SIAM J. NUMER. ANAL.
Vol. 46, No. 6, pp. 2843–2866
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ANALYSIS OF FINITE ELEMENT DOMAIN EMBEDDING METHODS FOR CURVED DOMAINS USING UNIFORM GRIDS*

SHENG ZHANG†

Abstract. We analyze the error of a finite element domain embedding method for elliptic equations on a domain ω with curved boundary. The domain is embedded in a rectangle R on which uniform mesh and linear continuous elements are employed. The numerical scheme is based on an extension of the differential equation from ω to R by regularization with a small parameter ϵ (for Neumann and Robin problems), or penalty with a large parameter ϵ^{-1} (for the Dirichlet problem), or a mixture of these (for a mixed boundary value problem). For Neumann and Robin problems, we prove that when $\epsilon \leq h$ (the mesh size), the error in the $H^1(\omega)$ norm is of the optimal order $\mathcal{O}(h)$. For the Dirichlet problem, when $\epsilon \leq h^{1/2}$, the error is $\mathcal{O}(h^{1/2})$ that is not optimal. If the mesh is adjusted around $\partial\omega$ to fit it, then the optimal convergence rate $\mathcal{O}(h)$ holds for the Dirichlet problem if ω is convex and $\epsilon \leq h$. If ω is not convex, then the convergence rate can only be improved to $\mathcal{O}(h^{2/3})$ by such mesh adjustment, with the parameter being $\epsilon = h^{2/3}$. In this latter case, a parameter smaller than $h^{2/3}$ thwarts the convergence rate, which is verified by a numerical result.

Key words. domain embedding, fictitious domain, curved boundary, uniform mesh, fast Poisson solver

AMS subject classifications. 65N30, 65N45, 65T50

DOI. 10.1137/060671681

1. Introduction.

In recent years, domain embedding, or fictitious domain, methods have developed into a general methodology of the numerical computations of partial differential equations that has the advantage of efficiently dealing with complex domain boundaries by using uniform, or slightly adjusted, grids on a larger rectangular domain [11]. For a second order elliptic boundary value problem on a complicated domain, the ultimate goal of domain embedding methods is to quickly obtain an accurate numerical solution by mainly solving the Poisson equation on uniform meshes on a rectangle by fast solvers, through, say, preconditioned iterations. A first step in these methods is to extend the boundary value problem to a larger rectangle. There are many ways to do this, one of which is using regularization for Neumann or Robin problem, or penalty for Dirichlet problem, or a combination of these for a mixed boundary value problem [14, 15, 17, 23]. A more general approach, including some additional treatments on the original domain boundary, can be found in [2]. This step results in an extended boundary value problem on the rectangle with large jumps in the coefficients of the differential equation. A finite element domain embedding method then is obtained as a straightforward discretization of the extended equation on a uniform mesh on the rectangle. The error of the finite domain embedding method has two sources. One is due to the replacement of the original problem by the one on the larger rectangular domain. This has been analyzed by many authors, and in [23, 24] the sharp estimates on the regularization/penalty error can be found. The other is due to the discretization of the extended equation, which seems not to have been completely understood. In the literature, there are numerous

*Received by the editors October 9, 2006; accepted for publication (in revised form) April 7, 2008; published electronically August 1, 2008. This work was partially supported by NSF grant DMS-0513559.
http://www.siam.org/journals/sinum/46-6/67168.html
†Department of Mathematics, Wayne State University, Detroit, MI 48202 (sheng@math.wayne.edu).

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works devoted to this subject. In [17], a thorough analysis is presented under the assumption that the original domain boundary aligns with the uniform mesh. This assumption seems to be a stringent restriction on the arbitrariness of the domain. For example, the domain boundary cannot be smoothly curved. The regularity of the solution and the alignment of the domain boundary with the uniform mesh required in this analysis are often not compatible. Furthermore, the results of [17] are actually not valid for general curved domains for Dirichlet problems. In [14, 15], an analysis for the Neumann problem can be found, which did not impose such restriction on the domain. But the error estimates established there are not sharp enough. A difficulty in this kind of error analysis for curved and complex domains is that the solution of the extended equation is not globally smooth. It has jumps in its normal derivative across the original domain boundary that inevitably cuts through some element if the mesh is uniform. In this paper, we present an analysis on such discretization errors for various boundary value problems. We first establish the estimates under the mere assumption that the original domain is Lipschitz. We then assume the domain boundary is smoothly curved, and we establish sharp estimates and use the aforementioned regularization/penalty error estimate to determine the balance between the mesh size and the regularization/penalty parameter such that the overall accuracy of the finite element domain embedding method is optimized [20, 19]. We shall see that the balance is rather delicate in some cases.

While the primary objective of this paper is analyzing the accuracy of the finite element domain embedding methods, there is, however, an important question that must be addressed before the methods can be claimed competitive. A domain embedding finite element method results in a discrete system that involves large or small parameters. The question is whether the fast Poisson solver can effectively precondition such a system so that a preconditioned iterative method could converge uniformly with respect to both the finite element mesh sizes and the large/small parameters. While the uniformity of such convergence with respect to the mesh sizes is obvious, the uniformity with respect to the parameters is by no means trivial. We shall not go into details with this issue, but refer to [5] for a strategy to resolve this. This strategy suggests to start the preconditioned iteration with some special initial vectors such that the whole process of the iteration is confined in a subspace in which the parameters have no effect. In our case, we need to choose initial vectors that are zero on the finite element nodes outside the original domain.

In view of the preconditioning, a slight distortion of the uniform mesh is acceptable, as long as the discrete Laplacian on the distorted mesh is spectrally equivalent to that on the uniform mesh. It turns out that no such adjustment is needed for Neumann or Robin problems, since the full accuracy is already achieved by uniform meshes. But for the Dirichlet problem, due to a locking phenomena, the accuracy offered by uniform meshes is rather poor. A slight adjustment of the uniform mesh around the original domain boundary by moving the nearby nodes onto the boundary efficiently reduces the locking effect and renders the full accuracy to the domain embedding finite element method when the domain is convex, or significantly improves the accuracy when the domain is not convex around the concave portion of accuracy.

For simplicity, we only present the results for the Poisson equation defined on a two-dimensional domain and for linear continuous finite elements. The results of this paper are, however, valid for general self-adjoint second order elliptic equations. Similar results can be derived for finite elements of higher orders. The theory is also possible to extend to some higher order equations like those of clamped and free (but not simply supported) plates. In the remainder of this introduction, we briefly

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summarize our results and compare them with existing results in the literature.

Let $\omega \subset \mathbb{R}^2$ be a bounded connected domain such that its boundary Γ is of C^2 class [13]. For a given function $f \in L^2(\omega)$, we consider the boundary value problem

$$-\Delta u = f \quad \text{in} \quad \omega$$

subject to the homogeneous Dirichlet boundary condition $u = 0$ on Γ; or the homogeneous Neumann condition $\partial u/\partial n = 0$, with n being the unit outward normal of Γ; or a Robin condition $ku + \partial u/\partial n = 0$, with k being a smooth, bounded, and a strictly positive function on Γ; or a mixed boundary condition. The homogeneity is not a real restriction for Neumann and Robin conditions. A nonhomogeneous Dirichlet condition could be reduced to a homogeneous one by defining a function satisfying the boundary condition and subtracting it from u.

The finite element domain embedding method is based on an approximation to u obtained by solving an ϵ-dependent boundary value problem on a larger rectangular domain $R \subset \mathbb{R}^2$ such that $\omega \subset R$. The formulation of the problem on R depends on the boundary condition in the original problem. The finite element domain embedding method is then a straightforward discretization of the extended one on R with mesh size h. We shall assume that $\omega \subset\subset R$. Then the boundary condition on ∂R is at one's disposal. We impose the homogeneous Dirichlet condition. However, Neumann, Robin, mixed, or a periodical boundary condition on ∂R works equally well. (If $\partial R \cap \Gamma \neq \emptyset$, the condition on ∂R must be subject to some restrictions [24].) We let $\Omega = R \setminus \overline{\omega}$ be the fictitious domain, and we extend the function f to a function $\bar{f}$ on R by defining $\bar{f} = 0$ on Ω. Henceforth, we shall use the notation $\mathcal{P} \lesssim \mathcal{Q}$, which means that there exists a constant C independent of $\mathcal{P}$, $\mathcal{Q}$, ϵ, and h such that $\mathcal{P} \leq C\mathcal{Q}$. The notation $\mathcal{P} \simeq \mathcal{Q}$ means $\mathcal{P} \lesssim \mathcal{Q}$ and $\mathcal{Q} \lesssim \mathcal{P}$. We use $H^s(D)$ to denote the L^2-based Sobolev space of order s on a domain D, in which the norm and semi-norm are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. In particular, the L^2 norm is denoted by $\|\cdot\|_{0,D}$ or $|\cdot|_{0,D}$.

We start with the Neumann problem of which the weak formulation seeks $u \in H^1(\omega)/\mathbb{R}$ such that

$$(1.1) \qquad (\nabla u, \nabla v)_{[L^2(\omega)]^2} = \int_\omega f v dx \quad \forall\, v \in H^1(\omega).$$

Here ∇ is the gradient operator, and the parentheses stand for the inner products in Hilbert spaces indicated by the subscripts. This problem is well posed under the assumption that $\int_\omega f dx = 0$. For $\epsilon > 0$, we determine that $u^\epsilon \in H_0^1(R)$ by a regularization such that

$$(1.2) \qquad (\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + \epsilon(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega)]^2} = \int_R \bar{f} v dx \quad \forall\, v \in H_0^1(R).$$

This is an elliptic problem that yields a unique $u^\epsilon \in H_0^1(R)$. As $\epsilon \to 0$, u^ϵ converges to a limit $u^0 \in H_0^1(R)$ at the sharp rate of ϵ as long as ω is a Lipschitz domain (which is assured by our assumption on Γ). The limit u^0 is harmonic on Ω, and $u^0|_\omega$ solves the Neumann problem (1.1). The equivalent estimate [24]

$$\|u^\epsilon - u^0\|_{1,R} \simeq |u^\epsilon - u^0|_{1,\omega} \simeq \epsilon$$

holds with the rare exception that $u^\epsilon \equiv u^0$, which occurs if and only if $u^0 = 0$ on Γ, i.e., a solution of the homogeneous Neumann problem also satisfies the homogeneous

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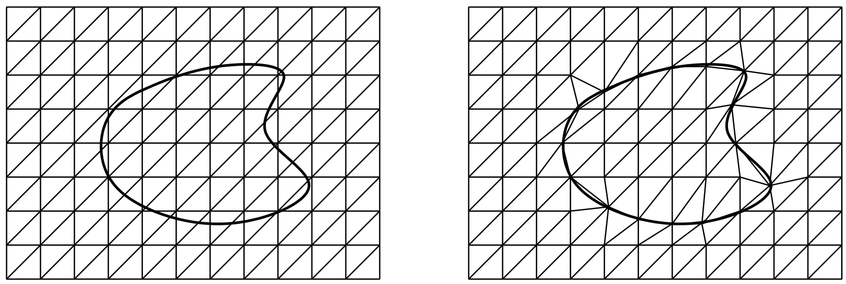

FIG. 1. *A domain ω embedded in a rectangle R with a uniform triangulation (left), and a triangulation adjusted around Γ (right).*

Dirichlet condition on Γ. Incidentally, this is also equivalent to $u^0 \in H^2(R)$. In this case, we can take $\epsilon = 1$ in (1.2), and the Poisson equation on R produces the exact solution of the original equation on ω. Except for this special case, we do not have $u^0 \in H^2(R)$, since there is a jump in its normal derivative across Γ.

We introduce a uniform triangulation of mesh size h on R; see Figure 1 (left), and let $H_h \subset H_0^1(R)$ be the subspace of piecewise linear continuous functions subordinate to this triangulation. The finite element domain embedding method is a straightforward discretization of (1.2). It determines $u_h^\epsilon \in H_h$ such that

$$(\nabla u_h^\epsilon, \nabla v_h)_{[L^2(\omega)]^2} + \epsilon(\nabla u_h^\epsilon, \nabla v_h)_{[L^2(\Omega)]^2} = \int_R \bar{f} v_h dx \quad \forall\, v \in H_h. \tag{1.3}$$

Then the restriction of u_h^ϵ on ω is the numerical solution offered by the finite domain embedding method. We prove that

$$|u_h^\epsilon - u^0|_{1,\omega} \lesssim \epsilon + \inf_{v_h \in H_h} \left(|u^0 - v_h|_{1,\omega} + \sqrt{\epsilon}|u^0 - v_h|_{1,\Omega} \right).$$

This estimate is valid as long as ω is a Lipschitz domain. When $\Gamma \in C^2$, we have the piecewise smoothness property of u^0 such that $u^0|_\omega \in H^2(\omega)$ and $u^0|_\Omega \in H^2(\Omega)$. Using this regularity, we construct an interpolation v_h of u^0 such that $|u^0 - v_h|_{1,\omega} \lesssim h$ and $|u^0 - v_h|_{1,\Omega} \lesssim \sqrt{h}$; thus to prove that

$$|u_h^\epsilon - u^0|_{1,\omega} \lesssim \epsilon + h + \sqrt{\epsilon\, h}. \tag{1.4}$$

From this we see that when $\epsilon \lesssim h$, we have a domain embedding finite element method that achieves the full accuracy of linear continuous finite elements with an error of $\mathcal{O}(h)$. This result is better than that of [14, 15], in which an estimate of the form $\sqrt{\epsilon} + h + \sqrt{\epsilon}$, instead of that in (1.4), was established. They, therefore, require $\epsilon = Ch^2$ to achieve the full accuracy. Taking smaller ϵ does not hurt the accuracy of the finite element domain embedding methods. From a computational point of view, one would like to avoid very small ϵ. In this sense, $\epsilon = Ch$ seems the best choice.

