MAT 5230 HOMEWORK AND ANSWERS

Homework 2, Due 2/18/2009

1(P.74:2). With the aid of the theorem in Sec. 20 (Cauchy–Riemann equation), show that each of the functions is nowhere analytic:

$$(a)\ f(z) = xy + iy; \quad (b)\ f(z) = 2xy + i(x^2 - y^2); \quad (c)\ f(z) = e^y e^{ix}.$$

answer. (a) In terms of $f = u + iv$, we have $u = xy$, $v = y$. Then CR means $y = 1$ and $x = 0$. This means that the CR is only satisfied at the single point $(0, 1)$. The function is nowhere analytic.

(b) We have $u = 2xy$ and $v = x^2 - y^2$. It follows from CR that $2y = -2y$ and $2x = -2x$. Thus the point $(0, 0)$ is the only one where CR holds. The function is nowhere analytic.

(c) We have $u = e^y \cos x$ and $v = e^y \sin x$. CR leads to $-e^y \sin x = e^y \sin x$ and $e^y \cos x = -e^y \cos x$. This is equivalent to $\sin x = 0$ and $\cos x = 0$. There is not a single point where CR holds.

2(P.74:7(a)). Let a function $f(z)$ be analytic in a domain D. Prove that $f(z)$ must be constant throughout D if (a) $f(z)$ is real valued for all z in D.
Answer. In the form of $f = u + iv$, we have $v = 0$. Thus $u_x = v_y = 0$ and $u_y = -v_x = 0$. Therefore, $u =$ constant.

3(P.90:9). Show that $\overline{\exp(iz)} = \exp(i\bar{z})$ if and only if $z = n\pi$ $(n = 0, \pm 1, \pm 2, \cdots)$.
Answer. Let $z = x + iy$. We have $\overline{\exp(iz)} = e^{-y}e^{-ix}$ and $\exp(i\bar{z}) = e^y e^{ix}$. Equate the modulus of these two expressions, we get $e^{-y} = e^y$. This is to say that $y = 0$. Then the equation $\overline{\exp(iz)} = \exp(i\bar{z})$ is reduced to $\cos x - i \sin x = \cos x + i \sin x$. This is to say that $\sin x = 0$. Thus $z = x = n\pi$ $(n = 0, \pm 1, \pm 2, \cdots)$.

4(P.95:11). Show that

$$\mathrm{Re}[\log(z - 1)] = \frac{1}{2} \ln[(x - 1)^2 + y^2] \quad (z \neq 1).$$

Why must this function satisfy Laplace's equation when $z \neq 1$?
Answer. Since $\log z = \ln |z| + i \arg z$, we have $\mathrm{Re}[\log z] = \ln |z|$. Therefore, $\mathrm{Re}[\log(z - 1)] = \ln |z - 1| = \frac{1}{2} \ln[(x - 1)^2 + y^2]$. Note that the real part of the log function is independent of which branch one uses for the multiple-valued function. Consider two branches of log defined with the branch cuts $x \leq 1$, and $x \geq 1$, one would be able to conclude that $\frac{1}{2} \ln[(x-1)^2 + y^2]$, as the real part of an analytic function, is harmonic on $\mathbb{C} \setminus \{1\}$.

5(P.99:1a). Show that when $n = 0, \pm 1, \pm 2, \cdots$,

$$(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right) \exp\left(\frac{i}{2}\ln 2\right).$$

Answer. The multiplly valued expression $(1+i)^i$ is defined as

$$(1+i)^i = \exp[i \log(1+i)] = \exp[i(\ln|1+i| + i \arg(1+i))].$$

Since $|1+i| = \sqrt{2}$ and $\arg(1+i) = \frac{\pi}{4} - 2n\pi$ $n = 0, \pm 1, \pm 2, \cdots$, we have the desired expression.

6(P.99:2b). Find the principal value of

$$\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i)}.$$

Answer. The principal value is defined in terms of the principal branch of the log function involved in the definition of such powers. Thus

$$\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i)} = \exp\{3\pi i \operatorname{Log}[\frac{e}{2}(-1-\sqrt{3}i)]\}.$$

Here $\operatorname{Log}[\frac{e}{2}(-1-\sqrt{3}i)] = \ln|\frac{e}{2}(-1-\sqrt{3}i)| + i \operatorname{Arg}[\frac{e}{2}(-1-\sqrt{3}i)]$. One needs to observe that $\operatorname{Arg}[\frac{e}{2}(-1-\sqrt{3}i)] = -\frac{2\pi}{3}$. It is straightforward to show that the final result is $-\exp(2\pi^2)$.

7(P.105:17). Find all roots of the equation $\sin z = \cosh 4$ by equating the real parts and the imaginary parts of $\sin z$ and $\cosh 4$.

Answer. Let $z = x + iy$. Then $\sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$. Equate this to $\cosh 4$, we find that $\sin x \cosh y = \cosh 4$ and $\cos x \sinh y = 0$. The latter condition requires either $y = 0$ or $\cos x = 0$. But $y = 0$ would force the equation $\sin x \cosh y = \cosh 4$ to $\sin x = \cosh 4$, which has no solution. We are left with $\cos x = 0$, which with the condition $\sin x \cosh y = \cosh 4$ determines that $y = \pm 4$ and $x = \frac{\pi}{2} + 2n\pi$, $n = 0, \pm 1, \pm 2, \cdots$.

8(P.110:7). Derive the formula for the inverse of cosh function.

$$\cosh^{-1} z = \log[z + (z^2-1)^{1/2}].$$

Answer. Let $\cosh^{-1} z = w$. We wish to determine an expression for w. This equation is equivalent to

$$z = \cosh w = \frac{e^w + e^{-w}}{2}.$$

This could be written as $(e^w)^2 - 2ze^w + 1 = 0$. Solve this equation for e^w, one gets $e^w = z + (z^2-1)^{1/2}$. Thus $w = \log[z + (z^2-1)^{1/2}]$, which is multiplly valued.