 8]. Despite its fundamental role in further developments, the theory for elliptic boundary value problems seems not completely satisfying. For example, convergence rates determined in the literature are not sharp; Dirichlet problem is often treated in very different manners than Neumann problem; It is hard to deal with mixed boundary value problems. A better understanding of

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these issues is not only interesting in its own right, it could also be desirable for designers of numerical methods based on the domain embedding methodology. For example, the sharp convergence rate of the domain embedding solution toward the original solution could provide valuable guides in choosing the parameter values to match up the mesh-size of, say, a finite element method to maximize the overall efficiency of the numerical method [20, 17].

We present an analysis on the continuous level on the convergence rate of domain embedding methods of the class that is based on small/large parameters. Our theory is simple and we obtain the sharp convergence rate for Dirichlet, Neumann, and Robin problems, for all of which the domain embedding methods involve a single, small or large, parameter. The theory is unified in a formulation for mixed boundary value problems which requires two parameters; One is small and the other large. This seems a new method. The requirement to obtain the sharp convergence rate is rather weak. For example, if one is willing to impose free condition on the artificial boundary, then for Neumann and Robin problems we merely need to assume that both the original and the larger domains are Lipschitz. If one is willing to impose Dirichlet condition on the artificial boundary, then for Dirichlet problem the best result can be obtained under the additional condition that the fictitious domain is also Lipschitz, which allows, for example, an L-shaped domain being tightly embedded in a rectangle.

When a boundary value problem is to be solved by a domain embedding method, there are many options one has to face in, say, choosing the larger domain, extending the equation to the fictitious domain, imposing conditions on the artificial boundary, etc. As a result, there could be a variety of methods that are applicable to a given boundary value problem. To handle such a situation, we shall establish a functional framework into which various methods fit. This framework is a model borrowed from the plates and shells community [6, 5, 23] and generalized and sharpened under a condition to well serve the purpose of this paper. All our estimates on the convergence of domain embedding methods directly follow from an analysis of such parameter-dependent models. This way, a theory of domain embedding methods can be established for general self-adjoint second order elliptic equations defined on spaces of any dimension. It can also be extended to higher order equations. The abstract result that is proved for resolving the domain embedding method of mixed boundary value problems might be useful to some other problems that involve multi-scale energies as well. For simplicity, we only present the results for Poisson equation defined on a two-dimensional domain. In the remainder of this introduction, we briefly summarize our results. For comparison, we will quote several existing results we found in the literature.

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary Γ [11]. (The domain could be multiply connected.) For a given function f defined on ω, which we assume to be in $L^p(\omega)$ for some $p > 1$, we seek a function u on ω such that

$$-\Delta u = f \quad \text{on} \quad \omega \tag{1.1}$$

subject to various boundary conditions on Γ. To make the presentation sufficiently general, we assume that Γ is divided into three parts, (one or two of which could be empty,) such that $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_R$, and on Γ_D, Γ_N, and Γ_R, homogeneous Dirichlet, Neumann, and Robin boundary conditions are imposed, respectively. I.e.,

$$u = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N, \quad \frac{\partial u}{\partial n} + ku = 0 \text{ on } \Gamma_R. \tag{1.2}$$

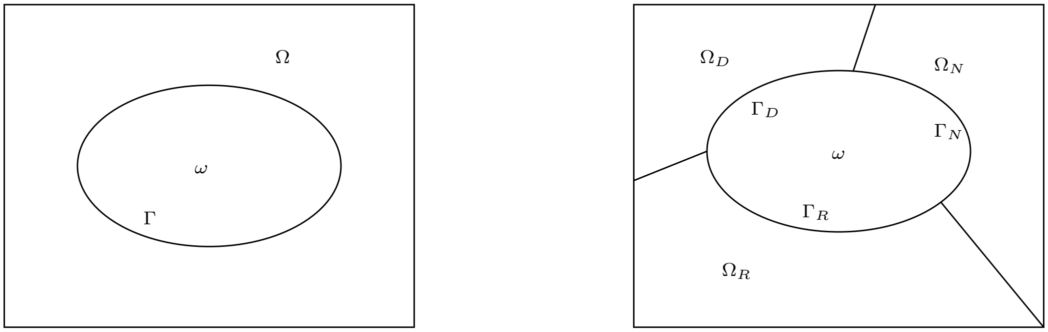

Figure 1. A domain ω embedded in a rectangular domain

Here n is the unit outward normal to Γ and k is a bounded and strictly positive function on Γ_R. (We could have viewed the Neumann condition as a special case of the Robin condition with $k = 0$.) The homogeneity is not a real restriction for Neumann and Robin conditions. For Dirichlet problem, a non-homogeneous condition could be reduced to a homogeneous one by defining a function satisfying the boundary condition, and subtracting it from u.

As usual, we let $H^1(\omega)$ be the first order Sobolev space of functions on ω and $H^1_D(\omega)$ be the subspace whose functions vanish on Γ_D. When $\Gamma = \Gamma_D$, we denote the latter by $H^1_0(\omega)$. The weak formulation of the boundary value problem (1.1) and (1.2) is as follows.

$$\begin{gathered}(\nabla u, \nabla v)_{[L^2(\omega)]^2} + (ku, v)_{L^2(\Gamma_R)} = \int_\omega f v dx \quad \forall\ v \in H^1_D(\omega),\\ u \in H^1_D(\omega),\end{gathered} \tag{1.3}$$

where ∇ is the gradient operator, and the parentheses stand for the inner products in Hilbert spaces indicated by the subscripts. If $\Gamma_D \neq \emptyset$ or $\Gamma_R \neq \emptyset$, then (1.3) allows for a unique solution as long as the right hand side integral in (1.3) defines a continuous functional on $H^1_D(\omega)$. For example, when $\Gamma = \Gamma_D$, we only need $f \in H^{-1}(\omega)$ for the problem to be well-posed. The assumption that $f \in L^p(\omega)$ for some $p > 1$ is a sufficient condition for the following domain embedding methods to work. If the problem is a pure Neumann problem, i.e., $\Gamma = \Gamma_N$, then we need to assume $\int_\omega f dx = 0$. Then u is uniquely determined in the quotient space $H^1(\omega)/\mathbb{R}$.

The domain embedding method determines an approximation to u by solving a boundary value problem on a larger domain $R \subset \mathbb{R}^2$ such that $\omega \subset R$. We let $\Omega = R \setminus \overline{\omega}$ be the fictitious domain. See the left figure in Figure 1. If we assume that $\omega \subset\subset R$, then the boundary condition on ∂R is at one's disposal. For simplicity, we impose the homogeneous Dirichlet condition. However, Neumann, Robin, mixed, or periodical boundary condition on ∂R works equally well. If we only have $\omega \subset R$, then on portions of ∂R that touch Γ_N or Γ_R, free boundary condition should be imposed, and on portions of ∂R that touch Γ_D Dirichlet condition should be imposed. Otherwise the domain embedding method is doomed to failure. We extend the function f to a functions $\bar{f}$ on R by defining $\bar{f} = 0$ on Ω. Other extensions are possible. For example, for Dirichlet problems, f could be extended to Ω arbitrarily. But for Neumann and Robin problems, there is no such freedom.

We first discuss the Dirichlet problem for which $\Gamma = \Gamma_D$. If ω is simply connected, then for small but positive ϵ, the domain embedding method determines $u^\epsilon \in H_0^1(R)$ such that

$$(\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + \epsilon^{-1}(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega)]^2} = \int_R \bar{f} v dx \quad \forall\ v \in H_0^1(R). \tag{1.4}$$

Since we have assumed $f \in L^p(\omega)$ for some $p > 1$, by the Sobolev embedding theorem [1], the right hand side integral defines a linear continuous functional on $H_0^1(R)$. Therefore, the problem (1.4) is a well-defined elliptic problem on R. The fundamental question is when $\epsilon \to 0$ whether or not u^ϵ converges to the solution u of (1.3) with Γ_N and Γ_R being empty. We analyze this problem in two steps:

1) Whether and at what rate does u^ϵ converge to a limiting function u^0 defined on R as $\epsilon \to 0$;
2) If there is a limit, whether the restriction of the limiting function on ω is actually a solution of the original boundary value problem on ω, which we denote by $u^0|_\omega = u$.

For the Dirichlet problem, we prove that 1) as $\epsilon \to 0$, u^ϵ always converges to a limit $u^0 \in H_0^1(R)$; 2) if Ω is a Lipschitz domain, (for example, when $\omega \subset\subset R$ or both ω and R are polygons such that $\Gamma \cap \partial R$ has no isolated point) the convergence occurs at the rate $\mathcal{O}(\epsilon)$; and 3) when ω is simply connected we have $u^0|_\omega = u$ and $u^0|_\Omega = 0$. Henceforth, we shall use the notation $\mathcal{P} \lesssim \mathcal{Q}$ which means that there exists a constant C independent of ϵ such that $\mathcal{P} \le C\mathcal{Q}$. (The constant C could be dependent on ω, Ω, f, and boundary conditions on Γ and ∂R. Detailed information on such dependence can be derived.) The notation $\mathcal{P} \simeq \mathcal{Q}$ means $\mathcal{P} \lesssim \mathcal{Q}$ and $\mathcal{Q} \lesssim \mathcal{P}$. In terms of these notations, we have the equivalent estimate

$$\|u^\epsilon - u^0\|_{H^1(R)} \simeq \|u^\epsilon\|_{H^1(\Omega)} \simeq \epsilon\,. \tag{1.5}$$

So the convergence rate is sharp. There is one exception to this equivalence in which $u^\epsilon \equiv u^0$. This occurs if and only if u^0 also satisfies the homogeneous Neumann condition on $\Gamma \setminus \partial R$, of which a trivial example is when $\Gamma \setminus \partial R = \emptyset$, i.e., $\omega = R$.