The piecewise smoothness of u^0 is a consequence of the assumption that $\Gamma \in C^2$ and $f \in L^2(\omega)$ [13]. It is not valid when ω is a polygon, since either $u^0|_\omega \in H^2(\omega)$ or $u^0|_\Omega \in H^2(\Omega)$ will be broken, depending on the convexity of ω. However, when ω is convex and Γ aligns with the mesh line, as assumed in [17], the result (1.4) remains valid, which needs a different argument.

If we fix h and let $\epsilon \to 0$, we have $u_h^\epsilon \to u_h^0 \in H_h$. The restriction of the limit u_h^0 on ω is identical to the solution given by a method proposed by Friedrichs and Keller

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in 1966 [12], of which an analogue for the Robin problem (see below) was used in the 1980s by the Boeing program `tranair` to circumvent the mesh generation problem around a complete aircraft and was also included in the software `freefem3d` [11]. The positive value of ϵ renders to the equation (1.3) the advantages of better data structure and faster resolution by using fast Poisson solvers on the rectangle R as preconditioners. This method also offers an alternative of dealing with the singularity of a discretized Neumann problem. See [6], where a survey of such various techniques can be found.

The weak formulation of the Robin problem seeks $u \in H^1(\omega)$ such that

$$(\nabla u, \nabla v)_{[L^2(\omega)]^2} + (ku, v)_{L^2(\Gamma)} = \int_\omega fv dx \quad \forall\, v \in H^1(\omega). \tag{1.5}$$

This equation can be extended to R in the following way. We determine $u^\epsilon \in H_0^1(R)$ such that

$$(\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + (ku^\epsilon, v)_{L^2(\Gamma)} + \epsilon(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega)]^2} = \int_R \bar{f} v dx \quad \forall\, v \in H_0^1(R). \tag{1.6}$$

This, once again, is a well-defined problem on R, and $\lim_{\epsilon\to 0} u^\epsilon = u^0 \in H_0^1(R)$. The limit u^0 is harmonic on Ω, and $u^0|_\omega$ solves the Robin problem (1.5). The equivalent estimate

$$\|u^\epsilon - u^0\|_{1,R} \simeq \|u^\epsilon - u^0\|_{1,\omega} + \|u^\epsilon - u^0\|_{0,\Gamma} \simeq \epsilon \tag{1.7}$$

holds as long as ω is Lipschitz [24]. There is an exception to the equivalent estimate (1.7) in which $u^\epsilon \equiv u^0$. This happens if and only if $u^0 = 0$ on Γ. This is the only case in which $u^0 \in H^2(R)$. Otherwise, u^0 has only the piecewise smoothness as for the Neumann problem. The finite element domain embedding method that determines $u_h^\epsilon \in H_h$ is a straightforward discretization of (1.6) on a uniform mesh. Using the very same argument as for Neumann problem, we obtain that

$$\|u_h^\epsilon - u^0\|_{1,\omega} \lesssim \epsilon + h + \sqrt{\epsilon\, h}.$$

The weak formulation of the Dirichlet problem seeks $u \in H_0^1(\omega)$ such that

$$(\nabla u, \nabla v)_{[L^2(\omega)]^2} = \int_\omega fv dx \quad \forall\, v \in H_0^1(\omega).$$

If ω is simply connected, then, on the continuous level, the domain embedding method determines $u^\epsilon \in H_0^1(R)$ such that

$$(\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + \epsilon^{-1}(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega)]^2} = \int_R \bar{f} v dx \quad \forall\, v \in H_0^1(R). \tag{1.8}$$

We have that $\lim_{\epsilon\to 0} u^\epsilon = u^0 \in H_0^1(R)$, that the limit is characterized by $u^0|_\omega = u$ and $u^0|_\Omega = 0$, and that the equivalent estimate

$$\|u^\epsilon - u^0\|_{1,R} \simeq \|u^\epsilon\|_{1,\Omega} \simeq \epsilon \tag{1.9}$$

holds, under the assumption that Ω is Lipschitz [24]. (This is true since we have assumed $\Gamma \in C^2$ and $\omega \subset\subset R$; it is also true when, for example, a L-shaped polygon is tightly embedded in a rectangle. It, however, is not true when a circular domain

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is tightly embedded in a square.) Furthermore, if we let $u^1 \in H_0^1(R)$ be the unique function determined by $\Delta u^1 = 0$ in ω, and let

$$(\nabla u^1, \nabla v)_{[L^2(\Omega)]^2} = \int_\omega f v dx - (\nabla u^0, \nabla v)_{[L^2(\omega)]^2} \quad \forall\ v \in H_0^1(R). \tag{1.10}$$

Then we have

$$\|u^\epsilon - u^0 - \epsilon\, u^1\|_{1,R} \simeq \epsilon^2\,. \tag{1.11}$$

An exception to (1.9) and (1.11) is that $u^\epsilon \equiv u^0$ and $u^1 = 0$. This occurs if and only if u also satisfies the homogeneous Neumann condition on Γ. This is equivalent to, say, $u^0 \in H^2(R)$. Except for this special case, we do not have $u^0 \in H^2(R)$, since the normal derivative of u^0 is not continuous across Γ.

The finite element domain embedding method determines $u_h^\epsilon \in H_h$ such that

$$(\nabla u_h^\epsilon, \nabla v_h)_{[L^2(\omega)]^2} + \epsilon^{-1}(\nabla u_h^\epsilon, \nabla v_h)_{[L^2(\Omega)]^2} = \int_R \bar{f} v_h dx \quad \forall\ v \in H_h. \tag{1.12}$$

We prove that

$$\|u_h^\epsilon - u\|_{1,\omega} \lesssim \epsilon + \inf_{v_h^1 \in H_h} \left(\epsilon\, \|u^1 - v_h^1\|_{1,\omega} + \sqrt{\epsilon}\|u^1 - v_h^1\|_{1,\Omega}\right) + \inf_{v_h^0 \in H_h} \left(\|u^0 - v_h^0\|_{1,\omega} + \frac{1}{\sqrt{\epsilon}}\|v_h^0\|_{1,\Omega}\right). \tag{1.13}$$

We construct a v_h^1 to interpolate u^1 in the same way as interpolating u^0 in a Neumann problem and bound the above second term by $\epsilon\sqrt{h} + h\sqrt{\epsilon}$. Our assumption of $\Gamma \in C^2$ and $f \in L^2(\omega)$ assures us that $u = u^0|_\omega \in H^2(\omega) \cap H_0^1(\omega)$. We construct an interpolation v_h^0 in such a way that it is zero on any open triangular element that is not entirely contained in ω. This makes $\|v_h^0\|_{1,\Omega} = 0$ and $\|u^0 - v_h^0\|_{1,\omega} \lesssim \sqrt{h}$. The above third term is then bounded by $\sqrt{h}$, which is a sharp estimate. We thus have

$$\|u_h^\epsilon - u\|_{1,\omega} \lesssim \epsilon + \epsilon\sqrt{h} + h\sqrt{\epsilon} + \sqrt{h}. \tag{1.14}$$

So the accuracy of the finite element domain embedding method is limited by $\mathcal{O}(\sqrt{h})$, which is assured when $\epsilon \lesssim \sqrt{h}$. The rather low order of accuracy is a kind of locking phenomena that is not unusual in the numerical computation of equations involving large parameters, like the Reissner–Mindlin plate and nearly incompressible elasticity [3].

The accuracy of the finite element domain embedding method for Dirichlet problem can be significantly improved by a slight adjustment of the nodes around Γ. One can move the nearby nodes onto Γ and reconnect some of the affected nodes such that no mesh-line segment has one end in ω and the other in Ω. This way, a polygonal interpolation of Γ will be formed in the mesh; see Figure 1 (right). This can be done by using Börger's algorithm [7] that results in a triangulation that is quasi uniform and shape regular, on which the discrete Laplacian is spectrally equivalent to that on the uniform mesh. On the adjusted triangulation, we then construct v_h^0 by simply interpolating u^0 at all of the nodes. When ω is convex, we have $\|u^0 - v_h^0\|_{1,\omega} \lesssim h$; see [10], and $\|v_h^0\|_{1,\Omega} = 0$ in (1.13). Thus, to obtain the estimate

$$\|u_h^\epsilon - u\|_{1,\omega} \lesssim \epsilon + \epsilon\sqrt{h} + h\sqrt{\epsilon} + h. \tag{1.15}$$

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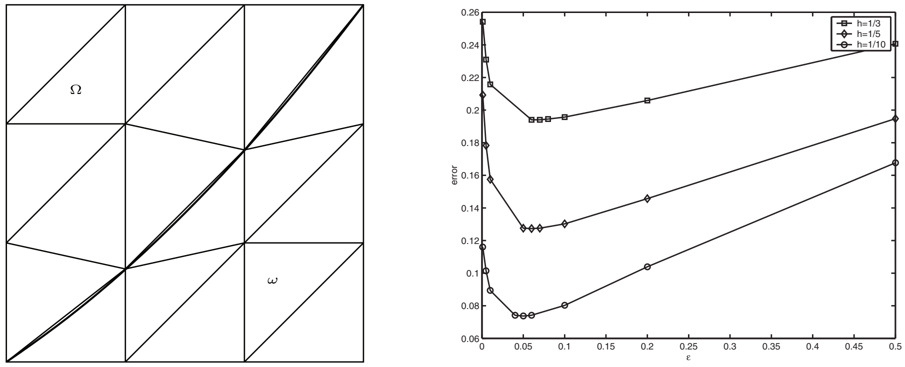

FIG. 2. *A nonconvex ω embedded in the unit square with adjusted mesh around its boundary (left), and the dependence of the error on ϵ and h (right).*

So when $\epsilon \lesssim h$ the error of the finite element domain embedding method in the $H^1(\omega)$ norm is of the optimal order $\mathcal{O}(h)$. If ω is not convex, we still have $\|u^0 - v_h^0\|_{1,\omega} \lesssim h$. However, $\|v_h^0\|_{1,\Omega}$ is no longer zero, but it is bounded by $\mathcal{O}(h)$. The error estimate becomes

$$\|u_h^\epsilon - u\|_{1,\omega} \lesssim \epsilon + \epsilon\sqrt{h} + h\sqrt{\epsilon} + h + \frac{h}{\sqrt{\epsilon}}. \tag{1.16}$$

This estimate is also sharp. The dependence of the error of a finite element domain embedding method on h and ϵ is rather complicated. From the above estimate we see that if we take $\epsilon = Ch^{2/3}$, the domain embedding finite element method has the error estimate $\mathcal{O}(h^{2/3})$ in $H^1(\omega)$ norm. It is interesting to note that, in this case, a smaller ϵ thwarts the convergence rate. This should be compared with the Neumann problem and the Dirichlet problem on convex domains, in which smaller ϵ does not improve but does not hurt the convergence rate of the finite element domain embedding method either. To confirm this observation, we include a numerical example here. We choose R to be the unit square $(0,1)\times(0,1)$ on the xy-plane, in which ω is the curvilinear triangle under the curved diagonal $y = x(x+2)/3$, which is not convex. The exact solution of the Dirichlet problem with a homogeneous boundary condition is $u(x,y) = [3y - x(x+2)]y(x-1)$. We first divide R by a uniform mesh of mesh size h, and then we move some nodes near the curved boundary of ω onto it to form the adjusted mesh; see Figure 2 (left), in which $h = 1/3$. On this mesh, we perform the computation with the method (1.12). Figure 2 (right) shows the dependence of $|u - u_h^\epsilon|_{1,\omega}$ on ϵ for $h = 1/3$, $1/5$, and $1/10$, respectively. It clearly shows that there is an optimal value of ϵ for each h, which decreases with h, and smaller ϵ thwarts the accuracy.

