MAT 5230 HOMEWORK AND ANSWERS

Homework 3, Due 3/4/2009

1(P.115:2). Evaluate the following integrals:

$$a)\ \int_1^2 \left(\frac{1}{t}-i\right)^2 dt;\quad b)\ \int_0^{\pi/6} e^{i2t}dt;\quad c)\ \int_0^{\infty} e^{-zt}dt\ (\operatorname{Re} z > 0).$$

Answer: These are complex valued but real variable function intergals. One treats them as two real functions, and could use the real FTC. Thus

a) $\int_1^2 \left(\frac{1}{t}-i\right)^2 dt = -t^{-1} - t - 2i\ln t]_1^2 = -\frac{1}{2} - i\ln 4.$

b) $\int_0^{\pi/6} e^{i2t}dt = \frac{1}{2i}e^{i2t}]_0^{\frac{\pi}{6}} = \frac{\sqrt{3}}{4} + \frac{1}{4}i.$

c) $\int_0^{\infty} e^{-zt}dt = -\frac{1}{z}e^{-zt}]_0^{\infty} = \frac{1}{z}$. Here, we used the assumption that $\operatorname{Re} z > 0$.

2(P.129:3). Use parametric representation for C, or legs of C, to evaluate $\int_C f(z)dz$. Here $f(z) = \pi\exp(\pi\bar{z})$ and C is the boundary of the square with vertices at 0, 1, $1+i$, and i, the orientation of C being in the counterclockwise direction.

Answer: We parameterize the four legs respectively by $z = x$ with $x \in [0,1]$, $z = 1 + iy$ with $y \in [0,1]$, $z = x + i$ with $x \in [0,1]$ (reversed), and $z = iy$ with $y \in [0,1]$ (reversed). Then

$$\begin{aligned}\int_C f(z)dz &= \int_0^1 \pi e^{\pi x}dx + \int_0^1 \pi e^{\pi(1-yi)}idy - \int_0^1 \pi e^{\pi(x-i)}dx - \int_0^1 \pi e^{\pi(-yi)}idy\\ &= (e^{\pi}-1) + e^{\pi}(1-e^{-i\pi}) - e^{-i\pi}(e^{\pi}-1) - (1-e^{-i\pi}) = 4e^{\pi} - 4.\end{aligned}$$

3(P.129:4). Do the same thing as in the last problem with

$$f(z) = \begin{cases} 1 & \text{when } y < 0,\\ 4y & \text{when } y > 0,\end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

Answer: We parameterize the curve by $z = x + x^3 i$. Then

$$\int_C f(z)dz = \int_{-1}^0 (1 + 3x^2 i)dx + \int_0^1 4x^3(1 + 3x^2 i)dx = 2 + 3i.$$

4(P.130:10). Let C_0 denote the circle $|z - z_0| = R$, taken counterclockwise. Use the parametric representation $z = z_0 + Re^{i\theta}$ $(pi \le \theta \le \pi)$ for C_0 to derive the following integral formulas:

$$(a)\ \int_{C_0} \frac{dz}{z - z_0} = 2\pi i;\quad (b)\ \int_{C_0} (z - z_0)^{n-1}dz = 0\ (n = \pm 1, \pm 2, \cdots).$$

Answer: Let's generally consider the integral $\int_{C_0}(z - z_0)^k dz$. By the suggested parameterization, we see

$$\int_{C_0} (z - z_0)^k dz = \int_0^{2\pi} (Re^{i\theta})^k Rie^{i\theta}d\theta = i\int_0^{2\pi} (Re^{i\theta})^{k+1}d\theta.$$

This is equal to $2\pi i$ for $k=-1$. For other integer k values this is equal to $R^{k+1}\frac{1}{k+1}e^{i(k+1)\theta}]_0^{2\pi}=0$.

5(P.133:2). Let C denote the line segment from i to 1. By observing that, of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$$

without evaluating the integral.
Answer: This is seen from the estimate that $|\int_C f(z)dz| \le LM$, in which L is the length of C and M is the maximum modulus of $f(z)$ for z on C.

6(P.142:5). Show that

$$\int_{-1}^{1} z^i dz = \frac{1+e^{-\pi}}{2}(1-i),$$

where z^i denote the principal branch

$$z^i = \exp(i \operatorname{Log} z) \quad (|z|>0, -\pi < \operatorname{Arg} z < \pi)$$

and where the path of integration is any contour from -1 to 1 that, except for its end points, lies above the real axis.
Answer: One could use the fundamental theorem because this integrant has the antiderivative $\frac{z^{i+1}}{i+1}$, in which z^{i+1} is defined on the same branch cut as z^i. That is $z^{i+1}=\exp((i+1)\operatorname{Log} z)$. To avoid the point -1 that is cut out, we use a limit process: One then uses

$$\int_{e^{i(\pi-\epsilon)}}^{1} z^i dz = \frac{1}{i+1} z^{i+1}]_{e^{i(\pi-\epsilon)}}^{1} = \frac{1}{i+1}\left[1-e^{-(\pi-\epsilon)}e^{i(\pi-\epsilon)}\right].$$

Then, we let $\epsilon \to 0$ to prove the result.