If ω is not simply connected, then we do not have $u^0|_\omega = u$. However, if the term $\epsilon^{-1}(u^\epsilon, v)_{L^2(\Omega)}$ were added to the left hand side of (1.4), then the estimate (1.5) holds and $u^0|_\omega = u$ regardless of the connectivity of ω. Actually, in case that ω is multiply connected, adding a term of the form $\epsilon^{-1}(u^\epsilon, v)_{L^2(\Omega^0)}$ in the left hand side of (1.4) is sufficient to guarantee the validity of the domain embedding method. Here Ω^0 is the union of isolated and simply connected components of Ω. If Ω is not a Lipschitz domain, then we still have $\lim_{\epsilon \to 0} u^\epsilon = u^0$. But the convergence rate is lower, which depends on, for example, acuteness of cusps of Ω [1]. Some other methods that are essentially different from, and converge more slowly than (1.4) will also be discussed in the following sections.

The method (1.4) is equivalent to the one proposed in [22], see [16]. The best convergence rate we found in the literature is [9]

$$\|u^\epsilon - u^0\|_{H^1(R)} = \mathcal{O}(\sqrt{\epsilon}).$$

For Neumann problem, we have $\Gamma = \Gamma_N$, and the domain embedding method determines $u^\epsilon \in H_0^1(R)$ such that

$$(\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + \epsilon(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega)]^2} = \int_R \bar{f} v dx \quad \forall\ v \in H_0^1(R). \tag{1.6}$$

This is an elliptic problem that yields a unique $u^\epsilon \in H_0^1(R)$. As $\epsilon \to 0$, u^ϵ converges to a limit $u^0 \in H_0^1(R)$ at the rate $\mathcal{O}(\epsilon)$ as long as ω is a Lipschitz domain, as we have assumed at the outset. In contrast to the Dirichlet problem, we do not need to assume that Ω to be a Lipschitz domain to obtain such a rate. We prove the equivalent estimate

$$\|u^\epsilon - u^0\|_{H^1(R)} \simeq |u^\epsilon - u^0|_{H^1(\omega)} \simeq \epsilon, \tag{1.7}$$

except the case that $u^\epsilon \equiv u^0$, which happens if and only if $u^0 = 0$ on Γ. The limit u^0 is harmonic on Ω, and if $\omega \subset\subset R$ then $u^0|_\omega$ solves the Neumann problem (1.3) with Γ_D and Γ_R being empty. The aforementioned special case of $u^\epsilon \equiv u^0$ occurs when a solution of the homogeneous Neumann boundary value problem also satisfies the homogeneous Dirichlet condition on Γ.

If we impose homogeneous Neumann condition on ∂R in the domain embedding equation (1.6), which is then well-posed in the quotient space $H^1(R)/\mathbb{R}$, then u^ϵ converges to u^0 at the sharp rate of ϵ in the $H^1(R)$ semi-norm, and $u^0|_\omega$ solves the the original Neumann problem on ω without assuming $\omega \subset\subset R$. And we have

$$|u^\epsilon - u^0|_{H^1(R)} \simeq |u^\epsilon - u^0|_{H^1(\omega)} \simeq \epsilon\,. \tag{1.8}$$

The same exception to the equivalent estimate (1.7) exists, in which $|u^\epsilon - u^0| \equiv 0$.

The Neumann problem is the most extensively studied; Various domain embedding methods have been proposed and analyzed in the literature. As far as we know, the existing best result on the convergence rate is [13]

$$|u^\epsilon - u^0|_{H^1(\omega)} = o(\sqrt{\epsilon}) \quad \text{and} \quad \lim_{\epsilon \to 0} |u^\epsilon - u^0|_{H^1(\Omega)} = 0.$$

For Robin problem, we have $\Gamma = \Gamma_R$. The domain embedding method determines $u^\epsilon \in H_0^1(R)$ such that

$$(\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + (k u^\epsilon, v)_{L^2(\Gamma)} + \epsilon(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega)]^2} = \int_R \bar{f} v dx \ \ \forall\ v \in H_0^1(R). \tag{1.9}$$

This, once again, is a well defined problem on R, and $\lim_{\epsilon\to 0} u^\epsilon = u^0 \in H_0^1(R)$. We prove that

$$\|u^\epsilon - u^0\|_{H^1(R)} \simeq \|u^\epsilon - u^0\|_{H^1(\omega)} + \|u^\epsilon - u^0\|_{L^2(\Gamma)} \simeq \epsilon\,. \tag{1.10}$$

The limit u^0 is harmonic on Ω, and if $\omega \subset\subset R$ then $u^0|_\omega$ solves the Robin problem (1.3) with Γ_D and Γ_N being empty. There is an exception to the equivalent estimate (1.10) in which $u^\epsilon \equiv u^0$. This happens if and only if $u^0 = 0$ on Γ.

If we impose free boundary condition in (1.9), then it is well defined in the space $H^1(R)$. By doing so, the condition $\omega \subset\subset R$ can be removed. As in the Neumann problem, we do not need to assume Ω to be a Lipschitz domain to obtain the convergence rate (1.10).

The ultimate result of this paper is on the domain embedding method for mixed boundary value problems that unifies all the above. Corresponding to the splitting of the boundary $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_R$, we divide Ω into three parts such that $\overline{\Omega} = \overline{\Omega}_D \cup \overline{\Omega}_N \cup \overline{\Omega}_R$, and $\partial\Omega_D \cap \Gamma = \overline{\Gamma}_D$, $\partial\Omega_N \cap \Gamma = \overline{\Gamma}_N$, and $\partial\Omega_R \cap \Gamma = \overline{\Gamma}_R$, see the right figure in Figure 1. The domain embedding method determines $u^\epsilon \in H_0^1(R)$ such that

$$(1.11)\quad (\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + (ku^\epsilon, v)_{L^2(\Gamma_R)} + \epsilon^{-1}(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega_D)]^2} + \epsilon(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega\setminus\overline{\Omega}_D)]^2} = \int_R \bar{f} v dx \quad \forall\ v \in H_0^1(R).$$

This is a well-defined problem on in $H_0^1(R)$. When $\epsilon \to 0$, u^ϵ converges to a limit $u^0 \in H_0^1(R)$. Under the assumption that *both* Ω_D *and* $\omega \cup \Gamma_D \cup \Omega_D$ *are Lipschitz domains*, we have the equivalent estimate

$$(1.12)\qquad \|u^\epsilon - u^0\|_{H^1(R)} \simeq \|u^\epsilon - u^0\|_{H^1(\omega)} + \|u^\epsilon\|_{H^1(\Omega_D)} \simeq \epsilon,$$

with one exception in which $u^\epsilon \equiv u^0$. Under the assumptions that $\overline{\Gamma}_N \cup \overline{\Gamma}_R \subset R$ and Ω_D has no isolated and simply connected components, we have that $u^0 = 0$ on Ω_D, $\Delta u^0 = 0$ on $\Omega \setminus \overline{\Omega}_D$, and the restriction of u^0 on ω is the solution of the mixed boundary value problem (1.3). The exception $u^\epsilon \equiv u^0$ occurs if and only if the solution of the original mixed boundary value problem on ω also satisfies homogeneous Dirichlet condition on $\Gamma_N \cup \Gamma_R$ and satisfies homogeneous Neumann condition on $\Gamma_D \setminus \partial R$.

The condition that Ω_D has no isolated and simply connected component can be removed by adding the term $\epsilon^{-1}(u^\epsilon, v)_{L^2(\Omega_D)}$ to the left hand side of (1.11). If in the domain embedding equation (1.11) we impose free boundary condition on portions of ∂R that touch Γ_N and Γ_R, then the restriction $\overline{\Gamma}_N \cup \overline{\Gamma}_R \subset R$ can be released. The condition that both Ω_D and $\omega \cup \Gamma_D \cup \Omega_D$ are Lipschitz domains is sufficient to ensure the convergence rate.

Before closing this introduction, we make a few remarks on numerical methods for the original boundary value problems based on the domain embedding equations on R. We assume that Γ and f are sufficiently smooth so that u (that is the solution of the original boundary value problem) has the necessary regularity. Let's introduce a triangulation of maximum mesh size h on R and apply the standard finite element (linear and continuous, e.g.) method to the equation (1.4), (1.6), (1.9), or (1.11), to obtain an approximation u_h^ϵ to u^ϵ. Then the restriction of u_h^ϵ on ω approximates u. For Neumann problem, it can be shown that an error estimate of the form

$$|u_h^\epsilon - u|_{H^1(\omega)} \lesssim \epsilon + \inf_{v_h} \left(\|u^0 - v_h\|_{H^1(\omega)} + \sqrt{\epsilon}\|u^0 - v_h\|_{H^1(\Omega)}\right)$$

holds, in which the infimum is taken over the finite element space. Thus a best choice of $\epsilon \simeq h$ gives a domain embedding finite element method that achieves the full accuracy with an error of $\mathcal{O}(h)$. This result is better than that of [13] virtually due to our sharp estimate (1.7), and there is no need of assuming the alignment of Γ with the finite element mesh as required in [15]. For Dirichlet problem, it can be shown that an error estimate of the form

$$\|u_h^\epsilon - u\|_{H^1(\omega)} \lesssim \epsilon + \inf_{v_h} \left(\|u^0 - v_h\|_{H^1(\omega)} + \frac{1}{\sqrt{\epsilon}}\|v_h\|_{H^1(\Omega)}\right)$$

holds. We see that if Γ does not intersect any mesh-line (that is viewed as an open straight line segment) then a best choice of $\epsilon \simeq h$ gives a domain embedding finite element method that has the optimal accuracy of $\mathcal{O}(h)$. If Γ crosses some mesh-line segments, the finite element convergence rate will be limited by $\mathcal{O}(\sqrt{h})$, no matter how small ϵ is. Thus a slight adjustment of the finite element mesh to accommodate Γ appears to be desirable. Similar results can be proved for mixed boundary value problems, and we shall discuss this in a future work. The condition number of the resulting stiffness matrix is $\mathcal{O}(\epsilon^{-1}\,h^{-2})$ for the Dirichlet, Neumann, and Robin problems, and $\mathcal{O}(\epsilon^{-2}\,h^{-2})$ for the mixed problem. This, of course, poses additional challenges in solving the discrete system. Some references in this direction can be found in [14, 15, 16].