In conclusion, a slight adjustment of the finite element mesh to accommodate Γ appears to be worthwhile for the Dirichlet problem, which enhances the convergence rate from $\sqrt{h}$ to h if ω is convex, and from $\sqrt{h}$ to $h^{2/3}$ if ω is not convex.

It is important to note that the discrete system (1.12) based on such adjusted mesh can be efficiently preconditioned by using the discrete Laplacian on the uniform mesh with the preconditioned iteration initiated with a vector of zero entries corresponding to nodes in $R \setminus \omega$ [7, 5]. If one wishes to achieve full accuracy for nonconvex ω, then the mesh could be refined around the concave portion of Γ. A local mesh size $\mathcal{O}(h^{1.5})$ would lead to the full accuracy of the order $\mathcal{O}(h)$ of the domain embedding finite

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element method in which $\epsilon = Ch$. This refinement is needed only around the concave portion of the boundary of ω. See [18] for such an example.

If ω is polygonal and Γ aligns with the finite element mesh, then if $u^0|_\omega \in H^2(\omega)$ (which is true when ω is a convex polygon, but is only an assumption otherwise), we construct v_h^0 and v_h^1 by nodal interpolation to u^0 and u^1 in (1.13) to obtain the estimate

$$\|u_h^\epsilon - u\|_{1,\omega} \lesssim \epsilon + \epsilon\sqrt{h} + \sqrt{\epsilon\, h} + h. \tag{1.17}$$

Here we have used the fact that $u^1|_\omega \in H^{1.5}(\omega)$ and $u^1|_\Omega \in H^{1.5}(\Omega)$ [16]. Thus when $\epsilon \lesssim h$, one has a domain embedding finite element method that has the optimal accuracy of $\mathcal{O}(h)$. This was mentioned in [17], where a rigorous analysis was presented for the Neumann problem under this alignment assumption.

If ω is not simply connected, a term like $\epsilon^{-1}(u^\epsilon, v)_{L^2(\Omega)}$ needs to be added to the left-hand side of (1.8). Then the estimate (1.9) holds, and $u^0|_\omega = u$ regardless of the connectivity of ω. Actually, in case that ω is multiply connected, adding a term of the form $\epsilon^{-1}(u^\epsilon, v)_{L^2(\Omega^0)}$ in the left-hand side of (1.8) is sufficient to guarantee the validity of the domain embedding method on the continuous level. Here Ω^0 is the union of the isolated components of Ω. The finite element domain embedding methods should then be based on such a modification of (1.8).

Finally, we make some remarks on the mixed boundary value problem. To make the presentation sufficiently general, we assume that Γ is split into three parts such that $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_R$, and on Γ_D, Γ_N, and Γ_R, homogeneous Dirichlet, Neumann, and Robin conditions are imposed, respectively. Corresponding to this splitting, we divide Ω into three parts such that $\overline{\Omega} = \overline{\Omega}_D \cup \overline{\Omega}_N \cup \overline{\Omega}_R$, and $\partial\Omega_D \cap \Gamma = \overline{\Gamma}_D$, $\partial\Omega_N \cap \Gamma = \overline{\Gamma}_N$, and $\partial\Omega_R \cap \Gamma = \overline{\Gamma}_R$. The weak formulation of the mixed problem determines $u \in H_D^1(\omega)$ (that is, the space of $H^1(\omega)$ functions that vanish on Γ_D) such that

$$(\nabla u, \nabla v)_{[L^2(\omega)]^2} + (ku, v)_{L^2(\Gamma_R)} = \int_\omega f v dx \quad \forall\, v \in H_D^1(\omega). \tag{1.18}$$

On the continuous level, the domain embedding method determines that $u^\epsilon \in H_0^1(R)$ such that

$$\begin{aligned}(\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} &+ (ku^\epsilon, v)_{L^2(\Gamma_R)} \\ &+ \epsilon^{-1}(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega_D)]^2} + \epsilon(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega\setminus\overline{\Omega}_D)]^2} = \int_R \bar{f} v dx \quad \forall\, v \in H_0^1(R).\end{aligned} \tag{1.19}$$

When $\epsilon \to 0$, u^ϵ converges to a limit $u^0 \in H_0^1(R)$. The limit satisfies (1) $u^0|_\omega$ solves (1.18) (assuming ω is simply connected), (2) $u^0 = 0$ on Ω_D, and (3) u^0 is harmonic on Ω_N and Ω_R. Under the assumption that *both* Ω_D *and* $\omega \cup \Gamma_D \cup \Omega_D$ *are Lipschitz domains*, the equivalent estimate

$$\|u^\epsilon - u^0\|_{1,R} \simeq \|u^\epsilon - u^0\|_{1,\omega} + \|u^\epsilon\|_{1,\Omega_D} \simeq \epsilon \tag{1.20}$$

holds, with the exception that $u^\epsilon \equiv u^0$, which occurs when and only when $u^0 \in H^2(R)$ [23]. The finite element domain embedding method then is again a straightforward discretization of (1.18) that determines $u_h^\epsilon \in H_h$ such that

$$\begin{aligned}(\nabla u_h^\epsilon, \nabla v_h)_{[L^2(\omega)]^2} &+ (ku_h^\epsilon, v_h)_{L^2(\Gamma_R)} \\ &+ \epsilon^{-1}(\nabla u_h^\epsilon, \nabla v_h)_{[L^2(\Omega_D)]^2} + \epsilon(\nabla u_h^\epsilon, \nabla v_h)_{[L^2(\Omega\setminus\overline{\Omega}_D)]^2} = \int_R \bar{f} v_h dx \quad \forall\, v_h \in H_h.\end{aligned} \tag{1.21}$$

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The convergence rate estimates on $\|u^0 - u_h^\epsilon\|_{1,\omega}$ can be obtained under some regularity assumption on the solution of the original problem (1.18), which is not guaranteed by our assumptions on Γ and f. For uniform meshes, Γ_D generally cuts through some elements, and the ends of Γ_D lie in the interior of some elements; the rate is only bounded by $\sqrt{h}$. By moving nearby nodes onto Γ_D and its ends, and reconnecting some mesh lines, as for the Dirichlet problem, we can improve the convergence rate to $\mathcal{O}(h)$ or $\mathcal{O}(h^{2/3})$, depending on the convexity of Γ_D with respect to ω. As for Neumann and Robin problems, no treatment is needed to accommodate Γ_N or Γ_R.

The paper is organized as follows. In section 2, as a preparation, we derive some estimates on the finite element interpolation error on a strip around Γ that cuts through some elements. We also include a result about the ϵ-dependent variational problem, which is a cornerstone in estimating the regularization/penalty error. This result is also needed in the discretization error estimate for the Dirichlet problem. In section 3, we estimate the error of the finite element domain embedding methods for Neumann and Dirichlet problems, and we determine the balance between the value of the regularization/penalty parameter and the finite element mesh size. Since no new technique is needed for the Robin problem and the mixed problem, we shall not go into details to prove the results mentioned above.

2. Technical preliminaries. We need to estimate the error of piecewise linear interpolation to functions whose restriction on both ω and Ω are in H^2, but the functions themselves do not belong to $H^2(R)$. Their normal derivatives are not continuous across the curve Γ, and Γ cuts through some of the triangular elements. For this purpose we need to estimate the H^1 norms on thin strips of piecewise H^2 functions and their finite element interpolations. The following lemmas will be used in the next section.

Let $y = \gamma(x)$ be a C^2 function on $[0, L]$. We consider a strip domain γ_δ that is bounded by $x = 0$, $x = L$, $y = \gamma(x)$, and $y = \gamma(x) + \delta$; see Figure 3. We assume that D is a given domain such that $\gamma_\delta \subset\subset D$. We have the following estimates on functions restricted on γ_δ.

LEMMA 2.1. *Let $w \in H^1(D)$. Then we have that*

$$\|w\|^2_{L^2(\gamma_\delta)} \le C\delta \|w\|^2_{1,D}.$$

Here, C is a constant independent of δ and w.

Proof. In terms of the curvilinear coordinates $X = x, Y = y - \gamma(x)$, we have that $\|w\|^2_{0,\gamma_\delta} \simeq \int_0^\delta \int_0^L w^2 dX dY$. For each $Y \in (0, \delta)$, by a trace theorem [1], we have that $\int_0^L w^2 dX \le C\|w\|^2_{1,D}$. The constant C can be chosen to be independent of Y. Integrating this estimate, with respect to Y from 0 to δ, gives the desired estimate. ∎

From this result, the following lemma easily follows.

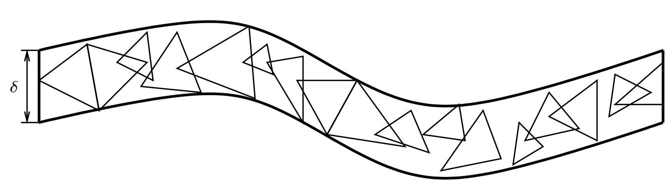

FIG. 3. *A strip γ_δ with a cluster of shape regular triangles.*

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LEMMA 2.2. *Let $w \in H^2(D)$. Then we have that*

$$\|w\|^2_{1,\gamma_\delta} \le C\delta \|w\|^2_{2,D}.$$

Here, C is independent of δ and w.

LEMMA 2.3. *Let $\mathcal{T}_\delta$ be the set of a cluster of shape regular, but not necessarily quasi uniform, open triangles contained in γ_δ, and suppose that any point in γ_δ can only be covered by at most a certain number of triangles in $\mathcal{T}_\delta$. Let $w \in H^2(D)$. For each $\tau \in \mathcal{T}_\delta$, we let $I_\tau w$ be the linear interpolation of w on the vertices of τ. Then we have that*

$$\sum_{\tau \in \mathcal{T}_\delta} |I_\tau w|^2_{1,\tau} \le C\delta \|w\|^2_{2,D}.$$

Here, C is a constant independent of δ and w.

Proof. Let $\tau \in \mathcal{T}_\delta$ be one such triangle. We affine map it onto a standard reference triangle T of unit size through the mapping F_τ. The restriction on τ of a function $w \in H^2(D)$ is mapped to a function $\hat{w}$ on T such that $\hat{w} \circ F_\tau = w$. Let I be the linear interpolation operator on the vertices of T. For any constant p_0, we have that

$$|I_\tau w|^2_{1,\tau} = |I_\tau(w - p_0)|^2_{1,\tau} \le C|I(\hat{w} - p_0)|^2_{1,T} \le C\|\hat{w} - p_0\|^2_{L^\infty(T)} \le C\|\hat{w} - p_0\|^2_{2,T}.$$

Thus, by the Bramble–Hilbert Lemma

$$|I_\tau w|^2_{1,\tau} \le C(|\hat{w}|^2_{1,T} + |\hat{w}|^2_{2,T}).$$

Scaling the right-hand side back from T to τ, we get that

$$|I_\tau w|^2_{1,\tau} \le C(|w|^2_{1,\tau} + h^2_\tau |w|^2_{2,\tau}).$$

Here h_τ is the diameter of τ, which does not exceed δ. Summing up the above inequality for all triangles in the cluster $\mathcal{T}_\delta$, we get that

$$\sum_{\tau \in \mathcal{T}_\delta} |I_\tau w|^2_{1,\tau} \le C \sum_{\tau \in \mathcal{T}_\delta} |w|^2_{1,\tau} + C\delta^2 \sum_{\tau \in \mathcal{T}_\delta} |w|^2_{2,\tau} \le C|w|^2_{1,\gamma_\delta} + C\delta^2 |w|^2_{2,\gamma_\delta}.$$

In the last step, we used the assumption that any point in γ_δ can only be covered by at most a certain number of triangles in $\mathcal{T}_\delta$. The desired result follows from Lemma 2.2. ◻

We shall apply these estimates to strips that can be covered by a finite number of (translated and rotated) strips as defined here. The strips of width δ referred to in the later sections are understood in this sense.

As we mentioned in the introduction, there are two sources of errors in the finite element domain embedding methods. The first one is due to the extension of the original boundary value problem from ω to R. The sharp estimates on this kind of error can be obtained by using the following equivalent estimates on an abstract ϵ-dependent equation. This estimate will also be directly used in estimating the finite element interpolation error for the Dirichlet problem in the next section.