7(P.153:1c). Apply the Cauchy–Goursat theorem to show that $\int_C f(z)dz=0$ where C is the circle $|z|=1$ in either direction and

$$f(z)=\frac{1}{z^2+2z+2}.$$

Answer: The function f is analytic at every point in $\mathcal{C}$, except the roots of the bottom polynomial. These are $-1+i$ and $-1-i$. This means that f is analytic on and in the interior of the unit circle. Therefore $\int_C f(z)dz=0$, according to Cauchy–Goursat.

8(P.153:2c). Let C_1 denote the positively oriented circle $|z|=4$ and C_2 the positively oriented boundary of the square whose sides lie along the lines $x=\pm 1, y=\pm 1$. Prove why

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz,$$

where $f(z)=z/(1-e^z)$.
Answer: The points where f is not analytic are $2k\pi i$ for all integer k. Thus on C_1 and C_2 and in the interior of the domain between them f is analytic. The claim then follows from a variant of Cauchy–Goursat.

9(P.154:4). Use the method described in the book to derive the integration formula

$$\int_0^\infty e^{-x^2}\cos 2bx dx = \frac{\sqrt{\pi}}{2}e^{-b^2} \quad (b > 0).$$

Answer: Consider the analytic function $\exp(-z^2) = e^{-x^2}e^{y^2}(\cos 2xy - i\sin 2xy)$ and the rectangular closed contour C defined by $x = \pm a$ and $y = 0$ and $y = b$, directed positively. Thus $\int_C \exp(-z^2)dz = 0$. We parameterize the horizontal sides by x and vertical sides by y, and write the closed contour integral as

$$\int_{-a}^{a} e^{-x^2}dx - \int_{-a}^{a} e^{-x^2}e^{b^2}(\cos 2xb - i\sin 2xb)dx + \int_0^b e^{-a^2}e^{y^2}(\cos 2ay - i\sin 2ay)idy$$
$$- \int_0^b e^{-a^2}e^{y^2}(\cos 2ay + i\sin 2ay)idy = 0.$$

Equating the real and imaginary parts of both sides of this equation, we get

$$\int_{-a}^{a} e^{-x^2}dx - \int_{-a}^{a} e^{-x^2}e^{b^2}\cos 2xbdx + 2\int_0^b e^{-a^2}e^{y^2}\sin 2aydy = 0.$$

We then let $a \to \infty$, invoke the well-known integral $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$, use the fact that $e^{-a^2}\int_0^b e^{y^2}\sin 2aydy \to 0$ as $a \to \infty$, and use symmetry to prove the result.

10(P.162:1c). Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the integral

$$\int_C \frac{zdz}{2z+1}.$$

Answer: We write

$$\int_C \frac{zdz}{2z+1} = \frac{1}{2}\int_C \frac{zdz}{z+\frac{1}{2}} = -\frac{1}{2}\pi i.$$

The last step follows from the Cauchy integral formula.

11(P.162:1e). For the same curve, evaluate

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz \ (-2 < x_0 < 2).$$

Answer: The function tan is not analytic at the zeros of cos. Thus $\tan(z/2)$ is singular at $\pi + 2k\pi$. This means tan is analytic on C and its interior. We can use Cauchy integral formula for derivatives to get

$$\int_C \frac{\tan(z/2)}{(z-x_0)^2}dz = 2\pi i \sec^2 x_0.$$

12(P.163:2a). Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when $g(z) = 1/(z^2 + 4)$.
Answer: The function g has two singular points, $z = \pm 2i$, of which only $2i$ is in the circle. Thus, using the Cauchy integral formula,

$$\int \frac{1}{z^2+4}dz = \int \frac{1}{z-2i}\frac{1}{z+2i}dz = 2\pi i\frac{1}{4i} = \frac{\pi}{2}.$$

13(P.163:2b). Do the same thing as in the above problem with $g(z) = 1/(z^2+4)^2$.
Answer:

$$\int \frac{1}{(z^2+4)^2} dz = \int \frac{1}{(z-2i)^2} \frac{1}{(z+2i)^2} dz = 2\pi i(-2)(4i)^{-3} = \frac{\pi}{16}.$$

14(P.163:4). Let C be any simple closed contour, described in the positive sense, and write

$$g(w) = \int_C \frac{z^3+2z}{(z-w)^3} dz.$$

Show that $g(w) = 6\pi i w$ when w is inside C and $g(w) = 0$ when w is outside C.
Answer: When w is inside C, we use Cauchy integral formula for the second derivative. Thus

$$g(w) = \int_C \frac{z^3+2z}{(z-w)^3} dz = \pi i \left. \frac{d^2(z^3+2z)}{dz^2} \right|_{z=w} = 6\pi i w.$$

When w is outside of C, by Cauchy–Goursat, we have $g(w) = 0$.