The paper is organized as follows. In Section 2, we introduce our parameter-dependent functional equations and derive sharp estimates on the dependence of its solution on the parameter. In Section 3, we prove the above estimates by fitting the domain embedding equations into the abstract framework and verifying the conditions enforced on the abstract problem.

2. A FUNCTIONAL FRAMEWORK

Let H, U, and V be Hilbert spaces, $A : H \to U$ a bounded linear operator, and $B : H \to V$ a bounded linear operator *with closed range*. We assume that

$$\|A\boldsymbol{v}\|_U + \|B\boldsymbol{v}\|_V \simeq \|\boldsymbol{v}\|_H \quad \forall\, \boldsymbol{v} \in H. \tag{2.1}$$

Thus the bilinear form

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathcal{H}} := (A\boldsymbol{u}, A\boldsymbol{v})_U + (B\boldsymbol{u}, B\boldsymbol{v})_V$$

defines an equivalent inner product on H. Furnished with this new inner product, the space H will be denoted by $\mathcal{H}$. Let $\ker B$ be the kernel of B in H. For any $\boldsymbol{f} \in H^*$, the dual space of H, we have either $\boldsymbol{f}|_{\ker B} = 0$ or $\boldsymbol{f}|_{\ker B} \neq 0$. If $\boldsymbol{f}|_{\ker B} = 0$, we consider the variational problem of finding $\boldsymbol{u}^\epsilon \in H$ such that

$$\epsilon(A\boldsymbol{u}^\epsilon, A\boldsymbol{v})_U + (B\boldsymbol{u}^\epsilon, B\boldsymbol{v})_V = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \;\forall \boldsymbol{v} \in H. \tag{2.2}$$

If $\boldsymbol{f}|_{\ker B} \neq 0$, we consider the variational problem of finding $\boldsymbol{u}^\epsilon \in H$ such that

$$(A\boldsymbol{u}^\epsilon, A\boldsymbol{v})_U + \epsilon^{-1}(B\boldsymbol{u}^\epsilon, B\boldsymbol{v})_V = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \;\forall \boldsymbol{v} \in H. \tag{2.3}$$

Both of the two problems are well-posed in H. The problem (2.2) applies to domain embedding methods for the Neumann and Robin problems, while (2.3) is suitable for the Dirichlet problem. These abstract problems have been extensively studied in the literature since the work of Lions [18], especially in the context of plates and shells in recent years. The problem (2.2) also represents models of stiff arches and membrane shells, and the problem (2.3) applies to Timoshenko beam, Reissner–Mindlin plate, Naghdi and Koiter shell models [2, 4, 5, 6]. However, for most of these applications, the condition that B has closed range can not be enforced [10, 21, 23]. The novelty of the following two lemmas is in their sharpness of estimates.

We first assume that $\boldsymbol{f}|_{\ker B} = 0$ and discuss (2.2). Without loss of generality, we can assume that the operator B maps H onto V. (If necessary, we replace V by the range of B in it.) Then B is an isomorphism between $(\ker B)^{\perp}_{\mathcal{H}}$ (the orthogonal complement of $\ker B$ with respect to the $\mathcal{H}$-norm) and V. Since we have assumed $\boldsymbol{f}|_{\ker B} = 0$, according the the closed range theorem and Riesz representation theorem, there exists a unique $\boldsymbol{u}^0 \in (\ker B)^{\perp}_{\mathcal{H}}$ such that

$$(B\boldsymbol{u}^0, B\boldsymbol{v})_V = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in H. \tag{2.4}$$

Lemma 2.1. *Under the assumptions that* $\boldsymbol{f}|_{\ker B} = 0$ *and* B *has closed range in* V*, as* $\epsilon \to 0$*,* $\boldsymbol{u}^\epsilon$*, the solution of* (2.2)*, converges to the limit* $\boldsymbol{u}^0 \in (\ker B)^{\perp}_{\mathcal{H}}$ *defined by* (2.4)*, and we have the equivalence estimate*

$$\|\boldsymbol{u}^\epsilon - u^0\|_H \simeq \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V \simeq \epsilon\,\|A\boldsymbol{u}^0\|_U. \tag{2.5}$$

It is obvious that $\|A\boldsymbol{u}^0\|_U \lesssim \|\boldsymbol{f}\|_{H^*}$*. Therefore, if* $A\boldsymbol{u}^0 = 0$ *then* $\boldsymbol{u}^\epsilon \equiv \boldsymbol{u}^0$*. Otherwise* $\boldsymbol{u}^\epsilon$ *converges to* $\boldsymbol{u}^0$ *at the sharp rate of* ϵ*.*

Proof. From (2.2) and (2.4), we see that

$$\epsilon(A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0), A\boldsymbol{v})_U + (B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0), B\boldsymbol{v})_V = -\,\epsilon(A\boldsymbol{u}^0, A\boldsymbol{v})_U \;\forall \boldsymbol{v} \in H. \tag{2.6}$$

Taking $\boldsymbol{v} = \boldsymbol{u}^\epsilon - \boldsymbol{u}^0$, we get

$$\epsilon\,\|A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_U^2 + \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V^2 = -\,\epsilon(A\boldsymbol{u}^0, A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0))_U. \tag{2.7}$$

It is easy to see that $\boldsymbol{u}^\epsilon$ lies in the subspace $(\ker B)^{\perp}_{\mathcal{H}}$. So does $\boldsymbol{u}^\epsilon - \boldsymbol{u}^0$. Because B is an isomorphism between $(\ker B)^{\perp}_{\mathcal{H}}$ and V, we have

$$\|A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_U \lesssim \|\boldsymbol{u}^\epsilon - \boldsymbol{u}^0\|_H \simeq \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V. \tag{2.8}$$

By the Cauchy–Schwarz inequality, there exists a constant C such that

$$|\,\epsilon(A\boldsymbol{u}^0, A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0))_U| \le C\,\epsilon^2\,\|A\boldsymbol{u}^0\|_U^2 + \frac{1}{2}\|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V^2.$$

This estimate and the equation (2.7) show that

$$\|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V \lesssim \epsilon\,\|A\boldsymbol{u}^0\|. \tag{2.9}$$

The equivalence (2.1) and (2.8) then lead to $\|\boldsymbol{u}^\epsilon - \boldsymbol{u}^0\|_H \lesssim \epsilon\,\|A\boldsymbol{u}^0\|_U$.

To see the lower bound, in the (2.6) we take $\boldsymbol{v} = \boldsymbol{u}^0$ to get

$$\epsilon(A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0), A\boldsymbol{u}^0)_U + (B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0), B\boldsymbol{u}^0)_V = -\,\epsilon\,\|A\boldsymbol{u}^0\|_U^2.$$

If $A\boldsymbol{u}^0 \neq 0$, then

$$\epsilon\,\|A\boldsymbol{u}^0\|_U \lesssim \epsilon\,\|A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_U + \frac{\|B\boldsymbol{u}^0\|_V}{\|A\boldsymbol{u}^0\|_U}\|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V \lesssim \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V.$$

The equivalence (2.5) then follows. ☐

Next, we assume $\boldsymbol{f}|_{\ker B} \neq 0$ and consider the problem (2.3). Under this assumption, there exists a unique non-zero element $\boldsymbol{u}^0 \in \ker B$ such that

$$(A\boldsymbol{u}^0, A\boldsymbol{v})_U = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \;\; \forall\, \boldsymbol{v} \in \ker B. \tag{2.10}$$

Thus $\langle \boldsymbol{f}, \boldsymbol{v}\rangle - (A\boldsymbol{u}^0, A\boldsymbol{v})_U = 0 \;\forall\, \boldsymbol{v} \in \ker B$. Therefore, there exists a unique $\boldsymbol{u}^1 \in (\ker B)^{\perp}_{\mathcal{H}}$ such that

$$(B\boldsymbol{u}^1, B\boldsymbol{v})_V = \langle \boldsymbol{f}, \boldsymbol{v}\rangle - (A\boldsymbol{u}^0, A\boldsymbol{v})_U \;\; \forall\, \boldsymbol{v} \in H. \tag{2.11}$$

Obviously $\|B\boldsymbol{u}^1\|_V \lesssim \|\boldsymbol{f}\|_{H^*}$.

