MAT 5230 MIDTERM EXAM II

Student Name: **Student Id:**

1. (10 pts.) Evaluate the integral $\int_C f(z)dz$, with $f(z) = (z+2)/z$ and C being the upper semi-circle of $|z| = 2$, directed from 2 to -2.
Answer: We parameterize the path by $z = 2e^{i\theta}$, $\theta \in [0, \pi]$, and write the integral as

$$\int_C f(z)dz = \int_0^{\pi} (1 + \frac{2}{2e^{i\theta}})2ie^{i\theta}d\theta = -4 + 2\pi i.$$

2. (10 pts.) Find the value of the integral $\int_0^{\pi} e^{(1+i)x}dx$. Use your result to determine the values of $\int_0^{\pi} e^x \cos x dx$ and $\int_0^{\pi} e^x \sin x dx$
Answer: This integral could be viewed as contour integral of $e^{(1+i)z}$ along the straight line from 0 to π on the $\mathbb{C}$-plane. Thus, by the fundamental theorem

$$\int_0^{\pi} e^{(1+i)x}dx = \int_0^{\pi} e^{(1+i)z}dz = \frac{1}{1+i}e^{(1+i)z}\Big|_0^{\pi} = (-1+i)(e^{\pi}+1)/2.$$

The integrals $\int_0^{\pi} e^x \cos x dx = -(e^{\pi}+1)/2$ and $\int_0^{\pi} e^x \sin x dx = (e^{\pi}+1)/2$, as, respectively, the real and imaginary part of the above number.

3. (10 pts.) Let C_R denote the upper half of the circle $|z| = R$ ($R > 2$), taken in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}dz\right| \le \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}.$$

Answer: We use the estimate that $|\int_C f(z)dz| \le LM$, with L being the length of C and M the maximum modulus of $f(z)$ on C. For this problem, obviously $L = \pi R$. The integrant is a fraction whose top is bounded in modulus from above by $|2z^2 - 1| \le 2R^2 + 1$, and the bottom is bounded from below by $|z^4 + 5z^2 + 4| = |(z^2+1)(z^2+4)| \ge (R^2-1)(R^2-4)$. Thus we have $|\frac{2z^2-1}{z^4+5z^2+4}| \le \frac{(2R^2+1)}{(R^2-1)(R^2-4)}$.

4. (10 pts.) Let C be an arbitrary contour from 1 to -1 that does not pass 0 on the complex plane. Find the value of the integral

$$\int_C \frac{1}{z}dz.$$

Note that there are two possibly different values.
Answer: Since $1/z$ is analytic except at 0. Any contour that goes above 0 from 1 to -1 gives the same integral value as the upper semi-circle. The latter could be parameterized as $z = e^{i\theta}$ for $\theta \in [0, \pi]$. Thus for such C, we have $\int_C (1/z)dz = \int_0^{\pi} id\theta = \pi i$.

If the conture goes under 0, we could use the lower semi-circle, directed from 1 to -1. A similar parameterization gives the value $-\pi i$.

5. (15 pts.) Let C be the positively oriented boundary of the square $x = \pm 2$ and $y = \pm 2$. Find the integral

$$\int_C \frac{\cos z}{z^3(z^2+10)} dz.$$

Answer: We use the Cauchy integral formula for derivatives that $f''(z_0) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz$. Taking $f(z) = \frac{\cos z}{z^2+10}$ and $z_0 = 0$, we see that

$$\int_C \frac{\cos z}{z^3(z^2+10)} dz = \pi i f''(0) = -\pi i \frac{12}{100}.$$

6. (15 pts.) For the function $f(z) = \frac{1}{1+z}$, (1) find its Taylor expansion about 0 and determine the radius of convergence, and (2) find its Laurent expansion on the annulus $1 < |z| < \infty$.

Answer: We use the identity that when $|z| < 1$ we have $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ to solve both. (1)

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-1)^n z^n.$$

The convergence radius is 1, because $\frac{1}{1+z}$ is analytic in the open unit circle centered at 0, but not on any bigger circle.

(2)

$$\frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1-(-z^{-1})} = \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n z^{-n} = \sum_{n=0}^{\infty} (-1)^n z^{-(n+1)} = \sum_{n=1}^{\infty} (-1)^{n+1} z^{-n}$$