Let H, U, and V be Hilbert spaces, $A : H \to U$ be a bounded linear operator, and let $B : H \to V$ be a bounded linear operator *with closed range*. We assume that

$$\|A\boldsymbol{v}\|_U + \|B\boldsymbol{v}\|_V \simeq \|\boldsymbol{v}\|_H \quad \forall\, \boldsymbol{v} \in H. \tag{2.1}$$

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Thus the bilinear form

$$(\boldsymbol{u},\boldsymbol{v})_{\mathcal{H}} := (A\boldsymbol{u},A\boldsymbol{v})_U + (B\boldsymbol{u},B\boldsymbol{v})_V$$

defines an equivalent inner product on H. Furnished with this new inner product, the space H will be denoted by $\mathcal{H}$. Let $\ker B \subset H$ be the kernel of the operator B. Let $\boldsymbol{f}$ be a linear continuous functional on H such that $\boldsymbol{f}|_{\ker B} = 0$. There is a unique $\boldsymbol{u}^\epsilon \in H$ such that

$$\epsilon(A\boldsymbol{u}^\epsilon, A\boldsymbol{v})_U + (B\boldsymbol{u}^\epsilon, B\boldsymbol{v})_V = \langle \boldsymbol{f},\boldsymbol{v}\rangle \;\forall \boldsymbol{v}\in H. \tag{2.2}$$

Without loss of generality, we assume that the operator B maps H onto V. (If necessary, we replace V by the range of B in it.) Then B is an isomorphism between $(\ker B)^\perp_{\mathcal{H}}$ (the orthogonal complement of $\ker B$ with respect to the $\mathcal{H}$-norm) and V. Since we have assumed that $\boldsymbol{f}|_{\ker B} = 0$, according the the closed range theorem and the Riesz representation theorem, there exists a unique $\boldsymbol{u}^0 \in (\ker B)^\perp_{\mathcal{H}}$ such that

$$(B\boldsymbol{u}^0, B\boldsymbol{v})_V = \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\, \boldsymbol{v}\in H. \tag{2.3}$$

LEMMA 2.4. *Under the assumptions that $\boldsymbol{f}|_{\ker B} = 0$ and B has a closed range in V as $\epsilon \to 0$, $\boldsymbol{u}^\epsilon$, the solution of* (2.2) *converges to the limit $\boldsymbol{u}^0 \in (\ker B)^\perp_{\mathcal{H}}$ defined by* (2.3)*, and we have the equivalence estimate*

$$\|\boldsymbol{u}^\epsilon - \boldsymbol{u}^0\|_H \simeq \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V \simeq \epsilon\, \|A\boldsymbol{u}^0\|_U. \tag{2.4}$$

Therefore, if $A\boldsymbol{u}^0 = 0$, then $\boldsymbol{u}^\epsilon \equiv \boldsymbol{u}^0$. Otherwise $\boldsymbol{u}^\epsilon$ converges to $\boldsymbol{u}^0$ at the sharp rate of ϵ.

Proof. From (2.2) and (2.3), we see that

$$\epsilon(A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0), A\boldsymbol{v})_U + (B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0), B\boldsymbol{v})_V = -\,\epsilon(A\boldsymbol{u}^0, A\boldsymbol{v})_U \;\forall \boldsymbol{v}\in H. \tag{2.5}$$

Taking $\boldsymbol{v} = \boldsymbol{u}^\epsilon - \boldsymbol{u}^0$, we get that

$$\epsilon\, \|A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_U^2 + \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V^2 = -\,\epsilon(A\boldsymbol{u}^0, A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0))_U. \tag{2.6}$$

It is easy to see that $\boldsymbol{u}^\epsilon$ lies in the subspace $(\ker B)^\perp_{\mathcal{H}}$. So does $\boldsymbol{u}^\epsilon - \boldsymbol{u}^0$. Because B is an isomorphism between $(\ker B)^\perp_{\mathcal{H}}$ and V, we have that

$$\|A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_U \lesssim \|\boldsymbol{u}^\epsilon - \boldsymbol{u}^0\|_H \simeq \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V. \tag{2.7}$$

By the Cauchy–Schwarz inequality, there exists a constant C such that

$$|\,\epsilon(A\boldsymbol{u}^0, A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0))_U| \le C\,\epsilon^2\, \|A\boldsymbol{u}^0\|_U^2 + \frac{1}{2}\|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V^2.$$

This estimate and the equation (2.6) show that

$$\|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V \lesssim \epsilon\, \|A\boldsymbol{u}^0\|. \tag{2.8}$$

The equivalence (2.1) and (2.7) then lead to $\|\boldsymbol{u}^\epsilon - \boldsymbol{u}^0\|_H \lesssim \epsilon\, \|A\boldsymbol{u}^0\|_U$.

To see the lower bound, in (2.5) we take $\boldsymbol{v} = \boldsymbol{u}^0$ to get

$$\epsilon(A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0), A\boldsymbol{u}^0)_U + (B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0), B\boldsymbol{u}^0)_V = -\,\epsilon\, \|A\boldsymbol{u}^0\|_U^2.$$

If $A\boldsymbol{u}^0 \neq 0$, then

$$\epsilon\, \|A\boldsymbol{u}^0\|_U \lesssim \epsilon\, \|A(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_U + \frac{\|B\boldsymbol{u}^0\|_V}{\|A\boldsymbol{u}^0\|_U}\|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V \lesssim \|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_V.$$

The equivalence (2.4) then follows. ∎

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3. Analysis of domain embedding methods. In this section, we derive the error estimates for the finite element domain embedding methods for various boundary conditions, as described in the Introduction. The main issues appear in Neumann and Dirichlet boundary value problems. In subsection 3.1, we analyze the Neumann problem and assume that the grid on R is uniform. However, all of the results hold for shape regular triangulations. We analyze the Dirichlet problem with uniform mesh in subsection 3.2. The results of this subsection motivate mesh adjustment around Γ for the Dirichlet problem, to which we devote subsection 3.4.

3.1. Neumann boundary condition. We first consider the Neumann problem. Let $\omega \subset \mathbb{R}^2$ be a bounded domain whose boundary Γ is in the C^2 class. Let $f \in L^2(\omega)$. We seek a function u on ω solving the homogeneous Neumann problem whose weak formulation seeks $u \in H^1(\omega)/\mathbb{R}$ such that

$$(\nabla u, \nabla v)_{[L^2(\omega)]^2} = \int_\omega f v dx \quad \forall\, v \in H^1(\omega). \tag{3.1}$$

We assume that $\int_\omega f(x)dx = 0$, so that this is well posed in the quotient space $H^1(\omega)/\mathbb{R}$.

Let $R \subset \mathbb{R}^2$ be a larger rectangular domain such that $\omega \subset\subset R$. Let $\Omega = R \setminus \overline{\omega}$ be the fictitious domain. We extend the function f to a function $\bar{f}$ on R by defining $\bar{f} = 0$ on Ω. The first step in the finite element domain embedding method is replacing (3.1) by the following problem defined on the rectangle R. For $\epsilon > 0$, one determines $u^\epsilon \in H_0^1(R)$ such that

$$(\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + \epsilon(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega)]^2} = \int_R \bar{f} v dx \;\forall\, v \in H_0^1(R). \tag{3.2}$$

This equation fits in the form of (2.2) in an obvious manner. The condition that B has closed range is assured as long as ω is a Lipschitz domain. All of the other conditions of Lemma 2.4 are also satisfied; see [24] for details. We have that $\lim_{\epsilon\to 0} u^\epsilon = u^0$. The limit $u^0 \in H_0^1(R)$ is characterized by the following two equations:

$$(\nabla u^0, \nabla v)_{[L^2(\omega)]^2} = \int_\omega f v dx \quad \forall\, v \in H^1(\omega), \tag{3.3}$$

i.e., the restriction of u^0 on ω solves the Neumann problem (3.1), and

$$\Delta u^0 = 0 \text{ in } \;\Omega, \quad \left\langle \frac{\partial u^0}{\partial n^-}, 1 \right\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} = 0. \tag{3.4}$$

Here n^- is the outward normal to Γ viewed as a boundary of Ω. By Lemma 2.4 we have that

$$\|u^\epsilon - u^0\|_{1,R} \simeq |u^\epsilon - u^0|_{1,\omega} \simeq \epsilon\, |u^0|_{1,\Omega}. \tag{3.5}$$

It is easy to see that $|u^0|_{1,\Omega} = 0$ if and only if $u^0 = 0$ on Γ, in which case $u^\epsilon \equiv u^0$. This is the exceptional case in which $u^0 \in H^2(R)$. Generally, u^0 is only piecewise smooth, with a normal derivative jump across Γ.

We introduce a uniform triangulation of mesh size h on R; see Figure 1 (left), and let $H_h \subset H_0^1(R)$ be the subspace of piecewise linear continuous functions subordinate to this triangulation. The finite element domain embedding method is a straightforward discretization of (1.2). It determines $u_h^\epsilon \in H_h$ such that

$$(\nabla u_h^\epsilon, \nabla v_h)_{[L^2(\omega)]^2} + \epsilon(\nabla u_h^\epsilon, \nabla v_h)_{[L^2(\Omega)]^2} = \int_R \bar{f} v_h dx \quad \forall\, v \in H_h. \tag{3.6}$$

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The numerical approximation of the solution of (3.1) offered by the finite element domain embedding method is the restriction of u_h^ϵ on ω. We have the following theorem.

THEOREM 3.1. *Let $u^0 \in H_0^1(R)$ be the limit of u^ϵ that is determined by the domain embedding equation* (3.2). *Then $u^0|_\omega$ solves the Neumann problem* (3.1). *Let $u_h^\epsilon \in H_h$ be the solution of the finite element domain embedding equation* (3.6). *We have the following estimate on the difference between $u_h^\epsilon|_\omega$ and $u^0|_\omega$:*

$$|u_h^\epsilon - u^0|_{1,\omega} \le C\,\epsilon\,|u^0|_{1,\Omega} + C \inf_{v_h \in H_h} \left(|u^0 - v_h|_{1,\omega} + \sqrt{\epsilon}|u^0 - v_h|_{1,\Omega}\right). \tag{3.7}$$

Here, C is a constant independent of h, ϵ, and f.

Proof. Since u_h^ϵ is the projection of u^ϵ in the subspace H_h with respect to the inner product defined by the bilinear form in the left-hand side of the domain embedding equation (3.2), we have that

$$|u^\epsilon - u_h^\epsilon|_{1,\omega} + \sqrt{\epsilon}|u^\epsilon - u_h^\epsilon|_{1,\Omega} \le |u^\epsilon - v_h|_{1,\omega} + \sqrt{\epsilon}|u^\epsilon - v_h|_{1,\Omega} \quad \forall\, v_h \in H_h.$$

Thus by (3.5)

$$\begin{aligned} |u^0 - u_h^\epsilon|_{1,\omega} &\le |u^0 - u^\epsilon|_{1,\omega} + |u^\epsilon - u_h^\epsilon|_{1,\omega} \\ &\le \epsilon\,|u^0|_{1,\Omega} + |u^\epsilon - v_h|_{1,\omega} + \sqrt{\epsilon}|u^\epsilon - v_h|_{1,\Omega} \\ &\le \epsilon\,|u^0|_{1,\Omega} + |u^0 - v_h|_{1,\omega} + |u^\epsilon - u^0|_{1,\omega} + \sqrt{\epsilon}|u^0 - v_h|_{1,\Omega} + \sqrt{\epsilon}|u^\epsilon - u^0|_{1,\Omega} \\ &\le (2\,\epsilon + \epsilon^{3/2})|u^0|_{1,\Omega} + |u^0 - v_h|_{1,\omega} + \sqrt{\epsilon}|u^0 - v_h|_{1,\Omega} \quad \forall\, v_h \in H_h. \end{aligned}$$

The estimate thus follows. □

This theorem is valid as long as ω is Lipschitz. Based on this result, we can establish the following error estimate on the finite element domain embedding method for the Neumann problem under our assumptions that $\Gamma \in C^2$ and $f \in L^2(\omega)$. Under these assumptions, by the regularity of elliptic differential equation [13], we have that $u^0|_\omega \in H^2(\omega)$ and $\|u^0\|_{2,\omega} \lesssim \|f\|_{0,\omega}$. Since $u^0|_\Omega$ is harmonic and it shares the value with $u^0|_\omega$ on Γ, we have that $\|u^0\|_{2,\Omega} \lesssim \|u^0\|_{1.5,\Gamma} \lesssim \|u^0\|_{2,\omega} \lesssim \|f\|_{0,\omega}$. We can extend $u^0|_\omega$ to R; see [21], to obtain a function $\bar{u}^0 \in H^2(R)$ such that $\bar{u}^0|_\omega = u^0|_\omega$ and $\|\bar{u}^0\|_{2,R} \lesssim \|u^0\|_{2,\omega}$. Similarly, we extend $u^0|_\Omega$ to R to obtain a function $\underline{u}^0 \in H^2(R)$ such that $\underline{u}^0|_\Omega = u^0|_\Omega$ and $\|\underline{u}^0\|_{2,R} \lesssim \|u^0\|_{2,\Omega}$. Note that $\bar{u}^0|_\Omega \neq u^0|_\Omega$, $\underline{u}^0|_\omega \neq u^0|_\omega$, and $\bar{u}^0 = \underline{u}^0 = u^0$ on Γ.