Lemma 2.2. *Under the assumption that* $\boldsymbol{f}|_{\ker B} \neq 0$ *and* B *has closed range in* V*, as* $\epsilon \to 0$*,* $\boldsymbol{u}^\epsilon$*, the solution of* (2.3)*, converges to the nonzero* $\boldsymbol{u}^0 \in \ker B$ *defined by* (2.10)*. We have the equivalent estimate*

$$\|\boldsymbol{u}^\epsilon - \boldsymbol{u}^0\|_H \simeq \|B\boldsymbol{u}^\epsilon\|_V \simeq \epsilon\, \|B\boldsymbol{u}^1\|_V. \tag{2.12}$$

Here $\boldsymbol{u}^1 \in (\ker B)^\perp_{\mathcal{H}}$ *is defined by* (2.11)*. Therefore, if* $\boldsymbol{u}^1 = 0$ *then* $\boldsymbol{u}^\epsilon \equiv \boldsymbol{u}^0$*. Otherwise,* $\boldsymbol{u}^\epsilon$ *converges to* $\boldsymbol{u}^0$ *at the sharp rate of* ϵ*.*

Proof. The key observation is that $\boldsymbol{u}^\epsilon - \boldsymbol{u}^0$ satisfies the equation

$$\epsilon(A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0), Av)_U + (B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0), Bv)_V = \epsilon[\langle \boldsymbol{f}, \boldsymbol{v}\rangle - (A\boldsymbol{u}^0, A\boldsymbol{v})_U] \quad \forall \boldsymbol{v} \in H.$$

Note that the right hand side of this equation is a functional that annihilates $\ker B$. We see $\boldsymbol{u}^\epsilon - \boldsymbol{u}^0 \in (\ker B)^\perp_{\mathcal{H}}$ and that Lemma 2.1 is applicable to estimating $\boldsymbol{u}^\epsilon - \boldsymbol{u}^0 - \epsilon\, \boldsymbol{u}^1$. The equivalence (2.5) leads to

$$\|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0 - \epsilon\, \boldsymbol{u}^1)\|_V \simeq \epsilon^2\, \|A\boldsymbol{u}^1\|_U.$$

Therefore,

$$\epsilon\, \|B\boldsymbol{u}^1\|_V - \epsilon^2\, \|A\boldsymbol{u}^1\|_U \lesssim \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V \lesssim \epsilon\, \|B\boldsymbol{u}^1\|_V + \epsilon^2\, \|A\boldsymbol{u}^1\|_U.$$

Since $\boldsymbol{u}^1 \in (\ker B)^\perp_{\mathcal{H}}$, we have $\|A\boldsymbol{u}^1\|_U \lesssim \|B\boldsymbol{u}^1\|_V$. As ϵ is sufficiently small, we have $\|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V \simeq \epsilon\, \|B\boldsymbol{u}^1\|_V$. The result (2.12) thus follows from $\|\boldsymbol{u}^\epsilon - \boldsymbol{u}^0\|_H \simeq \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V$ and $B\boldsymbol{u}^0 = 0$. ☐

To analyze the domain embedding method for mixed boundary condition, we need to consider a "three-terms" variational equation. Let H, U, V, and W be Hilbert spaces, $A : H \to U$ and $B : H \to V$ be bounded linear operators, and $C : H \to W$ be a bounded linear operator *with closed range*. Furthermore, we assume that $B \times C : H \to V \times W$ also *has closed range*, which is defined as $(B \times C)\boldsymbol{v} = (B\boldsymbol{v}, C\boldsymbol{v}) \in V \times W \;\; \forall\, \boldsymbol{v} \in H$. We assume that

$$\|A\boldsymbol{v}\|_U + \|B\boldsymbol{v}\|_V + \|C\boldsymbol{v}\|_W \simeq \|\boldsymbol{v}\|_H \quad \forall\, \boldsymbol{v} \in H. \tag{2.13}$$

Thus the bilinear form

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathcal{H}} := (A\boldsymbol{u}, A\boldsymbol{v})_U + (B\boldsymbol{u}, B\boldsymbol{v})_V + (C\boldsymbol{u}, C\boldsymbol{v})_W$$

defines an equivalent inner product in H. Furnished with this new inner product, we denote the space by $\mathcal{H}$. For small but positive ϵ, there exists a unique $\boldsymbol{u}^\epsilon \in H$ such that

$$\epsilon(A\boldsymbol{u}^\epsilon, A\boldsymbol{v})_U + (B\boldsymbol{u}^\epsilon, B\boldsymbol{v})_V + \epsilon^{-1}(C\boldsymbol{u}^\epsilon, Cv)_W = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \; \forall \boldsymbol{v} \in H. \tag{2.14}$$

Here $\boldsymbol{f} \in H^*$ is a functional such that

$$\boldsymbol{f}|_{\ker B \cap \ker C} = 0. \tag{2.15}$$

Since $B \times C$ has closed range in $V \times W$, we have

$$\|B\boldsymbol{u}\|_V + \|C\boldsymbol{u}\|_W \simeq \|\boldsymbol{u}\|_H \quad \forall\, \boldsymbol{u} \in (\ker B \cap \ker C)^\perp_{\mathcal{H}}. \tag{2.16}$$

So, $\|B\boldsymbol{u}\|_V \simeq \|\boldsymbol{u}\|_H \;\forall\; \boldsymbol{u} \in \ker C \cap (\ker B \cap \ker C)^\perp_{\mathcal{H}}$. Therefore, there exists a unique $\boldsymbol{u}^0 \in \ker C \cap (\ker B \cap \ker C)^\perp_{\mathcal{H}}$ such that

$$(B\boldsymbol{u}^0, B\boldsymbol{v})_V = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v} \in \ker C. \tag{2.17}$$

Since C has closed range in W and the functional $\langle \boldsymbol{f}, \boldsymbol{v}\rangle - (B\boldsymbol{u}^0, B\boldsymbol{v})_V$ vanishes for any $\boldsymbol{v} \in \ker C$, we have a unique $\boldsymbol{u}_0^1 \in (\ker C)_{\mathcal{H}}^{\perp}$ such that

$$(C\boldsymbol{u}_0^1, C\boldsymbol{v})_W = \langle \boldsymbol{f}, \boldsymbol{v}\rangle - (B\boldsymbol{u}^0, B\boldsymbol{v})_V \quad \forall\ \boldsymbol{v} \in H. \tag{2.18}$$

It is easy to see that $\|A\boldsymbol{u}^0\|_U + \|C\boldsymbol{u}_0^1\|_W \lesssim \|\boldsymbol{f}\|_{H^*}$.

Lemma 2.3. *Under the assumptions that* $\boldsymbol{f}|_{\ker B \cap \ker C} = 0$, *$C$ has closed range in W, and $B \times C$ has closed range in $V \times W$, we have* $\lim_{\epsilon \to 0} \boldsymbol{u}^\epsilon = \boldsymbol{u}^0$. *The limit belongs to* $\ker C \cap (\ker B \cap \ker C)_{\mathcal{H}}^{\perp}$, *and is defined by* (2.17). *We have the estimate*

$$\begin{aligned}\epsilon[\|C\boldsymbol{u}_0^1\|_W + \kappa(\|C\boldsymbol{u}_0^1\|_W)\|A\boldsymbol{u}^0\|_U] &\lesssim \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V + \|C\boldsymbol{u}^\epsilon\|_W \\ &\lesssim \|\boldsymbol{u}^\epsilon - \boldsymbol{u}^0\|_H \lesssim \epsilon[\|A\boldsymbol{u}^0\|_U + \|C\boldsymbol{u}_0^1\|_W].\end{aligned} \tag{2.19}$$

Here $\boldsymbol{u}_0^1 \in (\ker C)_{\mathcal{H}}^{\perp}$ *is defined by* (2.18). *The function κ is defined as $\kappa(x) = 1$ if $x = 0$ and $\kappa(x) = 0$ otherwise. When $\epsilon \to 0$, we have either $\boldsymbol{u}^\epsilon \equiv \boldsymbol{u}^0$ or it converges to $\boldsymbol{u}^0$ at the sharp rate of ϵ. The former occurs if and only if $A\boldsymbol{u}^0 = 0$ and $C\boldsymbol{u}_0^1 = 0$.*

Proof. Using the closed range theorem and Riesz representation theorem, together with the assumptions (2.15) and that $B \times C$ has closed range in $V \times W$, we see that there exists a unique $\boldsymbol{u}_0^\epsilon \in (\ker B \cap \ker C)_{\mathcal{H}}^{\perp}$ such that

$$(B\boldsymbol{u}_0^\epsilon, B\boldsymbol{v})_V + \epsilon^{-1}(C\boldsymbol{u}_0^\epsilon, C\boldsymbol{v})_W = \langle \boldsymbol{f}, \boldsymbol{v}\rangle \quad \forall\ \boldsymbol{v} \in H. \tag{2.20}$$

Since C has closed range in W, we can apply Lemma 2.2 to this equation to obtain

$$\|\boldsymbol{u}_0^\epsilon - \boldsymbol{u}^0\|_H \simeq \|C\boldsymbol{u}_0^\epsilon\|_W \simeq \epsilon\, \|C\boldsymbol{u}_0^1\|_W. \tag{2.21}$$

Subtracting equation (2.20) from equation (2.14), we get

$$\begin{aligned}\epsilon(A(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon), A\boldsymbol{v})_U + (B(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon), B\boldsymbol{v})_V + \epsilon^{-1}(C(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon), C\boldsymbol{v})_W& \\ = -\,\epsilon(A\boldsymbol{u}_0^\epsilon, A\boldsymbol{v})_U\ \forall \boldsymbol{v} \in H.&\end{aligned} \tag{2.22}$$

The condition (2.15) guarantees that $\boldsymbol{u}^\epsilon \in (\ker B \cap \ker C)_{\mathcal{H}}^{\perp}$. So $\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon \in (\ker B \cap \ker C)_{\mathcal{H}}^{\perp}$. Therefore, by (2.16), we have $\|A(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon)\|_U \lesssim \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon)\|_V + \|C(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon)\|_W$. Taking $\boldsymbol{v} = \boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon$ in (2.22), and using the Cauchy–Schwarz inequality, we get

$$\epsilon\, \|A(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon)\|_U^2 + \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon)\|_V^2 + \epsilon^{-1}\, \|C(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon)\|_W^2 \lesssim \epsilon^2\, \|A\boldsymbol{u}_0^\epsilon\|_U^2. \tag{2.23}$$

Thus

$$\|\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon\|_H \simeq \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon)\|_V + \|C(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon)\|_W \lesssim \epsilon\, \|A\boldsymbol{u}_0^\epsilon\|_U. \tag{2.24}$$

Using (2.21), we get the upper bound $\|\boldsymbol{u}^\epsilon - \boldsymbol{u}^0\|_H \lesssim \epsilon[\|A\boldsymbol{u}^0\|_U + \|C\boldsymbol{u}_0^1\|_W]$.

