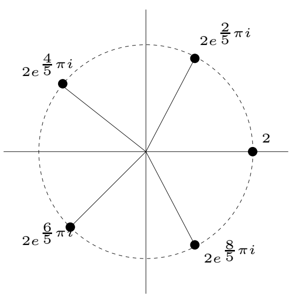
MAT 5230 MIDTERM I

Student Name: **Student Id:**

1. (10 pts.) Given $a = 4 + 3i$ and $b = 2e^{i\pi/2}$, find the values of $a + b$, ab, a/b, and a^4.
amswer: Note that $b = 2e^{i\pi/2} = 2i$. Thus $a + b = 4 + 5i$, $ab = -6 + 8i$, $a/b = \frac{3}{2} - 2i$. To calculate a^4, we start with $a^2 = 7 + 24i$. Then $a^4 = (7 + 24i)^2 = -527 + 336i$.

2. (10 pts.) Find and plot the 5 distinctive values of $32^{1/5}$.
amswer:

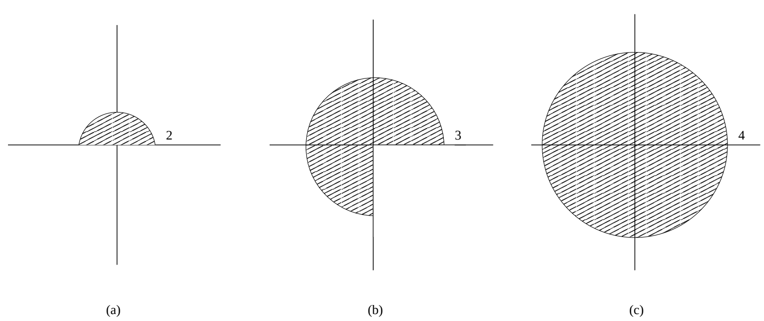


Generally, the n'th roots of a number $re^{i\theta}$ are

$$\sqrt[n]{r}e^{i\frac{\theta}{n}+i\frac{2k\pi}{n}} \quad k = 0, 1, \cdots, n-1.$$

For the particular number 32, the fifth roots are $2e^{i\frac{2k\pi}{5}}$, $k = 0, 1, 2, 3, 4$.

3. (10 pts.) Sketch the region onto which the section $r \le 1$, $0 \le \theta \le \pi/2$ is mapped by the transformation a) $w = 2z^2$, b) $w = 3z^3$, c) $w = 4z^4$.
amswer: It is better to represent numbers in the polar form $z = re^{i\theta}$. Then $2z^2 = 2r^2e^{2i\theta}$, etc. The mapped regions are various sectors of circles of varing radius.



4. (10 pts.) Let $f(z)$ be analytic in a domain D. Prove that if $|f(z)|$ is constant throughout D, then $f(z)$ must be constantly valued on D.
amswer: Let $f(z) = u(x, y) + iv(x, y)$. Then $|f(z)| = C$, a constant, means $u^2 + v^2 = C^2$. Differentiate this equation with respect to x and y, respectively, we get

$$2uu_x + 2vv_x = 0, \quad 2uu_y + 2vv_y = 0.$$

Time the first equation by u, and the second equation by v, add the result, and invoke the Cauchy–Riemann equation that $u_x = v_y$ and $u_y = -v_x$, we get $2(u^2 + v^2)u_x = 0$. Thus $u_x = v_y = 0$. Using a similar technique, playing around with the above two equations, one gets $u_y = -v_x = 0$. Therefore, both u and v are constants.

5. (15 pts.) Show that $f'(z)$ does not exist at any point z when a) $f(z) = \bar{z}$, b) $f(z) = x^2 + y^2$, c) $f(z) = i \sin y$.
amswer: All these can be answered by using the Cauchy–Riemann.

a) In terms of $f = u + iv$, we have $u = x$ and $v = -y$. The first C-R that $u_x = v_y$ is broken at every point.

b) We have that $u = x^2 + y^2$ while $v = 0$. The C-R that $u_x = v_y$ would mean $x = 0$, while $u_y = -v_x$ would mean $y = 0$. Such f may have derivative at the single point $z = 0$, but it can not be analytic there. Since analyticity requires existence of derivative in a neighbourhood of the point.

c) We have $u = 0$ and $v = \sin y$. The C-R that $u_x = v_y$ means $\cos y = 0$. This requires that $y = \frac{\pi}{2} + 2n\pi$ for $n = 0, \pm 1, \pm 2, \cdots$. The derivative of such f does not exist at points except on these horizontal lines. The function is not analytic anywhere.

6. (15 pts.) Find the multiple values of a) $\log(-1)$, b) $(i+1)^i$, and c) $\cos^{-1}(-2)$. Note that $\cos^{-1}$ is the inverse of cos.

amswer:

a) $\log(-1) = \ln 1 + i[\arg(-1) + 2k\pi] = i(1 + 2k)\pi$ for $k = 0, \pm 1, \pm 2, \cdots$.

b)

$$(i+1)^i = \exp[i \log(i+1)] = \exp\{i[\frac{1}{2}\ln 2 + i(\frac{\pi}{4} + 2k\pi)]\} = \exp(i\frac{1}{2}\ln 2)\exp[-(\frac{\pi}{4} + 2k\pi)].$$

c) Let $w = \cos^{-1}(-2)$. This means that $\cos w = -2$. Thus $e^{iw} + e^{-iw} = -4$. This can be written as $(e^{iw})^2 + 4e^{iw} + 1 = 0$. Solve this quadratic equation for e^{iw}, we get $e^{iw} = -2 \pm \sqrt{3}$. Therefore $iw = \log(-2 \pm \sqrt{3} = \ln(2 \pm \sqrt{3}) + i(2k+1)\pi$. And so, $w = (2k+1)\pi - i\ln(2 \pm \sqrt{3})$ for $k = 0, \pm 1, \pm 2, \cdots$.