THEOREM 3.2. *Under the assumption that $\Gamma \in C^2$ and $f \in L^2(\omega)$, there is a positive constant C that is independent of ϵ, h, and f such that the following error estimate of the finite element domain embedding method for the Neumann problems holds:*

$$|u_h^\epsilon - u^0|_{1,\omega} \le C\left(\epsilon + h + \sqrt{\epsilon\, h}\right)\|f\|_{0,\omega}. \tag{3.8}$$

Before proving this theorem, we introduce some notations to describe triangles, vertices, and their relative relations with Γ. Let $\mathcal{T}$ be the set of all the open triangular elements of the finite element partition of R. We divide $\mathcal{T}$ into three distinctive subsets such that $\mathcal{T} = \mathcal{T}_\omega \cup \mathcal{T}_\Gamma \cup \mathcal{T}_\Omega$, and

$$\mathcal{T}_\Gamma = \{\tau \in \mathcal{T}; \tau \cap \Gamma \neq \emptyset\},\ \mathcal{T}_\omega = \{\tau \in \mathcal{T}; \tau \subset \omega\},\ \text{and}\ \mathcal{T}_\Omega = \{\tau \in \mathcal{T}; \tau \subset \Omega\}.$$

Among the triangles of $\mathcal{T}_\omega$, we let $\mathcal{T}_\omega^0$ be the subset of those that share a vertex with some triangle in $\mathcal{T}_\Gamma$, and we define $\mathcal{T}_\Omega^1$ as a subset of $\mathcal{T}_\Omega$ similarly. See Figure 4 in

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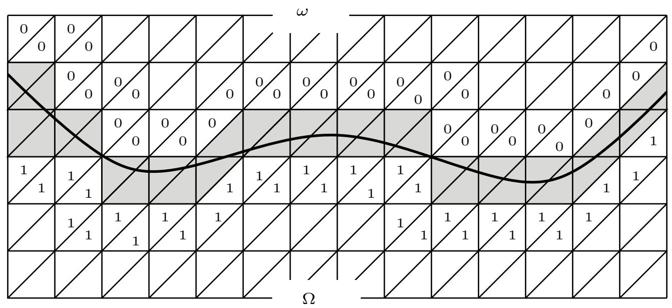

FIG. 4. *A portion of uniform mesh around* Γ.

which triangles in $\mathcal{T}_\Gamma$ are shaded, those in $\mathcal{T}_\omega^0$ are marked by 0, and those in $\mathcal{T}_\Omega^1$ are marked by 1.

With a slight abuse of notations, we shall also use $\mathcal{T}_\Gamma$ to denote the domain formed by the union of all triangles in $\mathcal{T}_\Gamma$, which is the interior of $\cup_{\tau\in\mathcal{T}_\Gamma}\bar{\tau}$. We use $\mathcal{T}$, $\mathcal{T}_\omega$, $\mathcal{T}_\Omega$, $\mathcal{T}_\omega^0$, and $\mathcal{T}_\Omega^1$ in the same way. Thus $\mathcal{T} = R$, and $\omega \subset \overline{\mathcal{T}}_\omega \cup \mathcal{T}_\Gamma$, etc.

Let $\mathcal{V}$ be the set of all the vertices in R of the triangulation. We divide $\mathcal{V}$ into three distinctive subsets such that $\mathcal{V} = \mathcal{V}_\omega \cup \mathcal{V}_\Gamma \cup \mathcal{V}_\Omega$, in which

$$\begin{aligned} \mathcal{V}_\Gamma &= \{\nu \in \mathcal{V}; \nu \text{ is a vertex of a triangle of } \mathcal{T}_\Gamma, \text{ or } \nu \in \Gamma\},\\ \mathcal{V}_\omega &= \{\nu \in \mathcal{V}; \nu \in \omega, \text{ and } \nu \text{ is not a vertex in } \mathcal{V}_\Gamma\},\\ \mathcal{V}_\Omega &= \{\nu \in \mathcal{V}; \nu \in \Omega, \text{ and } \nu \text{ is not a vertex in } \mathcal{V}_\Gamma\}. \end{aligned}$$

Note that $\mathcal{T}_\Gamma$ could be empty, for example, when Γ is straight and aligns with the triangulation, in which case $\mathcal{V}_\Gamma$ is composed of nodes on Γ. We let

$$\begin{aligned} \mathcal{V}_\omega^0 &= \{\nu \in \mathcal{V}_\omega;\ \nu \text{ is a vertex of a triangle in } \mathcal{T}_\omega^0\},\\ \mathcal{V}_\Omega^1 &= \{\nu \in \mathcal{V}_\Omega;\ \nu \text{ is a vertex of a triangle in } \mathcal{T}_\Omega^1\}. \end{aligned}$$

Proof of Theorem 3.2. What we need to do is to construct a finite element interpolation $v_h \in H_h$ to u^0 such that $|u^0 - v_h|_{1,\omega} \lesssim h$ and $|u^0 - v_h|_{1,\Omega} \lesssim \sqrt{h}$. We define v_h by requiring $v_h(\nu) = \bar{u}^0(\nu)$ for $\nu \in \mathcal{V}_\omega \cup \mathcal{V}_\Gamma$, and $v_h(\nu) = u^0(\nu)$ for $\nu \in \mathcal{V}_\Omega$. Note that the defined interpolation interpolates u^0 on all of the vertices except those in $\mathcal{V}_\Gamma \cap \Omega$. By the standard argument of finite element interpolation, we have that

$$|v_h - \bar{u}^0|_{1,\tau}^2 \le Ch^2 |\bar{u}^0|_{2,\tau}^2 \quad \forall\ \tau \in \mathcal{T}_\omega \cup \mathcal{T}_\Gamma.$$

Summing up these estimates, we get that

$$|v_h - \bar{u}^0|_{1,\omega\cup\mathcal{T}_\Gamma}^2 \le Ch^2 |\bar{u}^0|_{2,\omega\cup\mathcal{T}_\Gamma}^2 \le Ch^2 |\bar{u}^0|_{2,R}^2 \le Ch^2 |u^0|_{2,\omega}^2 \le Ch^2 \|f\|_{0,\omega}^2.$$

Thus

$$|v_h - u^0|_{1,\omega}^2 = |v_h - \bar{u}^0|_{1,\omega}^2 \le Ch^2 \|f\|_{0,\omega}^2. \tag{3.9}$$

By the standard argument of finite element interpolation, we have that

$$|v_h - u^0|_{1,\tau}^2 \le Ch^2 |u^0|_{2,\tau}^2 \quad \forall\ \tau \in \mathcal{T}_\Omega \setminus \mathcal{T}_\Omega^1.$$

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Summing up, we get that

$$|v_h - u^0|^2_{1,\mathcal{T}_\Omega\setminus\overline{\mathcal{T}}^1_\Omega} \le Ch^2|u^0|^2_{2,\Omega} \le Ch^2\|f\|^2_{0,\omega}. \tag{3.10}$$

The remaining part of Ω, i.e., $\mathcal{T}^1_\Omega \cup (\Omega \cap \mathcal{T}_\Gamma)$, is covered by a strip of width $\mathcal{O}(h)$, which meets the definition of a γ_δ of section 2, with $\delta = \mathcal{O}(h)$. Thus, according to Lemma 2.2, we have that

$$|u^0|^2_{1,\mathcal{T}^1_\Omega\cup(\Omega\cap\mathcal{T}_\Gamma)} \le Ch\|\underline{u}^0\|^2_{2,R} \le Ch\|f\|^2_{0,\omega}. \tag{3.11}$$

We are left with estimating $|v_h|^2_{1,\mathcal{T}^1_\Omega\cup(\Omega\cap\mathcal{T}_\Gamma)}$. We consider this separately on $\Omega\cap\mathcal{T}_\Gamma$ and $\mathcal{T}^1_\Omega$. We see that $|v_h|^2_{1,\Omega\cap\mathcal{T}_\Gamma} \le |v_h|^2_{1,\mathcal{T}_\Gamma}$. Since v_h interpolates $\bar{u}^0$ at every vertex of triangles in $\mathcal{T}_\Gamma$, which itself is a cluster of shape regular triangles contained in a strip of width $\mathcal{O}(h)$, by Lemma 2.3, we have that $|v_h|^2_{1,\mathcal{T}_\Gamma} \le Ch\|\bar{u}^0\|^2_{2,R}$. Thus

$$|v_h|^2_{1,\Omega\cap\mathcal{T}_\Gamma} \le Ch\|\bar{u}^0\|^2_{2,R} \le Ch\|f\|^2_{0,\omega}. \tag{3.12}$$

To estimate $|v_h|^2_{1,\mathcal{T}^1_\Omega}$, we introduce an auxiliary interpolation $I_{\mathcal{T}^1_\Omega}u^0$ that interpolates u^0 on every vertex of triangles in $\mathcal{T}^1_\Omega$. Since $\mathcal{T}^1_\Omega$ is composed of a cluster of shape regular triangles covered by a strip of width h, by Lemma 2.3, we have that

$$|I_{\mathcal{T}^1_\Omega}u^0|^2_{1,\mathcal{T}^1_\Omega} \le Ch\|\underline{u}^0\|^2_{2,R} \le Ch\|f\|^2_{0,\omega}. \tag{3.13}$$

From the definition we see that $v_h(\nu) - I_{\mathcal{T}^1_\Omega}u^0(\nu) = 0$ for all $\nu \in \mathcal{V}^1_\Omega$, and $v_h(\nu) - I_{\mathcal{T}^1_\Omega}u^0(\nu) = \bar{u}^0(\nu) - u^0(\nu)$ for all $\nu \in \mathcal{V}_\Gamma \cap \Omega$. Thus we have that

$$|v_h - I_{\mathcal{T}^1_\Omega}u^0|^2_{1,\mathcal{T}^1_\Omega} \le C\sum_{\nu\in\mathcal{V}_\Gamma\cap\Omega}(v_h(\nu) - I_{\mathcal{T}^1_\Omega}u^0(\nu))^2 = C\sum_{\nu\in\mathcal{V}_\Gamma\cap\Omega}(\bar{u}^0(\nu) - u^0(\nu))^2.$$

For each $\nu \in \mathcal{V}_\Gamma \cap \Omega$, we construct a shape regular triangle τ_ν, with ν being one of its vertices, and the other two vertices sitting on Γ. Let $I_{\tau_\nu}(\bar{u}^0 - \underline{u}^0)$ be the linear interpolation of $\bar{u}^0 - \underline{u}^0$ on τ_ν. Since $\bar{u}^0 = \underline{u}^0$ on Γ, we have that

$$|I_{\tau_\nu}(\bar{u}^0 - \underline{u}^0)|^2_{1,\tau_\nu} \simeq (\bar{u}^0(\nu) - u^0(\nu))^2.$$

Thus

$$\sum_{\nu\in\mathcal{V}_\Gamma\cap\Omega}(\bar{u}^0(\nu) - u^0(\nu))^2 \le C\sum_{\nu\in\mathcal{V}_\Gamma\cap\Omega}|I_{\tau_\nu}(\bar{u}^0 - \underline{u}^0)|^2_{1,\tau_\nu}.$$

The cluster of triangles $\{\tau_\nu; \nu \in \mathcal{V}_\Gamma \cap \Omega\}$ satisfies the condition of Lemma 2.3, with $\delta = h$. We therefore have that

$$\sum_{\nu\in\mathcal{V}_\Gamma\cap\Omega}|I_{\tau_\nu}(\bar{u}^0 - \underline{u}^0)|^2_{1,\tau_\nu} \le Ch\|\bar{u}^0 - \underline{u}^0\|^2_{2,R}.$$

From these estimates, we see that

$$|v_h - I_{\mathcal{T}^1_\Omega}u^0|^2_{1,\mathcal{T}^1_\Omega} \le Ch\|\bar{u}^0 - \underline{u}^0\|^2_{2,R} \le Ch\|f\|^2_{0,\omega}. \tag{3.14}$$

Therefore, by (3.13) and (3.14),

$$|v_h|^2_{1,\mathcal{T}^1_\Omega} \le |I_{\mathcal{T}^1_\Omega}u^0|^2_{1,\mathcal{T}^1_\Omega} + |v_h - I_{\mathcal{T}^1_\Omega}u^0|^2_{1,\mathcal{T}^1_\Omega} \le Ch(\|\underline{u}^0\|^2_{2,R} + \|\bar{u}^0 - \underline{u}^0\|^2_{2,R}) \le Ch\|f\|^2_{0,\omega}.$$

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Combining this and (3.12), we get that

$$|v_h|^2_{1,\mathcal{T}^1_\Omega \cup (\Omega \cap \mathcal{T}_\Gamma)} \le Ch \|f\|^2_{0,\omega}. \tag{3.15}$$

The theorem now follows from (3.9), (3.10), (3.11), and (3.15). □

From this theorem we see that, when $\Gamma \in C^2$ and $f \in L^2(\omega)$ and if we take $\epsilon = Ch$, we have a domain embedding finite element method on uniform mesh that achieves the full accuracy with an error of $\mathcal{O}(h)$. Although the triangulation was assumed to be uniform, it is clear that Theorem 3.2 holds for any triangulation that is shape regular, (but not necessarily quasi uniform,) with a maximum mesh size h. The result of Theorem 3.1 is valid as long as ω is a Lipschitz domain, which does not guarantee the piecewise regularity of u^0. For example, when ω is a convex polygon, then $u^0|_\omega \in H^2(\omega)$, but $u^0|_\Omega$ cannot be expected to have the H^2 regularity. If ω is a convex polygon and Γ aligns with the mesh, one can define v_h by simply requiring it to interpolate u^0 at every node in $\mathcal{V}$. Then one uses the regularity $u^0|_\omega \in H^2(\omega)$ and $u^0|_\Omega \in H^{1.5}(\Omega)$ [16] to show the validity of the estimate in Theorem 3.2. If ω is a nonconvex polygon, then $u^0|_\omega \in H^2(\omega)$ may not hold, in which case, a local refinement may be needed at reentrant corners of ω to make $|u^0 - v_h|_{1,\omega} \lesssim h$. This can be done by taking a small rectangle around a corner, which conforms to the uniform mesh, and on which one uses a fine uniform mesh. Thus doing, fast Fourier transforms on the global uniform mesh and local uniform fine mesh can be combined in preconditioning the discrete system, in view of the additive multilevel Schwarz method or the Chimera method [9].