To see the lower bound in (2.19), we assume that $\|A\boldsymbol{u}^0\|_U + \|C\boldsymbol{u}_0^1\|_W \neq 0$ and separately consider the cases of whether $C\boldsymbol{u}_0^1 = 0$ or not. If $C\boldsymbol{u}_0^1 = 0$, from (2.21) we see that $\boldsymbol{u}_0^\epsilon = \boldsymbol{u}^0$ and $C\boldsymbol{u}^0 = C\boldsymbol{u}_0^\epsilon = 0$. In equation (2.22), we take $\boldsymbol{v} = \boldsymbol{u}^0$ to get

$$\epsilon(A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0), A\boldsymbol{u}^0)_U + (B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0), B\boldsymbol{u}^0)_V = -\,\epsilon(A\boldsymbol{u}^0, A\boldsymbol{u}^0)_U. \tag{2.25}$$

We thus get

$$\begin{aligned}\epsilon\, \|A\boldsymbol{u}^0\|_U &\lesssim \epsilon\, \|A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_U + \frac{\|B\boldsymbol{u}^0\|_V}{\|A\boldsymbol{u}^0\|_U}\|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V \\ &\lesssim \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V + \|C\boldsymbol{u}^\epsilon\|_W \lesssim \|\boldsymbol{u}^\epsilon - \boldsymbol{u}^0\|_H.\end{aligned} \tag{2.26}$$

If $C\boldsymbol{u}_0^1 \neq 0$, since $\boldsymbol{u}^0 \in \ker C$ we have

$$\|\boldsymbol{u}^\epsilon - \boldsymbol{u}^0\|_H \gtrsim \|C\boldsymbol{u}^\epsilon\|_W \gtrsim \|C\boldsymbol{u}_0^\epsilon\|_W - \|C(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon)\|_W.$$

From (2.21) we know that $\|C\boldsymbol{u}_0^\epsilon\|_W \simeq \epsilon \|C\boldsymbol{u}_0^1\|_W$, while from (2.23) and (2.21) we see $\|C(\boldsymbol{u}^\epsilon - \boldsymbol{u}_0^\epsilon)\|_W \lesssim \epsilon^{3/2} \|A\boldsymbol{u}_0^\epsilon\|_U \lesssim \epsilon^{3/2} \|A\boldsymbol{u}^0\|_U + \epsilon^{5/2} \|C\boldsymbol{u}_0^1\|_W$. Thus $\|C\boldsymbol{u}^\epsilon\|_W \gtrsim \epsilon \|C\boldsymbol{u}_0^1\|_W - \epsilon^{3/2} \|A\boldsymbol{u}^0\|_U$. So when ϵ is sufficiently small, we have $\|C\boldsymbol{u}^\epsilon\|_W \gtrsim \epsilon \|C\boldsymbol{u}_0^1\|_W$. ☐

3. Domain embedding methods

In this section, we prove convergence rates of domain embedding methods for all kinds of boundary value problems as introduced in the introduction. What we need to do is fitting an ϵ-dependent domain embedding equation into a suitable functional framework established in the previous section and verifying the conditions needed for the abstract result to hold. The following lemmas will be used several times in verifying the range closeness of some operators. The first one is a basic result [11] on function extension and the second one is due to Peetre [19].

Lemma 3.1. *Let $\Omega_1 \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary, and $\Omega_1 \subset\subset \Omega_2$ with Ω_2 being an open domain. Then there is a bounded linear extension operator E from $H^1(\Omega_1)$ to $H_0^1(\Omega_2)$ such that for any $v \in H^1(\Omega_1)$, $E(v)|_{\Omega_1} = v$ and $\|E(v)\|_{H^1(\Omega_2)} \leq C\|v\|_{H^1(\Omega_1)}$. Here C is a constant depending on Ω_1 and Ω_2.*

This result shows that the restriction of functions from Ω_2 on Ω_1 is an onto mapping from the space $H^1(\Omega_2)$ to $H^1(\Omega_1)$, if Ω_1 is a bounded Lipschitz domain. We will also need the Rellich–Kondrachov compactness theorem that says the inclusion of $H^1(\Omega_1)$ in $L^2(\Omega_1)$ is compact, if Ω_1 is a bounded Lipschitz domain [11].

Lemma 3.2. *Let X, Y_1, and Y_2 be Banach spaces, $L_1 : X \to Y_1$ be a bounded linear operator, and $L_2 : X \to Y_2$ a linear compact operator such that $\|\boldsymbol{v}\|_X \simeq \|L_1\boldsymbol{v}\|_{Y_1} + \|L_2\boldsymbol{v}\|_{Y_2}$ $\forall\ \boldsymbol{v} \in X$. Then the dimension of $\ker L_1$ is finite; the mapping L_1 is an isomorphism from $X/\ker L_1$ to its range in Y_1; the range of L_1 is a closed subspace of Y_1.*

3.1. Neumann and Robin boundary conditions.

We first consider the Neumann problem. Let $\omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary Γ. (The domain could be multiply connected.) Let $f \in L^p(\omega)$ for some $p > 1$, we seek function u on ω solving the homogeneous Neumann problem

$$-\Delta u = f \text{ on } \omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma. \tag{3.1}$$

Here n is the unit outward normal to Γ. The weak formulation of the boundary value problem is as follows.

$$\begin{gathered}(\nabla u, \nabla v)_{[L^2(\omega)]^2} = \int_\omega f v dx \quad \forall\ v \in H^1(\omega), \\ u \in H^1(\omega).\end{gathered} \tag{3.2}$$

We assume that $\int_\omega f(x)dx = 0$. Then this is a well-posed problem in the quotient space $H^1(\omega)/\mathbb{R}$.

Let $R \subset \mathbb{R}^2$ be a larger Lipschitz domain such that $\omega \subset R$. Let $\Omega = R \backslash \overline{\omega}$ be the fictitious domain. We extend the function f to a function $\bar{f}$ on R by defining $\bar{f} = 0$ on Ω. For $\epsilon > 0$, the domain embedding method determines $u^\epsilon \in H_0^1(R)$ such that

$$(\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + \epsilon(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega)]^2} = \int_R \bar{f} v dx \ \forall \ v \in H_0^1(R). \tag{3.3}$$

This equation fits in the form of (2.2), in which we have $H = H_0^1(R)$, $U = [L^2(\Omega)]^2$, and $V = [L^2(\omega)]^2$. The operator are defined as follows. For a $v \in H_0^1(R)$, $Av = \nabla(v|_\Omega)$ and $Bv = \nabla(v|_\omega)$. They are obviously continuous. The subspace $\ker B$ is composed of functions that are constant on ω. Therefore, $\int_R \bar{f} v dx = 0 \ \forall \ v \in \ker B$. To use Lemma 2.1, we also need to verify that the range of B is closed in V. Since

$$\|v\|_{H^1(\omega)} \simeq \|v\|_{L^2(\omega)} + \|\nabla v\|_{[L^2(\omega)]^2} \ \ \forall \ \in H^1(\omega)$$

and ω is a Lipschitz domain, by the Rellich–Kondrachov compactness theorem and Lemma 3.2, we see that ∇, as a mapping from $H^1(\omega)$ to $[L^2(\omega)]^2$, has closed range. Let's take a larger domain $D \subset \mathbb{R}^2$ such that $R \subset\subset D$. According to Lemma 3.1, the restriction from D to ω is an onto mapping from $H_0^1(D)$ to $H^1(\omega)$. Thus the mapping defined by $\nabla(v|_\omega)$ $\forall \ v \in H_0^1(D)$ is a linear continuous operator from $H_0^1(D)$ to $[L^2(\omega)]^2$ that has closed range. The range closeness of B then follows from that fact that $H_0^1(R)$ is identified with the closed subspace of $H_0^1(D)$ of which functions vanish on $D \setminus R$. The equivalence of the form (2.1) is obvious, and the equivalent $\mathcal{H}$ norm on $H_0^1(R)$ is just the usual semi-norm. It is easy to see that if $\Gamma \cap \partial R = \emptyset$,

$$(\ker B)_{\mathcal{H}}^\perp = \{v \in H_0^1(R); \Delta v = 0 \text{ in } \Omega, \langle \frac{\partial v}{\partial n^-}, 1\rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)} = 0\}.$$

Here n^- is the outward normal to Γ as a boundary of Ω. If $\Gamma \cap \partial R \neq \emptyset$ then $(\ker B)_{\mathcal{H}}^\perp = \{v \in H_0^1(R); \Delta v = 0 \text{ in } \Omega\}$. All the conditions of Lemma 2.1 are satisfied. Therefore, as $\epsilon \to 0$, u^ϵ converges to a limit $u^0 \in (\ker B)_{\mathcal{H}}^\perp$. The limiting equation (2.4) reads

$$(\nabla u^0, \nabla v)_{[L^2(\omega)]^2} = \int_\omega f v dx \ \ \forall \ v \in H^1_{\Gamma \cap \partial R}(\omega). \tag{3.4}$$

By Lemma 2.1 we have

$$\|u^\epsilon - u^0\|_{H^1(R)} \simeq |u^\epsilon - u^0|_{H^1(\omega)} \simeq \epsilon \, |u^0|_{H^1(\Omega)}. \tag{3.5}$$

It is easy to see that $|u^0|_{H^1(\Omega)} = 0$ if and only if $u^0 = 0$ on Γ, in which case $u^\epsilon \equiv u^0$. However, if $\Gamma \cap \partial R \neq \emptyset$ then $u^0|_\omega$ does not solve the Neumann problem (3.2), since $u^0 = 0$ on $\Gamma \cap \partial R$. To avoid this, we only need to assume that $\omega \subset\subset R$. Then the restriction of u^0 on ω is a specific solution of the Neumann problem. In summary, we have

Theorem 3.3. *Under the assumption that $\omega \subset\subset R$ and both ω and R are Lipschitz domains, the solution u^ϵ of the domain embedding equation* (3.3) *converges to the limit $u^0 \in H_0^1(R)$ whose restriction on ω solves the Neumann problem* (3.2)*. The limit u^0 is harmonic on the fictitious domain Ω. The convergence has a sharp rate of ϵ, except in the spacial case that a solution of the Neumann problem* (3.2) *is constant on Γ, in which case $u^\epsilon \equiv u^0$.*

Remark. Had the Neumann condition been imposed on ∂R in the domain embedding equation (3.3), we would only need to require that $\omega \subset R$ rather than $\omega \subset\subset R$. Then all the condition of Lemma 2.1 could be verified and the convergence rate of the domain embedding method would not be changed. More importantly, we would still have $u^0|_\omega = u$.

