MAT 5230 HOMEWORK AND ANSWERS

Homework 4, Due 4/1/2009

1(P.171:1). Let f be an entire function such that $|f(z)| \le A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant.
Answer: We show that $f'' = 0$ so that $f(z) = a_1 z + a_0$. Centered at a $z \in \mathbb{C}$, we choose a big circle of radius R, denoted by C_R. Use the Cauchy's formula for second derivatives,

$$|f''(z)| = |\frac{1}{\pi i}\int_{C_R} \frac{f(w)}{(w-z)^3}dw| \le 2R\frac{|z|+R}{R^3}.$$

Since R could be taken arbitarily large, we see $f'' = 0$. Thus, $f(z) = a_1 z + a_0$. The condition $|f(z)| \le A|z|$ obviously forces $a_0 = 0$.

2(P.172:6). Consider the function $f(z) = (z+1)^2$ and the closed triangular region R with vertices at 0, 2, and i. Find points in R where $|f(z)|$ has its maximum and minimum values.
Answer: Obviously $|f(z)| = |z-(-1)|^2$. Thus we need to find the point z on R that maximizes or minimizes $|z-(-1)|$, which is the distance from z to -1 in the metric of $\mathbb{C}$. The maximum point is 2, and the minimum is 0.

3(P.181:4). Write $z = re^{i\theta}$, where $0 < r < 1$, in the summation formula $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ when $|z| < 1$. Then show that

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta + r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta+r^2}$$

when $0 < r < 1$.
Answer: Note that when $0 < r < 1$ we have $|z| = |re^{i\theta}| < 1$. Therefore, $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ when $|z| < 1$. Splitting the real and imaginary parts of both sides of this equation, we get the desired the result.

4(P.189:3). Find the Maclaurin series expansion of the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9}\cdot\frac{1}{1+(z^4/9)}.$$

Also determine the convergence radius.
Answer:

$$\frac{1}{1+(z^4/9)} = \sum_{n=0}^{\infty}(-z^4/9)^n, \quad |z^4/9| < 1.$$

Thus

$$f(z) = \frac{z}{9}\cdot\frac{1}{1+(z^4/9)} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+1}}{9^{n+1}}.$$

This is the Maclaurin series expansion, whose radius of convergence is determined by $|z^4/9| = 1$. That is $\sqrt{3}$.

5(P.189:7). Derive the Taylor series representation of $\frac{1}{1-z}$ about i, and determine the convergence radius.
Answer: We often use the fact $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ when $|z| < 1$ to derive series expansions, which is a Taylor series about 0 of $\frac{1}{1-z}$. To do the expansion of the same function but about i, we write

$$\frac{1}{1-z} = \frac{1}{1-i-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-\frac{z-i}{1-i}} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}.$$

This is convergent when $|(z-i)/(1-i)| < 1$. Thus the convergence radius is $\sqrt{2}$.

6(P.190:13). Show that when $0 < |z| < 4$,

$$\frac{1}{4z-z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

Answer:

$$\frac{1}{4z-z^2} = \frac{1}{4z} \cdot \frac{1}{1-(z/4)} = \frac{1}{4z} \sum_{n=0}^{\infty} \frac{z^n}{4^n} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

7(P.198:3). Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+(1/z)}$$

in negative powers of z that is valid when $1 < |z| < \infty$.
Answer:

$$\frac{1}{z} \cdot \frac{1}{1+(1/z)} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} (-1/z)^n = \sum_{n=1}^{\infty} (-1)^{n-1} z^{-n}.$$

8(P.199:8). (a) Let a denote a real number in the interval $(-1, 1)$, and derive the Laurent series representation

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \quad (|a| < |z| < \infty).$$

(b) Write $z = e^{i\theta}$ in the above equation, and then equate the real parts and imaginary parts on each side to derive the summation formula

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}$$

when $-1 < a < 1$. Compare this with that of number 3 in this homework set.
Answer: Part (a) is easy. For $a \in (-1, 1)$ and $z = e^{i\theta}$, we have $|a| < |z|$, thus the validity of the Laurent expansion in part (a). Part be is done by comparing the real and imaginary parts of the identity in part (a). This result is more general than that of number 3 in this homework set.

9(P.200:11). (a) Let f be analytic in some annular domain containing the unit circle. By taking that circle as the path of integration in the Laurent expansion, show that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\phi})d\phi + \frac{1}{2\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} f(e^{i\phi})\left[\left(\frac{z}{e^{i\phi}}\right)^n + \left(\frac{e^{i\phi}}{z}\right)^n\right]d\phi$$

when z is any point in the annular domain.

(b) Write $u(\theta) = \mathrm{Re}(f(e^{i\theta})$, and show how it follows from this expansion that

$$u(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\phi)d\phi + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} u(\phi)\cos[n(\theta-\phi)]d\phi.$$

This is one form of the Fourier series.

Answer: The function f can be expanded as a Laurent in the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \quad c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}dz.$$

Here C is the unit circle. We parameterize C by $z = e^{i\phi}$ with $\phi \in [-\pi, \pi]$. With this, we see

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(e^{i\phi})}{e^{in\phi}}d\phi.$$

Part (a) is proved by a rearrangement of terms in the expansion.

Restricted on the unit circle, we write $f(e^{i\phi}) = u(\phi) + iv(\phi)$, and $z = e^{i\theta}$. The real part of the identity in part (a) is the desired result of part (b).

10(P.219:3). Use division to obtain the Laurent series representation

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \cdots \quad (0 < |z| < 2\pi).$$

Answer: The function $\frac{1}{e^z-1}$ is analytic except at $z = 2k\pi i$. It is, therefore, analytic on the annulus $0 < |z| < 2\pi$, on which it must have a laurent expansion. We view the function as a quotient of 1 and $e^z - 1 = \sum_{n=1}^{\infty} \frac{z^n}{n!}$. Do a long division of these two expansions, we would find the first few terms, as given in the problem.