For a Robin boundary condition, the theory is similar to that of the Neumann problem, so we will be very brief. We seek a function u on ω solving the homogeneous Robin problem

$$-\Delta u = f \text{ on } \omega, \quad \frac{\partial u}{\partial n} + ku = 0 \text{ on } \Gamma. \tag{3.16}$$

Here k is a smooth, bounded, and positively valued function on Γ. The weak formulation is

$$\begin{gathered}(\nabla u, \nabla v)_{[L^2(\omega)]^2} + \int_\Gamma kuvds = \int_\omega fvdx \quad \forall\ v \in H^1(\omega),\\ u \in H^1(\omega).\end{gathered} \tag{3.17}$$

On the continuous level, the domain embedding method determines $u^\epsilon \in H^1_0(R)$ such that

$$(\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + \int_\Gamma kuvds + \epsilon(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega)]^2} = \int_R \bar{f} vdx\ \forall\ v \in H^1_0(R). \tag{3.18}$$

This equation also fits in the form of (2.2), in which the B operator has closed range in V if ω is Lipschitz. Therefore, as $\epsilon \to 0$, u^ϵ converges to a limit $u^0 \in H^1_0(R)$. The limit is harmonic on Ω, and $u^0|_\omega$ solves the Robin problem (3.17). By Lemma 2.4 we have that

$$\|u^\epsilon - u^0\|_{1,R} \simeq |u^\epsilon - u^0|_{1,\omega} + \|u^\epsilon - u^0\|_{0,\Gamma} \simeq \epsilon\, |u^0|_{1,\Omega}. \tag{3.19}$$

The finite element domain embedding method is a straightforward discretization of (3.18) on a uniform mesh on R. The same results of Theorems 3.1 and 3.2 hold for the Robin problem, and the argument is exactly the same.

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The Robin problem provides an alternative approach to designing domain embedding for the Dirichlet problem, other than the one we analyze below. In the equation (3.18), we can fix ϵ and take k as a parameter. When $k \to \infty$, the solution of (3.18) then converges to a function (at a rate much lower than $1/k$ [22]) whose restriction on ω solves the homogeneous Dirichlet problem. This method is essentially that of [4]. But the above theory of the domain embedding method for the Robin problem does not extend to this approach for the Dirichlet problem.

3.2. Dirichlet boundary condition. We seek a function u on ω solving the homogeneous Dirichlet problem whose weak formulation is

$$(\nabla u, \nabla v)_{[L^2(\omega)]^2} = \int_\omega f v dx \quad \forall\, v \in H_0^1(\omega), \qquad u \in H_0^1(\omega). \tag{3.20}$$

Under the assumption that ω is simply connected, on the continuous level, the domain embedding method determines $u^\epsilon \in H_0^1(R)$ such that

$$(\nabla u^\epsilon, \nabla v)_{[L^2(\omega)]^2} + \epsilon^{-1}(\nabla u^\epsilon, \nabla v)_{[L^2(\Omega)]^2} = \int_R \bar{f} v dx \quad \forall\, v \in H_0^1(R). \tag{3.21}$$

This is a penalty formulation. We extend u to a function u^0 on R such that $u^0|_\omega = u$ and $u^0|_\Omega = 0$. It is easy to see that, for all $v \in H_0^1(R)$,

$$\epsilon(\nabla(u^\epsilon - u^0), \nabla v)_{[L^2(\omega)]^2} + (\nabla(u^\epsilon - u^0), \nabla v)_{[L^2(\Omega)]^2} = \epsilon \left[\int_R \bar{f} v dx - (\nabla u^0, \nabla v)_{[L^2(\omega)]^2} \right]. \tag{3.22}$$

This equation fits in the form of (2.2) by properly defining the operators and spaces. We see that

$$\ker B = \{v \in H_0^1(R); v = 0 \text{ on } \Omega\},$$

and, more importantly, the right-hand side of (3.22), as a functional, annihilates this subspace. Thus there is a unique $u^1 \in H_0^1(R)$ such that $\Delta u^1 = 0$ on ω (since $u^1 \in (\ker B)^\perp_{\mathcal{H}}$), and

$$(\nabla u^1, \nabla v)_{[L^2(\Omega)]^2} = \left[\int_R \bar{f} v dx - (\nabla u^0, \nabla v)_{[L^2(\omega)]^2} \right] \quad \forall\, v \in H_0^1(R). \tag{3.23}$$

In strong form, this function satisfies

$$-\Delta u^1 = 0 \text{ in } \Omega, \quad \frac{\partial u^1}{\partial n^-} = \frac{\partial u^0}{\partial n} \text{ on } \Gamma, \quad \text{and } u^1 = 0 \text{ on } \partial R. \tag{3.24}$$

Here n is the unit outward normal to Γ viewed as a boundary of ω, and n^- is the outward normal of Ω, which is opposite to n. When Ω is a Lipschitz domain, the operator B arising from fitting (3.22) into (2.2) has closed range. By Lemma 2.4 we have that

$$\|u^\epsilon - u^0 - \epsilon\, u^1\|_{1,R} \simeq |u^\epsilon - u^0 - \epsilon\, u^1|_{1,\Omega} \simeq \epsilon^2\, |u^1|_{1,\omega}. \tag{3.25}$$

From (3.24) we see that $|u^1|_{1,\omega} = 0$, which is equivalent to $|u^1|_{1,\Omega} = 0$ if and only if u^0 satisfies the homogeneous Neumann condition on Γ, or equivalently, $u^0 \in H^2(R)$. In this case $u^\epsilon \equiv u^0$, otherwise u^ϵ converges to u^0 at the sharp rate of ϵ.

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Remark. If ω is not simply connected, then Ω could have isolated components whose union is, say, Ω_0. In this case, the limit u^0 of u^ϵ does not satisfy the homogeneous Dirichlet condition on Γ, and the domain embedding method (3.21) fails. A remedy is adding, for example, the term $\epsilon^{-1}(u^\epsilon, v)_{L^2(\Omega_0)}$ or $\epsilon^{-1}(u^\epsilon, v)_{L^2(\Gamma)}$ (or something else as long as $\ker B$ can be identified with $H_0^1(\omega)$) to the left-hand side of the domain embedding equation (3.21). Then the method would be convergent, and the convergence rate would be retained.

The finite element domain embedding method is a straightforward discretization of (3.21) in the finite element space $H_h \subset H_0^1(R)$. In this subsection, as for the Neumann and Robin problems, we assume the triangulation is uniform. The numerical method determines $u_h^\epsilon \in H_h$ such that

$$(\nabla u_h^\epsilon, \nabla v_h)_{[L^2(\omega)]^2} + \epsilon^{-1}(\nabla u_h^\epsilon, \nabla v_h)_{[L^2(\Omega)]^2} = \int_R \bar{f} v_h dx \quad \forall\ v \in H_h. \tag{3.26}$$

We have the following estimate on the error of this method.

THEOREM 3.3. *Let $u^0 \in H_0^1(R)$ be the limit of u^ϵ determined by the domain embedding equation* (3.21). *Then $u^0|_\omega$ solves the Dirichlet problem* (3.20)*, and $u^0|_\Omega = 0$. Let $u_h^\epsilon \in H_h$ be the solution of the finite element domain embedding equation* (3.26). *We have that*

$$\|u_h^\epsilon - u^0\|_{1,\omega} \le C\,\epsilon\,|u^1|_{1,R} + C\sqrt{\epsilon} \inf_{v_h^1 \in H_h} \left(\sqrt{\epsilon}|u^1 - v_h^1|_{1,\omega} + |u^1 - v_h^1|_{1,\Omega}\right) + C \inf_{v_h^0 \in H_h} \left(|u^0 - v_h^0|_{1,\omega} + \frac{1}{\sqrt{\epsilon}}|v_h^0|_{1,\Omega}\right). \tag{3.27}$$

Here u^1 is defined by (3.23)*, and C is a constant independent of h, ϵ, and f.*

Proof. Since u_h^ϵ is the projection of u^ϵ in the subspace H_h with respect to the inner product defined by the bilinear form in (3.21), we have that

$$|u^\epsilon - u_h^\epsilon|_{1,\omega} + \frac{1}{\sqrt{\epsilon}}|u^\epsilon - u_h^\epsilon|_{1,\Omega} \le |u^\epsilon - v_h|_{1,\omega} + \frac{1}{\sqrt{\epsilon}}|u^\epsilon - v_h|_{1,\Omega} \quad \forall\ v_h \in H_h.$$

Thus, by using (3.25), we see that

$$\begin{aligned}\|u^0 - u_h^\epsilon\|_{1,\omega} le\ C|u^0 - u_h^\epsilon|_{1,R} &\le C\Big(|u^0 - u^\epsilon|_{1,R} + |u^\epsilon - u_h^\epsilon|_{1,R}\Big)\\ &\le C\left(\epsilon\,|u^1|_{1,R} + |u^\epsilon - v_h|_{1,\omega} + \frac{1}{\sqrt{\epsilon}}|u^\epsilon - v_h|_{1,\Omega}\right)\\ &\le C\Bigg(\epsilon\,|u^1|_{1,R} + |u^\epsilon - u^0 - \epsilon\,u^1|_{1,\omega} + |u^0 + \epsilon\,u^1 - v_h|_{1,\omega}\\ &\qquad + \frac{1}{\sqrt{\epsilon}}(|u^\epsilon - u^0 - \epsilon\,u^1|_{1,\Omega} + |\,\epsilon\,u^1 - v_h|_{1,\Omega})\Bigg) \quad \forall\ v_h \in H_h.\end{aligned}$$

We write any $v_h \in H_h$ as $v_h = v_h^0 + \epsilon\,v_h^1$ and use (3.25) again to obtain that

$$\begin{aligned}\|u^0 - u_h^\epsilon\|_{1,\omega} \le C[(\epsilon + \epsilon^{1.5} + \epsilon^2)|u^1|_{1,R} + \epsilon\,|u^1 - v_h^1|_{1,\omega} + \sqrt{\epsilon}|u^1 - v_h^1|_{1,\Omega}\\ + |u^0 - v_h^0|_{1,\omega} + \frac{1}{\sqrt{\epsilon}}|v_h^0|_{1,\Omega}].\end{aligned}$$

The desired estimate then follows. ∎

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This result is valid as long as Ω is a Lipschitz domain. In the following, we assume that $\omega \subset\subset R$, $\Gamma \in C^2$, and $f \in L^2(\omega)$. Under this assumption, we have the regularity that $u^0|_\omega \in H^2(\omega) \cap H_0^1(\omega)$. Thus $\partial u^0/\partial n \in H^{1/2}(\Gamma)$. This in turn means that $\partial u^1/\partial n^- \in H^{1/2}(\Gamma)$; cf. (3.24). Therefore, $u^1|_\Omega \in H^2(\Omega)$. Since $u^1|_\omega$ is harmonic and shares values with $u^1|_\Omega$ on Γ, so $u^1|_\omega \in H^2(\omega)$. Furthermore, we have that the estimates $\|u^0\|_{2,\omega} \le C\|f\|_{0,\omega}$, $\|u^1\|_{2,\omega} \le C\|f\|_{0,\omega}$, and $\|u^1\|_{2,\Omega} \le C\|f\|_{0,\omega}$. We shall also use the fact that $u^0|_\omega$ has a H^2 extension onto R [21], denoted by $\bar{u}^0$, such that $\|\bar{u}^0\|_{2,R} \le C\|u^0\|_{2,\omega} \le C\|f\|_{0,\omega}$.