Remark. The extension of f by zero to Ω is very important. Otherwise, the crucial condition $\boldsymbol{f}|_{\ker B} = 0$ of Lemma 2.1 would not hold, and the solution u^ϵ of the domain embedding equation would blow up to infinity in $H^1(R)$ at the rate of $\mathcal{O}(\epsilon^{-1})$.

For Robin boundary condition, the theory is similar to that of Neumann problem, so we will be very brief. We seek function u on ω solving the homogeneous Robin problem

$$-\Delta u = f \text{ on } \omega, \quad \frac{\partial u}{\partial n} + ku = 0 \text{ on } \Gamma. \tag{3.6}$$

Here k is a bounded and positively valued function on Γ. The weak formulation is

$$\begin{gathered}(\nabla u, \nabla v)_{[L^2(\omega)]^2} + \int_\Gamma kuvds = \int_\omega fvdx \ \ \forall\, v \in H^1(\omega), \\ u \in H^1(\omega).\end{gathered} \tag{3.7}$$

This problem is well-posed without the further requirement on f like that in the Neumann problem. For any $f \in L^p(\omega)$, u is uniquely determined in $H^1(\omega)$. The domain embedding method determines $u^\epsilon \in H^1_0(R)$ such that

$$(\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + \int_\Gamma kuvds + \epsilon(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega)]^2} = \int_R \bar{f}vdx \ \forall\, v \in H^1_0(R). \tag{3.8}$$

This equation also fits in the form of (2.2), in which we have $H = H^1_0(R)$, $U = [L^2(\Omega)]^2$, and $V = [L^2(\omega)]^2 \times L^2_k(\Gamma)$. Here $L^2_k(\Gamma)$ is a weighted L^2 space on Γ with an equivalent inner product defined by $(u, v)_{L^2_k(\Gamma)} = \int_\Gamma kuvds$. The operator A is the same as in the Neumann problem. And for a $v \in H^1_0(R)$, $Bv = (\nabla(v|_\omega), v|_\Gamma)$ in which the second component is the trace of v on Γ. The operator B is obviously continuous. We need to show that B has closed range in V. The argument is the same as before, except that now we use the equivalence

$$\|v\|_{H^1(\omega)} \simeq \|v\|_{L^2(\omega)} + \|\nabla v\|_{[L^2(\omega)]^2} + \|v\|_{L^2_k(\Gamma)} \ \ \forall \ \in H^1(\omega)$$

together with Lemma 3.1 and Lemma 3.2. We also have the equivalence of the form (2.1). The subspace $\ker B$ is composed of functions that are of zero value in $\bar{\omega}$. And we have $(\ker B)^\perp_{\mathcal{H}} = \{v \in H^1_0(R); \Delta v = 0 \text{ in } \Omega\}$. We obviously have $\int_R \bar{f}vdx = 0 \ \forall\, v \in \ker B$. Thus all the conditions of Lemma 2.1 are satisfied. Therefore, as $\epsilon \to 0$, u^ϵ converges to a limit $u^0 \in (\ker B)^\perp_{\mathcal{H}}$. The limiting equation (2.4) reads

$$(\nabla u^0, \nabla v)_{[L^2(\omega)]^2} + \int_\Gamma ku^0vds = \int_\omega fvdx \ \ \forall\, v \in H^1_{\Gamma\cap\partial R}(\omega). \tag{3.9}$$

So, when $\omega \subset\subset R$ we have $u^0|_\omega = u$, the solution of the Robin problem (3.7). By Lemma 2.1 we have

$$\|u^\epsilon - u^0\|_{H^1(R)} \simeq |u^\epsilon - u^0|_{H^1(\omega)} + \|u^\epsilon - u^0\|_{L^2(\Gamma)} \simeq \epsilon\, |u^0|_{H^1(\Omega)}. \tag{3.10}$$

It is easy to see that $u^\epsilon \equiv u^0$ if and only if $|u^0|_{H^1(\Omega)} = 0$ that is equivalent to $u^0 = 0$ on Γ. I.e., the solution of the Robin problem (3.7) also satisfies the homogeneous Dirichlet condition and homogeneous Neumann condition on Γ. We summarize this as

Theorem 3.4. *Under the assumption that* $\omega \subset\subset R$ *and both* ω *and* R *are Lipschitz domains, the solution* u^ϵ *of the domain embedding equation* (3.8) *converges to the limit* $u^0 \in H_0^1(R)$ *whose restriction on* ω *solves the Robin problem* (3.7)*. The limit* u^0 *is harmonic on the fictitious domain* Ω*. The convergence has a sharp rate of* ϵ*, except in the spacial case that the solution of the Robin problem* (3.2) *is zero on* Γ*, in which case* $u^\epsilon \equiv u^0$*.*

The same remarks following Theorem 3.3 hold for the Robin problem.

3.2. **Dirichlet boundary condition.** We seek a function u on ω solving the homogeneous Dirichlet problem

$$-\Delta u = f \text{ on } \omega, u = 0 \text{ on } \Gamma. \tag{3.11}$$

The weak formulation of the boundary value problem is

$$\begin{gathered}(\nabla u, \nabla v)_{[L^2(\omega)]^2} = \int_\omega f v dx\rangle \quad \forall\ v \in H_0^1(\omega),\\ u \in H_0^1(\omega).\end{gathered} \tag{3.12}$$

The domain embedding method determines $u^\epsilon \in H_0^1(R)$ such that

$$(\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + \epsilon^{-1}(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega)]^2} = \int_R \bar{f} v dx \quad \forall\ v \in H_0^1(R). \tag{3.13}$$

Remark. For the Dirichlet problem (3.12) to be well-posed, we only need $f \in H^{-1}(\omega)$. But for (3.13) to be well-posed, we need more regularity on f. According to the Sobolev embedding theorem, the condition $f \in L^p(\omega)$ for some $p > 1$ is sufficient.

The equation (3.13) fits in the form of (2.3), in which we have $H = H_0^1(R)$, $U = [L^2(\omega)]^2$, and $V = [L^2(\Omega)]^2$. For a $v \in H_0^1(R)$, we define $Av = \nabla(v|_\omega)$ and $Bv = \nabla(v|_\Omega)$. Note that roles of Ω and ω in defining A and B are reversed from that in Neumann problem. The operators are obviously continuous, and the equivalence (2.1) holds. The equivalent $\mathcal{H}$ norm on $H_0^1(R)$ is the usual semi-norm. To ensure the range closeness of the B operator, we now need to make an extra requirement: The fictitious domain Ω is a Lipschitz domain. This condition is not satisfied when, for example, a circular domain is tightly embedded in a square. But it is satisfied if both ω and R are polygonal such that $\Gamma \cap \partial R$ has no isolated point, or $\omega \subset\subset R$ and both ω and R are Lipschitz. Under this condition, the range closeness of the B operator then follows from the same reasoning we employed in the Neumann problem. The subspace $\ker B$ is composed of functions that are constant in Ω. If we assume that ω is simply connected, then Ω has no isolated component, and

$$\ker B = \{v \in H_0^1(R); v = 0 \text{ on } \Omega\},$$

which is identified with $H_0^1(\omega)$ in a obvious manner. And we have that $(\ker B)^\perp_{\mathcal{H}}$ is composed of functions that are harmonic in ω. All the conditions of Lemma 2.2 are satisfied. Therefore, as $\epsilon \to 0$, u^ϵ converges to a limit $u^0 \in \ker B$. The limiting equation (2.10) reads

$$(\nabla u^0, \nabla v)_{[L^2(\omega)]^2} = \int_\omega f v dx \quad \forall\ v \in H_0^1(\omega). \tag{3.14}$$

This is to say that the restriction of u^0 on ω is the solution of the Dirichlet problem (3.12). Since B has closed range, there is a unique $u^1 \in (\ker B)^{\perp}_{\mathcal{H}}$ such that

$$(\nabla u^1, \nabla v)_{[L^2(\Omega)]^2} = \int_\omega f v dx - (\nabla u^0, \nabla v)_{[L^2(\omega)]^2} \quad \forall\, v \in H^1_0(R). \tag{3.15}$$

This is just a specific form of the equation (2.11). By Lemma 2.2 we have

$$\|u^\epsilon - u^0\|_{H^1(R)} \simeq \|u^\epsilon\|_{H^1(\Omega)} \simeq \epsilon\, |u^1|_{H^1(\Omega)}. \tag{3.16}$$

From (3.15) we see that $|u^1|_{H^1(\Omega)} = 0$ if and only if u^0 satisfies the homogeneous Neumann condition on Γ, in which case $u^\epsilon \equiv u^0$. Otherwise, u^ϵ converges to u^0 at the sharp rate of ϵ. In summary, we have

Theorem 3.5. *Under the assumption that ω is simply connected, $\omega \subset R$, and $\Omega = R \setminus \overline{\omega}$ has Lipschitz boundary, the solution u^ϵ of the domain embedding equation* (3.13) *converges to the limit $u^0 \in H^1_0(R)$ that is identically equal to zero on Ω and whose restriction on ω solves the Dirichlet problem* (3.12). *The convergence has a sharp rate of ϵ, except in the spacial case that the solution of the Dirichlet problem* (3.12) *also satisfies the homogeneous Neumann boundary condition, in which case $u^\epsilon \equiv u^0$.*

Remark. If ω is not simply connected, then Ω could have isolated and simply connected components whose union is, say, Ω_0. In this case, $\ker B = \{v \in H^1_0(R); v = 0 \text{ on } \Omega \setminus \Omega_0, v = \text{constant on } \Omega_0\}$, which can not be identified with $H^1_0(\omega)$. As a consequence, u^0 does not satisfy the homogeneous Dirichlet condition on $\Gamma \cap \partial\Omega_0$, and the domain embedding method (3.13) fails. A remedy is adding, for example, the term $\epsilon^{-1}(u^\epsilon, v)_{L^2(\Omega_0)}$ or $\epsilon^{-1}(u^\epsilon, v)_{L^2(\Gamma)}$ (or something else as long as $\ker B$ can be identified with $H^1_0(\omega)$) to the left hand side of the domain embedding equation (3.13). Then the method would be convergent and the convergence rate would be retained.

