

MAT 5230 HOMEWORK AND ANSWERS

HOMEWORK 1

1(P.35:4). Write the function $f(z) = z + \frac{1}{z}$ ($z \neq 0$) in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

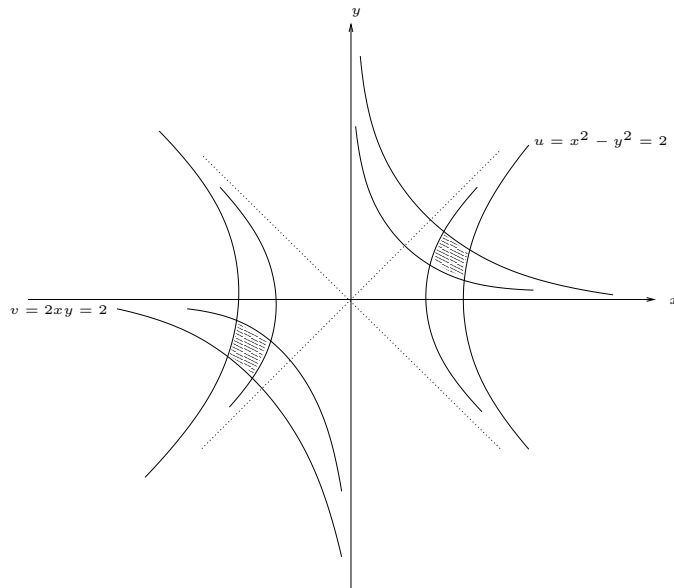
Answer:

$$f(z) = z + \frac{1}{z} = re^{i\theta} + \frac{1}{r}e^{-i\theta}.$$

Using the Euler equation that $e^{i\theta} = \cos \theta + i \sin \theta$, one sees this is equal to $(r + \frac{1}{r}) \cos \theta + i(r - \frac{1}{r}) \sin \theta$.

2(P.42:1). Find a domain in the z -plane whose image under the transformation $w = z^2$ is the square domain in the w -plane bounded by the lines $u = 1$, $u = 2$, $v = 1$, $v = 2$.

Answer: In terms of its real and imaginary parts, the mapping $w = z^2$ is $u + iv = (x^2 - y^2) + 2xyi$. Thus the preimage of the w -plane curve $u = a$ ($a \in \mathbb{R}$) in the z -plane is the hyperbola $x^2 - y^2 = a$, and the preimage of the curve $v = b$ ($b \in \mathbb{R}$) is the hyperbola $2xy = b$. Let $a \in [1, 2]$ and $b \in [1, 2]$, one will see the preimage of the square in the z -plane, which comprises two distinct parts, and is bounded by hyperbolas on the z -plane.



3(P.42:4). Show that the lines $ay = x$ ($a \neq 0$) are mapped onto the spiral $\rho = \exp(a\phi)$ under the transformation $w = e^z$, where $w = \rho e^{i\phi}$.

Answer: The image of the line $x = ay$ is the set in the w -plane $\{w = e^{ay+iy} = e^{ay} e^{iy}; y \in \mathbb{R}\}$. In the polar form, we write $w = \rho e^{i\phi}$. Comparing, we see the relationship between the two polar coordinates that $\rho = e^{a\phi}$, which is a spiral.

4(P.42:5). By considering the image of horizontal line segments, verify that the image of the rectangular region $a \leq x \leq b$, $c \leq y \leq d$ under the transformation $w = e^z$ is the region $e^a \leq \rho \leq e^b$, $c \leq \phi \leq d$.

Answer: A horizontal line $y = h$ is mapped to the half ray $e^x e^{ih}$, $x \in \mathbb{R}$. When $c \leq y \leq d$, these rays sweep the sector $c \leq \phi \leq d$. The restriction $a \leq x \leq b$ then defines the fan-shaped domain.

5(P.53:5). Show that the limit of the function

$$f(z) = \left(\frac{z}{\bar{z}} \right)^2$$

as $z \rightarrow 0$ does not exist.

Answer: As hinted in the book, we take two paths for $z \rightarrow 0$. When $z = (x, 0)$, $\bar{z} = z$, thus $f(z) = 1$. The limit along this path is 1. When $z = (x, x)$, $\bar{z} = (x, -x)$, and $z/\bar{z} = (x + ix)/(x - ix) = i(x - ix)/(x - ix) = i$. Thus $f(z) = -1$, which is the limit along the diagonal. Since the limits in the paths are different, $\lim_{z \rightarrow 0} f(z)$ does not exist.

6(P.60:7). Prove that $\frac{d}{dz} z^n = n z^{n-1}$ for n being negative integers, suppose this is true when n is a positive integer.

Answer: Use quotient rule of differentiation.

7(P.60:8). Show that $f'(z)$ does not exist at any point z , when

$$(a) f(z) = \bar{z}; \quad (b) f(z) = \operatorname{Re} z; \quad (c) f(z) = \operatorname{Im} z.$$

Answer: You need to do this by using the definition of derivatives. One can also use the Cauchy–Riemann equation. (a). In terms of its real and imaginary parts, we have $u = x$, $v = -y$. The Cauchy–Riemann $u_x = v_y$ is broken at every point, although the second equation $u_y = -v_x$ is valid every where.

The other two problems are similar.

8(P.60:9). Let

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{when } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases}$$

Show that this function has no derivative at $z = 0$.

Answer: We calculate the divided difference $\frac{\Delta w}{\Delta z}$ at 0. First, we choose $\Delta z = \Delta x$, we see $\frac{\Delta w}{\Delta z} = (\Delta x)^2 / (\Delta x)^2 = 1$. Thus the limit is 1.

Then, we choose $\Delta z = \Delta x + i\Delta x$. We see $\frac{\Delta w}{\Delta z} = \bar{\Delta z}^2 / \Delta z^2 = -1$. The divided difference converges to different limits along the two paths. Thus $f'(0)$ does not exist.