THEOREM 3.4. *Under the assumption that $\omega \subset\subset R$, $\Gamma \in C^2$, and $f \in L^2(\omega)$, for the Dirichlet problem, the following error estimate of the finite element domain embedding method based on uniform mesh holds:*

$$\|u^0 - u_h^\epsilon\|_{1,\omega} \le C(\epsilon + \epsilon\sqrt{h} + h\sqrt{\epsilon} + \sqrt{h})\|f\|_{0,\omega}.$$

Here, C is a constant independent of ϵ, h, and f.

Proof. We need to estimate the three terms in the right-hand side of (3.27). The first one is trivial. For the other two, we need to construct finite element interpolation v_h^1 to u^1 and v_h^0 to u^0, respectively. The construction of v_h^1 aims to estimate the infimum

$$\inf_{v_h^1 \in H_h} \left(\sqrt{\epsilon}|u^1 - v_h^1|_{1,\omega} + |u^1 - v_h^1|_{1,\Omega}\right).$$

We observe that this is in the same form as the second term in the right-hand side of the estimate for the Neumann problem; c.f. (3.7), except that the roles of Ω and ω are switched. We use exactly the same technique of proving Theorem 3.2 to estimate this term. We define $v_h^1 \in H_h$ such that $v_h^1(\nu) = \underline{u}^1(\nu)$ for all $\nu \in \mathcal{V}_\Omega \cup \mathcal{V}_\Gamma$, and $v_h^1(\nu) = u^1(\nu)$ for all $\nu \in \mathcal{V}_\omega$. Here $\underline{u}^1 \in H^2(R)$ is an extension of $u^1|_\Omega$ to R such that $\underline{u}^1|_\Omega = u^1|_\Omega$. With such defined interpolation, we have that

$$|u^1 - v_h^1|_{1,\Omega} \le Ch\|f\|_{0,\omega} \text{ and } |u^1 - v_h^1|_{1,\omega} \le C\sqrt{h}\|f\|_{0,\omega}.$$

Thus

$$\inf_{v_h^1 \in H_h} \left(\sqrt{\epsilon}|u^1 - v_h^1|_{1,\omega} + |u^1 - v_h^1|_{1,\Omega}\right) \le C(\sqrt{\epsilon h} + h)\|f\|_{0,\omega}. \tag{3.28}$$

We now turn to the last term in (3.27); i.e.,

$$\inf_{v_h^0 \in H_h} \left(|u^0 - v_h^0|_{1,\omega} + \frac{1}{\sqrt{\epsilon}}|v_h^0|_{1,\Omega}\right). \tag{3.29}$$

The best choice of the interpolation to estimate this one is taking $v_h^0 \in H_h$ such that

$$v_h^0(\nu) = u^0(\nu) \;\; \forall\, \nu \in \mathcal{V}_\omega, \quad v_h^0(\nu) = 0 \;\; \forall\, \nu \in \mathcal{V}_\Gamma \cup \mathcal{V}_\Omega.$$

(See the remark below for the reason why this is the best.) This choice makes $|v_h^0|_{1,\Omega} = 0$. To estimate the other term in (3.29), we observe that

$$|u^0 - v_h^0|_{1,\omega}^2 = |u^0 - v_h^0|_{1,\mathcal{T}_\omega \setminus \mathcal{T}_\omega^0}^2 + |u^0 - v_h^0|_{1,\mathcal{T}_\omega^0}^2 + |u^0|_{1,\omega\cap\mathcal{T}_\Gamma}^2. \tag{3.30}$$

Note that v_h^0 interpolates u^0 at all of the vertices of triangles in $\mathcal{T}_\omega \setminus \mathcal{T}_\omega^0$. On each of the triangle $\tau \in \mathcal{T}_\omega \setminus \mathcal{T}_\omega^0$, we have that $|u^0 - v_h^0|_{1,\tau}^2 \le Ch^2|u^0|_{2,\tau}^2$. Summing up, we get that

$$|u^0 - v_h^0|_{1,\mathcal{T}_\omega \setminus \mathcal{T}_\omega^0}^2 \le Ch^2|u^0|_{2,\mathcal{T}_\omega \setminus \mathcal{T}_\omega^0}^2 \le Ch^2|u^0|_{2,\omega}^2. \tag{3.31}$$

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Since $\mathcal{T}_\omega^0$ is covered by a strip of width $\mathcal{O}(h)$ that satisfies the condition of Lemma 2.2, so we have that

$$|u^0|_{1,\mathcal{T}_\omega^0}^2 \le Ch\|\bar{u}^0\|_{2,R}^2. \tag{3.32}$$

Here $\bar{u}^0$ is an H^2 extension of $u^0|_\omega$ to R such that $\|\bar{u}^0\|_{2,R} \le C\|u^0\|_{2,\omega}$. Note that a vertex of any triangle in $\mathcal{T}_\omega^0$ either belongs to $\mathcal{V}_\omega^0$ or belongs to $\omega \cap \mathcal{V}_\Gamma$. Since $v_h^0(\nu) = u^0(\nu)$ for all $\nu \in \mathcal{V}_\omega^0$ and $v_h^0(\nu) = 0$ for all $\nu \in \omega \cap \mathcal{V}_\Gamma$, we have that $|v_h^0|_{1,\mathcal{T}_\omega^0}^2 \le C\sum_{\nu\in\mathcal{V}_\omega^0}(u^0(\nu))^2$. For each $\nu \in \mathcal{V}_\omega^0$, we construct a shape regular triangle τ_ν with ν being a vertex, and the other two vertices sitting on Γ. Let $I_{\tau_\nu}\bar{u}^0$ be the linear interpolation of $\bar{u}^0$ on this triangle. We have that $(u^0(\nu))^2 \simeq |I_{\tau_\nu}\bar{u}^0|_{1,\tau_\nu}^2$. Thus

$$|v_h^0|_{1,\mathcal{T}_\omega^0}^2 \le C\sum_{\nu\in\mathcal{V}_\omega^0}|I_{\tau_\nu}\bar{u}^0|_{1,\tau_\nu}^2 \le Ch\|\bar{u}^0\|_{2,R}^2. \tag{3.33}$$

In the last step, we used Lemma 2.3. Finally, according to Lemma 2.2, we have that

$$|u^0|_{1,\omega\cap\mathcal{T}_\Gamma}^2 \le Ch\|\bar{u}^0\|_{2,R}^2. \tag{3.34}$$

The estimates (3.32), (3.33), and (3.34) together lead to

$$|u^0 - v_h^0|_{1,\mathcal{T}_\omega^0}^2 + |u^0|_{1,\omega\cap\mathcal{T}_\Gamma}^2 \le Ch\|\bar{u}^0\|_{2,R}^2. \tag{3.35}$$

This and (3.31) show that $|u^0 - v_h^0|_{1,\omega} \le C\sqrt{h}\|\bar{u}^0\|_{2,R}$, which together with (3.27) and (3.28) complete the proof. ∎

3.3. Remark on the sharpness of the estimate on (3.29). The estimated upper bound $C\sqrt{h}$ on (3.29) is rather big. This estimate was established by taking the interpolation v_h^0 to u^0. One of the reasons for the estimate to be so big is that the chosen v_h^0 does not offer any approximation to u^0 on $\omega \cap \mathcal{T}_\Gamma$, where v_h^0 is identically equal to zero. Even so, the estimate is sharp. To see this, we consider an example of a homogeneous Dirichlet boundary value problem in which ω is the unit circle, $f = 1$, and the solution is $u^0|_\omega = (1 - x^2 - y^2)/2$. We embed ω in a larger rectangle R, on which we introduce a uniform triangulation of mesh size h. The unit circle inevitably cuts through Ch triangles. The area of $\omega \cap \mathcal{T}_\Gamma$ is of the magnitude h. We have that $|u^0|_{1,\omega\cap\mathcal{T}_\Gamma} \simeq \sqrt{h}$. The interpolation v_h^0, defined in the proof, is zero on $\omega \cap \mathcal{T}_\Gamma$, which makes the error bound to be at least $C\sqrt{h}$. But the estimate $C\sqrt{h}$ is sharp. One may try to construct a "better" piecewise linear approximation to u^0 to make $|u^0 - v_h^0|_{1,\omega\cap\mathcal{T}_\Gamma}$ smaller. However, if the v_h^0 were to offer any approximation to u^0, one would at least have that $|v_h^0|_{1,\omega\cap\mathcal{T}_\Gamma} \simeq |u^0|_{1,\omega\cap\mathcal{T}_\Gamma}$. Since v_h^0 is piecewise linear on $\mathcal{T}_\Gamma$, we have that $|v_h^0|_{1,\Omega\cap\mathcal{T}_\Gamma} \simeq |v_h^0|_{1,\omega\cap\mathcal{T}_\Gamma} \simeq |u^0|_{1,\omega\cap\mathcal{T}_\Gamma} \simeq \sqrt{h}$. The bound for (3.29) thus obtained would exceed $\sqrt{h}/\sqrt{\epsilon}$, which is much bigger than $\sqrt{h}$.

This argument shows that the estimate of Theorem 3.4 on the error of the finite element domain embedding method for the Dirichlet problem is sharp. The optimal convergence rate $\mathcal{O}(\sqrt{h})$ can be achieved by taking the penalty parameter as $\epsilon = \mathcal{O}(\sqrt{h})$. Smaller ϵ does not help in improving the accuracy, but it does not hurt the accuracy either. From a computational point of view, the best value of ϵ should be $\epsilon = C\sqrt{h}$. Since this accuracy is rather poor, it seems desirable to find ways to enhance it.

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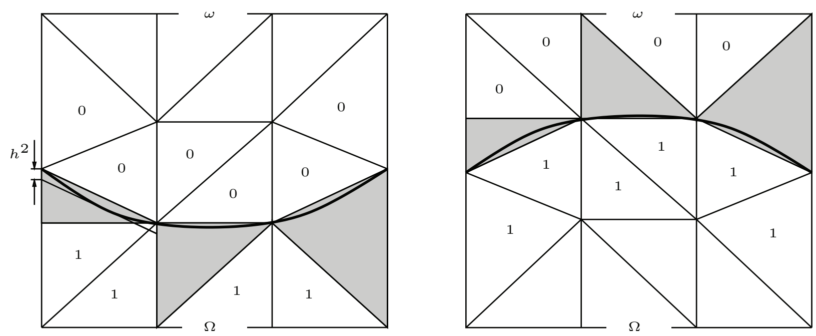

Convex ω Non-convex ω

FIG. 5. *A portion of the adjusted mesh around* Γ.

3.4. Dirichlet boundary condition with adjusted mesh. One way to reduce the magnitude of (3.29) is to adjust the uniform mesh around Γ by moving the nearby nodes onto Γ and reconnecting some mesh lines to form a polygonal interpolation to Γ in the mesh; see Figure 1. The adjusted mesh must fulfill the following requirements: (1) The total number of nodes keeps unchanged. (2) No mesh-line segment has one end in ω and the other in Ω. (3) The adjusted triangulation is shape regular and quasi uniform. Fulfilling these requirements ensures, on one hand, that the discrete Laplacian on the adjusted mesh is spectrally equivalent to the one on the uniform triangulation thus retains the effectiveness of preconditioning the discrete system by fast Poisson solvers. On the other hand, the magnitude of (3.29) is significantly reduced, and thus the accuracy of the finite element domain embedding method is accordingly enhanced. The Börgers's algorithm [7] exactly carries out such adjustment. It is interesting to note that the algorithm was proposed in a context of solving a Neumann problem by domain embedding methods [8]. Our above analysis shows that the mesh adjustment is actually not necessary for Neumann problems, while it has a significant effect for the Dirichlet problem.

It turns out that the effect of this mesh adjustment is different whether ω is convex. For convex ω, we prove that the full accuracy of $\mathcal{O}(h)$ is achieved by the domain embedding finite element method on such adjusted mesh if one takes $\epsilon = \mathcal{O}(h)$. Smaller ϵ does not hurt the accuracy. If ω is not convex, then the accuracy can only be improved to $\mathcal{O}(h^{2/3})$ by taking $\epsilon = Ch^{2/3}$. But in this case, smaller ϵ could diminish the improvement and sets the accuracy back to that of the uniform mesh. This is a case in which the balance between ϵ and h is delicate.

We keep the notations of triangles and vertices introduced for uniform triangulation. For example, $\mathcal{T}_\Gamma$ still comprises those triangles that intersect Γ, which are shaded in Figure 5. In addition, we shall use $\mathcal{P}$ to denote the set of all triangles enclosed by the polygonal interpolation of Γ. In consistence to our earlier convention, $\mathcal{P}$ also stands for the enclosed polygonal domain.