Remark. In contrast to the Neumann and Robin problems, the extension of f by zero to Ω is not necessary for the Dirichlet problem. An arbitrary extension is allowed and the convergence rate of ϵ remains unchanged.

Remark. The range-closeness of the B operator is crucial to the convergence rate $\mathcal{O}(\epsilon)$ of the domain embedding method (3.13) or its modification for multiply connected domains discussed above. But such requirement is not necessary if one merely wants a convergence. For example, if ω is a circular domain and R is a square whose four sides are tangential to the circle, then Ω has sharp cusps, on which both the extension theorem and Rellich–Kondrachov compactness theorem fail, and the operator B has no closed range in V. In this case, it can be shown that we still have $\lim_{\epsilon \to 0} u^\epsilon = u^0$ and $u^0|_\omega = u$, so the domain embedding method can be justified. But the convergence rate would be dependent on the acuteness of cusps of the fictitious domain Ω [1].

Another example in which the B operator has no closed range in V is the following alternative of the domain embedding equation (3.13). The method seeks $u^\epsilon \in H^1_0(R)$ such that

$$(\nabla u^\epsilon, \nabla v)_{[L^2(R)]^2} + \epsilon^{-1}(u^\epsilon, v)_{L^2(\Omega)} = \int_R \bar{f}(x) v(x) dx \quad \forall\, v \in H^1_0(R). \tag{3.17}$$

This is a convergent domain embedding method. This equation can also be fitted into the equation (2.3), in which the $V = L^2(\Omega)$, and for $v \in H^1_0(R)$, $Bv = v|_\Omega$. The operator B obviously has no closed range in V. This method has the advantage that u^ϵ is smoother (but not uniformly smoother as $\epsilon \to 0$), and $\lim_{\epsilon\to 0} u^\epsilon = u^0$ with u^0 being the same. However, the convergence rate is much lower. Actually, we can show that $\epsilon^{\frac{1}{4}+\delta} \lesssim \|u^\epsilon - u^0\|_{H^1(R)} \lesssim \epsilon^{\frac{1}{4}}$ for any $\delta > 0$. To prove this, the framework in Section 2 is not useful. A proper theory can be found in [23]. As an illustration, we apply these methods to an ordinary differential equation: Finding $u \in H^1_0(-1, 0)$ such that $-D^2u = 1$ on $(-1, 0)$. We embed the interval $(-1, 0)$ in the larger interval $(-1, 1)$, extend the right hand side function by 1, and solve the following variants of the domain embedding methods (3.13) and (3.17) to get $u^\epsilon \in H^1_0(-1, 1)$.

$$(Du^\epsilon, Dv)_{L^2(-1,0)} + \epsilon^{-1}(Du^\epsilon, Dv)_{L^2(0,1)} = (1, v)_{L^2(-1,1)} \quad \forall\, v \in H^1_0(-1, 1). \tag{3.18}$$

$$(Du^\epsilon, Dv)_{L^2(-1,1)} + \epsilon^{-1}(u^\epsilon, v)_{L^2(0,1)} = (1, v)_{L^2(-1,1)} \quad \forall\, v \in H^1_0(-1, 1). \tag{3.19}$$

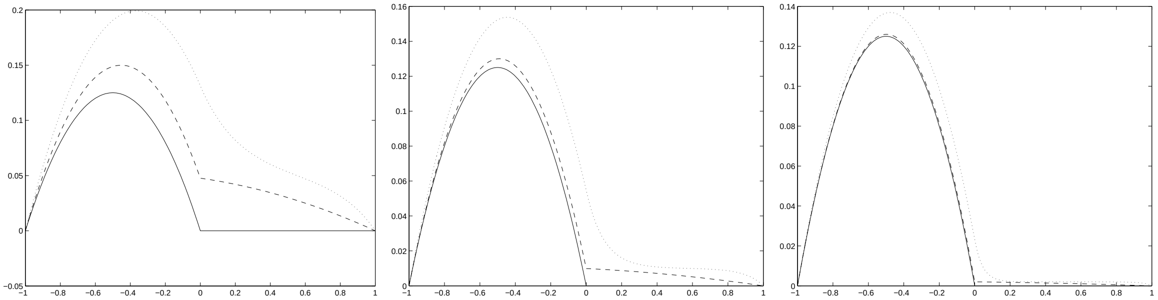

FIGURE 2. Solutions u^ϵ of the embedding methods (3.18) (dashed line), (3.19) (dotted line), and the limit u^0 (solid line), for $\epsilon = 0.05, 0.01, 0.002$ (from left to right).

The solutions of these equation are plotted in Figure 2. The method (3.18) obviously converges faster than the method (3.19). But the solution of the latter is smoother.

3.3. **Mixed boundary condition.** We seek a function u on ω such that

$$-\Delta u = f \quad \text{on} \quad \omega \tag{3.20}$$

subject to a mixed boundary condition on the boundary Γ. We assume that Γ is divided into three parts, such that $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_R$, and on Γ_D, Γ_N, and Γ_R, homogeneous Dirichlet, Neumann, and Robin boundary conditions are imposed, respectively. I.e.,

$$u = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N, \quad \frac{\partial u}{\partial n} + ku = 0 \text{ on } \Gamma_R, \tag{3.21}$$

here n is the unit outward normal to Γ and k is a bounded and positive valued function defined on Γ_R. The weak formulation of the boundary value problem (3.20) and (3.21) is

$$\begin{gathered}(\nabla u, \nabla v)_{[L^2(\omega)]^2} + (ku, v)_{L^2(\Gamma_R)} = \int_\omega fv dx \quad \forall\, v \in H^1_D(\omega), \\ u \in H^1_D(\omega).\end{gathered} \tag{3.22}$$

If $\Gamma_D \neq \emptyset$ or $\Gamma_R \neq \emptyset$, then (3.22) allows for a unique solution. If they both are empty, the problem is reduced to a pure Neumann problem that is resolved in Section 3.1 in which we need to assume $\int_\omega f dx = 0$.

Let $R \subset \mathbb{R}^2$ be a regular domain such that $\omega \subset R$. We extend the function f to a function $\bar{f}$ on R by defining $\bar{f} = 0$ on the fictitious domain $\Omega = R \setminus \overline{\omega}$. Corresponding to the division of Γ, we divide Ω into three parts such that $\overline{\Omega} = \overline{\Omega}_D \cup \overline{\Omega}_N \cup \overline{\Omega}_R$, and $\partial\Omega_D \cap \Gamma = \overline{\Gamma}_D$, $\partial\Omega_N \cap \Gamma = \overline{\Gamma}_N$, and $\partial\Omega_R \cap \Gamma = \overline{\Gamma}_R$, see the right figure in Figure 1. The domain embedding method determines $u^\epsilon \in H_0^1(R)$ such that

$$
\begin{aligned}
(3.23)\quad & (\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + (ku^\epsilon, v)_{L^2(\Gamma_R)} \\
& \qquad + \epsilon^{-1}(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega_D)]^2} + \epsilon(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega \setminus \overline{\Omega}_D)]^2} = \int_R \bar{f} v dx \quad \forall\ v \in H_0^1(R).
\end{aligned}
$$

This equation fits into the form of the equation (2.14) with

$$
H = H_0^1(R),\ U = [L^2(\Omega \setminus \overline{\Omega}_D)]^2,\ V = [L^2(\omega)]^2 \times L_k^2(\Gamma_R),\ W = [L^2(\Omega_D)]^2.
$$

Here $L_k^2(\Gamma_R)$ is the weighted L^2 space of functions on Γ_R in which the inner product is defined as $(u, v)_{L_k^2(\Gamma_R)} = (ku, v)_{L^2(\Gamma_R)}$, which is equivalent to the usual L^2 space. The operators are defined as follows. For $v \in H_0^1(R)$

$$
Au = \nabla(v|_{\Omega\setminus\overline{\Omega}_D}),\ Bv = \{\nabla(v|_\omega), v|_{\Gamma_R}\},\ Cv = \nabla(v|_{\Omega_D}).
$$

It is easy to see that A, B, and C are continuous operators from H to U, V, and W, respectively, and the equivalence $\|Av\|_U + \|Bv\|_V + \|Cv\|_W \simeq \|v\|_H\ \forall\ v \in H$ holds. Thus the problem (3.23) is well-posed in H, on which the equivalent $\mathcal{H}$ norm induced by these operators is given by

$$
\|v\|_{\mathcal{H}}^2 = |v|_{H^1(R)}^2 + \|v|_{\Gamma_R}\|_{L_k^2(\Gamma_R)}^2 \quad \forall\ v \in H_0^1(R).
$$

To apply Lemma 2.3, we need to verify that the operator C has closed range in W and the operator $B \times C$ has closed range in $V \times W$. Under the assumption that Ω_D is a Lipschitz domain, the range closeness of C can be argued in exactly the same way as we did for the B operator of Section 3.2. For the operator $B \times C$ to have a closed range in $V \times W$, we need to assume that the domain $\omega \cup \Gamma_D \cup \Omega_D$ is a Lipschitz domain as well. Then we observe that this operator, as a mapping from $H^1(\omega \cup \Gamma_D \cup \Omega_D)$ to $V \times W$, has closed range. This follows from Lemma 3.2, the Rellich–Kondrachov compactness theorem, and the equivalence

$$
\begin{aligned}
\|Bv\|_V + \|Cv\|_W + \|v\|_{L^2(\omega\cup\Gamma_D\cup\Omega_D)} &\simeq |v|_{H^1(\omega)} + \|v\|_{L^2(\Gamma_R)} + |v|_{H^1(\Omega_D)} + \|v\|_{L^2(\omega\cup\Gamma_D\cup\Omega_D)} \\
&\simeq \|v\|_{H^1(\omega\cup\Gamma_D\cup\Omega_D)} \quad \forall\ v \in H^1(\omega \cup \Gamma_D \cup \Omega_D).
\end{aligned}
$$

We choose a larger domain D such that $R \subset\subset D$. Then under the assumption that $\omega \cup \Gamma_D \cup \Omega_D$ has Lipschitz boundary, by using Lemma 3.1, we see that the restriction is an onto mapping from $H_0^1(D)$ to $H^1(\omega \cup \Gamma_D \cup \Omega_D)$. Thus $B \times C$, as a mapping from $H^1(D)$ to $V \times W$, has closed range. Since $H_0^1(R)$ is identified with the closed subspace of $H_0^1(D)$ by zero extension, $B \times C$ thus, as a mapping from $H_0^1(R)$ to $V \times W$, has closed range.