THEOREM 3.5. *We assume that* $\omega \subset\subset R$, $\Gamma \in C^2$, *and* $f \in L^2(\omega)$. *The triangulation* $\mathcal{T}$ *of* R *is obtained by adjusting a uniform mesh around* Γ *in the way described above. The finite element domain embedding method* (3.26) *is a straightforward discretization of the penalty formulation* (3.21) *on the adjusted mesh. Then if* ω *is convex, we have that*

$$\|u^0 - u_h^\epsilon\|_{1,\omega} \leq C(\epsilon + \epsilon\sqrt{h} + h\sqrt{\epsilon} + h)\|f\|_{0,\omega}. \tag{3.36}$$

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If ω is not convex, we have that

$$\|u^0 - u_h^\epsilon\|_{1,\omega} \le C\left(\epsilon + \epsilon\sqrt{h} + h\sqrt{\epsilon} + h + \frac{h}{\sqrt{\epsilon}}\right)\|f\|_{0,\omega}. \tag{3.37}$$

Here, C is a constant independent of ϵ, h, and f.

Proof. The proof is based on the estimate (3.27) that was established in Theorem 3.3, which is still valid even when $\mathcal{T}$ is not uniform. There is nothing new in estimating the first two terms in the right-hand side of (3.27); i.e., we have $\epsilon\,|u^1|_{1,R} \le C\,\epsilon\,\|f\|_{0,R}$ and $\sqrt{\epsilon}\inf_{v_h^1\in H_h}\left(\sqrt{\epsilon}|u^1 - v_h^1|_{1,\omega} + |u^1 - v_h^1|_{1,\Omega}\right) \le C(\epsilon\sqrt{h} + h\sqrt{\epsilon})\|f\|_{0,R}$. We need only to analyze the third term

$$\inf_{v_h^0\in H_h}\left(|u^0 - v_h^0|_{1,\omega} + \frac{1}{\sqrt{\epsilon}}|v_h^0|_{1,\Omega}\right). \tag{3.38}$$

We let $v_h^0 \in H_h$ be such a piecewise linear function that interpolates u^0 for all the vertices $\nu \in \mathcal{V}$. When ω is convex, this interpolation is consistent with that defined in the proof of Theorem 3.4. It is, however, different if ω is not convex, since the latter was required to be zero on the shaded $\mathcal{T}_\Gamma$; see Figure 5.

When ω is convex, we have $\mathcal{P} \subset \omega$, and

$$|u^0 - v_h^0|_{1,\omega}^2 = |u^0|_{1,\omega\setminus\overline{\mathcal{P}}}^2 + |u^0 - v_h^0|_{1,\mathcal{P}}^2.$$

Since $\omega \setminus \overline{\mathcal{P}}$ is covered by a strip γ_δ, with $\delta = Ch^2$; see Figure 5 (left), by Lemma 2.2, we have that $|u^0|_{1,\omega\setminus\overline{\mathcal{P}}}^2 \le Ch^2\|\bar{u}^0\|_{2,R}^2$. Here $\bar{u}^0 \in H^2(R)$ is a H^2 extension of $u^0|_\omega$ such that $\|\bar{u}^0\|_{2,R} \le C\|u^0\|_{2,\omega}$. Since v_h^0 interpolates u^0 at the vertices of every triangle $\tau \in \mathcal{P}$, by the standard argument, we see that $|u^0 - v_h^0|_{1,\mathcal{P}}^2 \le Ch^2|u^0|_{2,\mathcal{P}}^2$. Therefore, $|u^0 - v_h^0|_{1,\omega} \le Ch\|f\|_{0,\omega}$. Since $v_h^0 = 0$ on Ω, we get that

$$|u^0 - v_h^0|_{1,\omega} + \frac{1}{\sqrt{\epsilon}}|v_h^0|_{1,\Omega} \le Ch\|f\|_{0,\omega}.$$

The estimate (3.36) then follows from this and Theorem 3.3.

When ω is not convex, we have that

$$|u^0 - v_h^0|_{1,\omega}^2 = |u^0|_{1,\omega\cap\mathcal{P}^c}^2 + |u^0 - v_h^0|_{1,\omega\cap\mathcal{P}}^2.$$

Here $\mathcal{P}^c = R\setminus\overline{\mathcal{P}}$. Since $\omega\cap\mathcal{P}^c$ is contained in a strip of width $\mathcal{O}(h^2)$, by Lemma 2.2, we have that $|u^0|_{1,\omega\cap\mathcal{P}^c}^2 \le Ch^2\|\bar{u}^0\|_{2,R}^2$. We see that $|u^0 - v_h^0|_{1,\omega\cap\mathcal{P}}^2 \le |\bar{u}^0 - v_h^0|_{1,\mathcal{P}}^2$. Since v_h^0 interpolates $\bar{u}^0$ on the vertices of every triangle in $\mathcal{P}$, by the standard argument of finite element interpolation, we have that $|\bar{u}^0 - v_h^0|_{1,\mathcal{P}}^2 \le Ch^2\|\bar{u}^0\|_{2,\mathcal{P}}^2$. Therefore,

$$|u^0 - v_h^0|_{1,\omega} \le Ch\|f\|_{0,\omega}. \tag{3.39}$$

Now, we consider $|v_h^0|_{1,\Omega}$. Since v_h^0 is zero on $\Omega \cap \mathcal{P}^c$, we have that $|v_h^0|_{1,\Omega}^2 = |v_h^0|_{1,\Omega\cap\mathcal{P}}^2$. Let $\mathcal{P}_\Omega = \{\tau \in \mathcal{P}; \tau \cap \Omega \ne \emptyset\}$, which is a cluster of triangles contained in a strip of width $\mathcal{O}(h)$ around Γ. On every $\tau \in \mathcal{P}_\Omega$, v_h^0 interpolates $\bar{u}^0$. According to Lemma 2.3, we have that $\sum_{\tau\in\mathcal{P}_\Omega}|v_h^0|_{1,\tau}^2 \le Ch\|\bar{u}^0\|_{2,R}^2$. For each $\tau \in \mathcal{P}_\Omega$, the portion $\tau \cap \Omega$ is either contained in a trapezoid of height $\mathcal{O}(h^2)$ by one side of τ, or a smaller triangle of diameter $\mathcal{O}(h^2)$ at a vertex of τ. In any case, we have that $|v_h^0|_{1,\tau\cap\Omega}^2 \le Ch|v_h^0|_{1,\tau}^2$. We thus have that

$$|v_h^0|_{1,\Omega\cap\mathcal{P}}^2 = \sum_{\tau\in\mathcal{P}_\Omega}|v_h^0|_{1,\tau\cap\Omega}^2 \le Ch\sum_{\tau\in\mathcal{P}_\Omega}|v_h^0|_{1,\tau}^2 \le Ch^2\|\bar{u}^0\|_{2,R}^2.$$

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Therefore, $|v_h^0|_{1,\Omega} \le Ch\|\bar{u}^0\|_{2,R}$. This together with (3.39) shows that

$$\inf_{v_h^0 \in H_h} \left(|u^0 - v_h^0|_{1,\omega} + \frac{1}{\sqrt{\epsilon}} |v_h^0|_{1,\Omega} \right) \le C \left(h + \frac{h}{\sqrt{\epsilon}} \right) \|\bar{u}^0\|_{2,R}.$$

The proof of (3.37) is complete. □

The estimates of this theorem are sharp. The sharpness argument is similar to that which follows Theorem 3.4.

From (3.36) we see that when ω is convex, the finite element domain embedding method (3.26) with the adjusted mesh achieves the full accuracy $\mathcal{O}(h)$ in the H^1 norm by taking $\epsilon = \mathcal{O}(h)$. From a computational point of view, the best value then is $\epsilon = Ch$. But smaller ϵ won't hurt the accuracy.

From (3.37) we see that if ω is not convex, then the full accuracy of $\mathcal{O}(h)$ cannot be achieved by the mesh adjustment. In this case, there is an optimal value for ϵ. It is $\epsilon = Ch^{2/3}$, which leads to an overall accuracy of order $h^{2/3}$ in the H^1 norm. Unlike the Neumann problem with a uniform mesh or the Dirichlet problem on a convex domain with adjusted mesh, for this case, a smaller ϵ thwarts the accuracy of the finite element domain embedding. It could reduce the order back to $h^{1/2}$, that is, the order of accuracy achieved by the uniform mesh.

REFERENCES

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] P. Angot, *A unified fictitious domain model for general embedded boundary conditions*, C. R. Acad. Sci. Paris, Ser. I, 341 (2005), pp. 683–688.

[3] D.N. Arnold and R. Falk, *A uniformly accurate finite element method for the Reissner–Mindlin plate*, SIAM J. Numer. Anal., 26 (1989), pp. 1276–1290.

[4] I. Babuška, *The finite element method with penalty*, Math. Comp., 27 (1973), pp. 221–228.

[5] N.S. Bakhvalov and A.V. Knyazev, *Preconditioned iterative methods in a subspace for linear algebraic equations with large jumps in the coefficients*, in Domain Decomposition Methods in Science and Engineering, Contemp. Math. 180, D. Keyes and J. Xu, eds., AMS, Providence, RI, 1994.

[6] P. Bochev and R.B. Lehoucq, *On the finite element solution of the pure Neumann problem*, SIAM Rev., 47 (2005), pp. 50–66.

[7] C. Börgers, *A triangulation algorithm for fast elliptic solvers based on domain embedding*, SIAM J. Numer. Anal., 27 (1990), pp. 1187–1196.

[8] C. Börgers and O.B. Widlund, *On finite element domain imbedding methods*, SIAM J. Numer. Anal., 27 (1990), pp. 963–978.

[9] F. Brezzi, J.-L. Lions, and O. Pironneau, *The Chimera method for a model problem*, in Numerical Mathematics and Advanced Applications -ENUMATH 2001, F. Brezzi, A. Buffa, S. Corsaro, and A. Murli, eds., Springer, New York, 2003, pp. 817–826.

[10] P.G. Ciarlet, *Finite Element Method for Elliptic Equations*, North-Holland, Amsterdam, 1978.

[11] S. Del Pino and O. Pironneau, *A fictitious domain based general PDE solver*, in Numerical Methods for Scientific Computing: Variational Problems and Applications, Y. Kuznetsov, P. Neittanmaki, and O. Pironneau, eds., Internat. Center Numer. Methods Eng., Barcelona, 2003.

[12] K.O. Friedrichs and H.B. Keller, *A finite difference scheme for generalized Neumann problems*, in Numerical Solutions of Partial Differential Equations, J.H. Bramble, ed., Academic Press, New York and London, 1966.

[13] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1983.

[14] R. Glowinski and T.W. Pan, *Error estimate for fictitious domain/penalty/finite element methods*, Calcolo, 29 (1992), pp. 125–141.

[15] R. Glowinski, T.W. Pan, R. O. Wells, Jr., and X. Zhou, *Wavelet and finite element solutions for the Neumann problem using fictitious domains*, J. Comput. Phys., 126 (1996), pp. 40–51.

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[16] P. Grisvard, *Singularities in Boundary Value Problems*, Masson, Springer, New York, 1992.

[17] A. Knyazev and O. Widlund, *Lavrentiev regularization + Ritz approximation = uniform finite element error estimates for differential equations with rough coefficients*, Math. Comp., 72 (2003), pp. 17–40.

[18] Yu. A. Kuznetsov, *Domain decomposition and fictitious domain methods with distributed Lagrange multipliers*, in Domain Decomposition Methods in Science and Engineering, N. Debit, M. Garbey, R. Hoppe, D. Keyes, Y. Kuznetsov, and J. Periaux, eds., Internat. Center Numer. Methods Eng., Barcelona, 2002.

[19] R. Lazarov, S. Lu, and S.V. Pereverzev, *On the balancing principle for some problems of numerical analysis*, Numer. Math., 106 (2007), pp. 659–689.

[20] S. Pereverzev and E. Schock, *On the adaptive selection of the parameter in regularization of ill-posed problems*, SIAM J. Numer. Anal., 43 (2005), pp. 2060–2076.

[21] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, NJ, 1970.

[22] S. Zhang, *Equivalence estimates for a class of singular perturbation problems*, C. R. Acad. Sci. Paris Ser. I, 342 (2006), pp. 285–288.

[23] S. Zhang, *A domain embedding method for mixed boundary value problems*, C. R. Acad. Sci. Paris Ser. I, 343 (2006), pp. 287–290.

[24] S. Zhang, *Sharp Convergence Rate of Domain Embedding Methods for Various Boundary Conditions*, http://www.math.wayne.edu/~sheng/embedding.pdf (July 14, 2008).

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