We need to identify the subspaces involved in Lemma 2.3. We assume that Ω_D has no isolated and simply connected component. Then we see that

$$\begin{aligned}
\ker B &= \{v \in H_0^1(R);\ v = 0,\ \text{or constant if } \Gamma_R = \emptyset \text{ and } \Gamma \cap \partial R = \emptyset,\ \text{on } \omega\},\\
\ker C &= \{v \in H_0^1(R);\ v = 0 \text{ on } \Omega_D\},\\
(\ker C)_{\mathcal{H}}^{\perp} &= \{v \in H_0^1(R);\ \Delta v = 0 \text{ on } R \setminus \overline{\Omega}_D\},\\
\ker B \cap \ker C &= \{v \in H_0^1(R);\ v = 0 \text{ on } \omega \cup \Gamma_D \cup \Omega_D\},\\
(\ker B \cap \ker C)_{\mathcal{H}}^{\perp} &= \{v \in H_0^1(R);\ \Delta v = 0 \text{ on } \Omega \setminus \overline{\Omega}_D\},\\
(\ker B \cap \ker C)_{\mathcal{H}}^{\perp} \cap \ker C &= \{v \in H_0^1(R);\ v = 0 \text{ on } \Omega_D,\ \Delta v = 0 \text{ on } \Omega \setminus \overline{\Omega}_D\}.
\end{aligned}$$

Obviously, $\int_R \bar{f} v dx = 0\ \forall\ v \in \ker B \cap \ker C$. This verifies the condition (2.15). All the conditions of Lemma 2.3 are thus verified. As $\epsilon \to 0$, the solution of the domain embedding equation u^ϵ converges to the limit $u^0 \in (\ker B \cap \ker C)_{\mathcal{H}}^{\perp} \cap \ker C$ at the rate of $\mathcal{O}(\epsilon)$. The limit is uniquely determined by

$$(\nabla u^0, \nabla v)_{[L^2(\omega)]^2} + (k u^0, v)_{L^2(\Gamma_R)} = \int_R \bar{f} v dx \quad \forall\ v \in \ker C. \tag{3.24}$$

The equation (2.18) now takes the form

$$(\nabla u_0^1, \nabla v)_{[L^2(\Omega_D)]^2} = \int_R \bar{f} v dx - [(\nabla u^0, \nabla v)_{[L^2(\omega)]^2} + (k u^0, v)_{L^2(\Gamma_R)}] \quad \forall\ v \in H_0^1(R). \tag{3.25}$$

The solution $u_0^1 \in H_0^1(R)$ is uniquely determined in $(\ker C)_{\mathcal{H}}^{\perp}$, i.e., it is harmonic on $R \setminus \overline{\Omega}_D$. By (2.19) we get

$$\begin{aligned}
\epsilon[|u_0^1|_{1,\Omega_D} + \kappa(|u_0^1|_{1,\Omega_D})|u^0|_{1,\Omega\setminus\overline{\Omega}_D}] &\lesssim \|u^\epsilon - u^0\|_{H^1(\omega)} + \|u^\epsilon\|_{H^1(\Omega_D)}\\
&\lesssim \|u^\epsilon - u^0\|_{H^1(R)} \lesssim \epsilon[|u^0|_{1,\Omega\setminus\overline{\Omega}_D} + |u_0^1|_{1,\Omega_D}]
\end{aligned} \tag{3.26}$$

We recall that $\kappa(0) = 1$ and $\kappa(x) = 0\ \forall\ x \neq 0$. We see that if $|u^0|_{1,\Omega\setminus\overline{\Omega}_D} = |u_0^1|_{1,\Omega_D} = 0$, then $u^\epsilon \equiv u^0$. Otherwise, u^ϵ converges to u^0 at the sharp rate of ϵ in the space $H_0^1(R)$. Since u^0 is harmonic on $\Omega \setminus \overline{\Omega}_D$, the condition $|u^0|_{1,\Omega\setminus\overline{\Omega}_D} = 0$ is equivalent to $u^0 = 0$ on $\Gamma_N \cup \Gamma_R$. From (3.25), we see that $|u_0^1|_{1,\Omega_D} = 0$ is equivalent to that u^0 also satisfies the homogeneous Neumann condition on $\Gamma_D \setminus \partial R$.

For the restriction of u^0 on ω to actually be the solution of the mixed boundary value problem (3.22) we need to impose a condition on the domain R. Namely, the homogeneous Dirichlet boundary condition on ∂R should not be enforced on Γ_N and Γ_R. This can be guaranteed by assuming that $\overline{\Gamma}_N \cup \overline{\Gamma}_R \subset R$. For the domain embedding method for the mixed boundary value problem, we thus have the following theorem.

Theorem 3.6. *Let $\omega \subset \mathbb{R}^2$ be a Lipschitz domain. For $f \in L^p(\omega)$ for some $p > 1$, we consider the Poisson equation $-\Delta u = f$ subject to a mixed boundary condition. Let the boundary Γ of ω be divided into three parts, such that $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_R$, and on Γ_D, Γ_N, and Γ_R, homogeneous Dirichlet, Neumann, and Robin boundary conditions are imposed, respectively. We assume $\Gamma_D \cup \Gamma_R \neq \emptyset$ such that the mixed boundary value problem is well posed.*

We embed ω in a larger and regular domain R, and let $\Omega = R \setminus \overline{\omega}$ be the fictitious domain. Corresponding to the splitting of Γ, we divide Ω into three parts such that $\overline{\Omega} = \overline{\Omega}_D \cup \overline{\Omega}_N \cup \overline{\Omega}_R$, and $\partial\Omega_D \cap \Gamma = \overline{\Gamma}_D$, $\partial\Omega_N \cap \Gamma = \overline{\Gamma}_N$, and $\partial\Omega_R \cap \Gamma = \overline{\Gamma}_R$. We assume that both Ω_D and

$\omega \cup \Gamma_D \cup \Omega_D$ are Lipschitz domains. Further more we assume that $\overline{\Gamma}_N \cup \overline{\Gamma}_R \subset R$ and Ω_D has no isolated and simply connected component. We extend the function f to a function $\bar{f}$ on R by defining $\bar{f} = 0$ on Ω. For $\epsilon > 0$, the embedding equation (3.23) has a unique solution $u^\epsilon \in H_0^1(R)$.

As $\epsilon \to 0$, u^ϵ converges to the limit $u^0 \in H_0^1(R)$. The restriction of u^0 on ω is the solution of the mixed boundary value problem (3.22). We have the sharp convergence rate $\|u^\epsilon - u^0\|_{H^1(R)} \simeq \epsilon$. More specifically, there exist positive constants C_1 and C_2 that may depend on ω, Ω_D, Ω_N, Ω_R, and f, but are independent of ϵ such that

$$C_1\,\epsilon \leq \|u^\epsilon - u^0\|_{H^1(\omega)} + \|u^\epsilon\|_{H^1(\Omega_D)} \leq \|u^\epsilon - u^0\|_{H^1(R)} \leq C_2\,\epsilon\,.$$

There is one exception in which $u^\epsilon \equiv u^0$ that occurs when and only when the solution u of the mixed boundary value problem (3.22) simultaneously satisfies homogeneous Dirichlet, Neumann, and thus also Robin conditions on the portion of the boundary $\Gamma \setminus \partial R$. The limit u^0 is zero on Ω_D, and it is harmonic on $\Omega \setminus \overline{\Omega}_D$.

Remark. The restriction $\overline{\Gamma}_N \cup \overline{\Gamma}_R \subset R$ is necessary because we have imposed homogeneous Dirichlet boundary condition on ∂R. If free condition is imposed on portions of ∂R that touch Γ_N or Γ_R, then this restriction can be removed.

Remark. If Ω_D has isolated and simply connected components, then $u^0|_\omega \neq u$. In this case, a term of the form $\epsilon^{-1} \int_{\Omega_0} u^\epsilon v dx$ can be added to the left hand side of (3.23) to correct the model. Here Ω_0 is the union of the isolated and simply connected components of Ω_D.

Remark. There is some freedom in the selection of Ω_D, Ω_N, and Ω_R. But to ensure the convergence rate given above, both Ω_D and $\omega \cup \Gamma_D \cup \Omega_D$ should be Lipschitz domains. These requirements ensure the range closeness of the C operator and $B \times C$ operator, which are crucial for the validity of Lemma 2.3. For more information on the dependences of C_1 and C_2 on ω, Ω_D, Ω_N, Ω_R, and f, one could consult Lemma 2.3.

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