

## MAT 5230 HOMEWORK AND ANSWERS

### HOMEWORK 5, NOT TO BE COLLECTED

**1(P.230:3(a)).** Evaluate the integral

$$\int_C \frac{z^5}{1-z^3} dz.$$

Here  $C$  is the circle  $|z| = 2$  directed positively.

*Answer:* Since all the 3 singular points of the integrand are in the interior of  $C$ , we'd better use the theorem on page 228 to write

$$\int_C \frac{z^5}{1-z^3} dz = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} \frac{(1/z)^5}{1-(1/z)^3} = -2\pi i.$$

The reason is that this is equal to  $2\pi i \operatorname{Res}_{z=0} \frac{1}{z^4(z^3-1)}$ . We use the standard expansion  $\frac{1}{z^3-1} = -\sum_{n=0}^{\infty} z^{3n}$  to determine that the  $c_{-1}$  coefficient in the Laurent expansion of  $\frac{1}{z^4(z^3-1)}$  is  $-1$ , which is the desired residual.

**2(P.234:4).** Write the function

$$f(z) = \frac{8a^3 z^2}{(z^2 + a^2)^3} = \frac{\phi(z)}{(z - ai)^3},$$

with  $\phi(z) = \frac{8a^3 z^2}{(z+ai)^3}$ . Point out why  $\phi$  has a Taylor expansion about  $ai$ , and then use it to show that the principal part of  $f$  at the point is

$$\frac{\phi''(ai)/2}{z - ai} + \frac{\phi'(ai)}{(z - ai)^2} + \frac{\phi(ai)}{(z - ai)^3} = \frac{i/2}{z - ai} - \frac{a/2}{(z - ai)^2} - \frac{a^2 i}{(z - ai)^3}.$$

*Answer:* The function  $\phi$  is analytic at  $ai$ , since its singularities are at  $-ai$ , so  $\phi$  has a Taylor expansion about  $ai$ . The principal part in the Laurent expansion of  $f$  then must be the one given in the problem. One needs to calculate the derivatives of  $\phi$  at  $ai$  to see the equation.

**3(P.238:3).** Find the value of the integral

$$\int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} dz,$$

taken counterclockwise around the circle (a)  $|z - 2| = 2$ ; (b)  $|z| = 4$ . *Answer:* The integrand  $f$  has singularities at 1 and  $\pm 3i$ . For part (a), only 1 is in the interior of  $C$ , so

$$\int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} dz = 2\pi i \operatorname{Res}_{z=1} f(z) = 2\pi i \cdot \frac{1}{2} = \pi i.$$

For part (b), all the 3 singular points are in the interior of  $C$ , therefore,

$$\int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} dz = 2\pi i \operatorname{Res}_{z=1, z=\pm 3i} f(z) = 2\pi i \cdot \left(\frac{1}{2} + \frac{150}{60}\right) = 6\pi i.$$

**4(P.245:4(b)).** Let  $C$  denote the positively oriented circle  $|z| = 2$ , and evaluate the integral

$$\int_C \frac{dz}{\sinh 2z}.$$

*Answer:* The  $\sinh 2z$  function has zeros at  $z = \pm \frac{k}{2}\pi i$ , of which  $0, \pm \frac{\pi}{2}i$  are in the interior of the contour. We need to calculate the residuals of  $\frac{1}{\sinh 2z}$  at all the three points. These are the values of  $\frac{1}{2 \cosh 2z}$  at these points. They are  $\frac{1}{2}$ ,  $-\frac{1}{2}$ , and  $-\frac{1}{2}$ , the sum of which is  $-\frac{1}{2}$ . Therefore

$$\int_C \frac{dz}{\sinh 2z} = -\pi i.$$

**5(P.257:7).** Use the residues to find the Cauchy principal value of the integral

$$\int_{-\infty}^{\infty} \frac{x dx}{(x^2 + 1)(x^2 + 2x + 2)}.$$

*Answer:* We use the close contour integral  $\int_{-R}^R + \int_{C_R}$ , with  $C_R$  being the upper semicircle of radius  $R$ . In the upper half plane, there are two singularities of the complexified integrand at  $i$  and  $1 + i$ . We choose  $R$  so big that encloses both of these. Therefore  $\int_{-R}^R + \int_{C_R} = 2\pi i \operatorname{Res}_{z=i, z=-1+i} \frac{z}{(z^2+1)(z^2+2z+2)}$ . We write

$$\frac{z}{(z^2 + 1)(z^2 + 2z + 2)} = \frac{z}{(z + i)(z - i)(z + 1 + i)(z + 1 - i)}.$$

The residue at  $i$  is the value of  $\frac{z}{(z+i)(z+1+i)(z+1-i)}$  there. It is  $\frac{1-2i}{10}$ . Similarly, the residue at  $-1 + i$  is  $\frac{-1+3i}{10}$ . Therefore,  $\int_{-R}^R + \int_{C_R} = -\frac{\pi}{5}$ . Then, we note that  $\int_{C_R} \rightarrow 0$  as  $R \rightarrow \infty$ .

**6(P.265:8).** Use the residues to find the value of

$$\int_0^{\infty} \frac{x^3 \sin x dx}{(x^2 + 1)(x^2 + 9)}.$$

*Answer:* First, the integrand is even, so the value of the integral is half of the integral on the whole real number line. We consider the complex integral

$$\int_{-\infty}^{\infty} \frac{z^3 e^{iz} dz}{(z^2 + 1)(z^2 + 9)},$$

of which the imaginary part gives us the desired result. When  $R$  is sufficiently big, we have

$$\int_{-R}^R + \int_{C_R} = 2\pi i \operatorname{Res}_{z=i, 3i} \frac{z^3 e^{iz} dz}{(z^2 + 1)(z^2 + 9)} = \frac{27e^{-3} - 3e^{-1}}{24} \pi i.$$

We note that  $\int_{C_R} \rightarrow 0$  as  $R \rightarrow \infty$ . Thus

$$\int_0^{\infty} \frac{x^3 \sin x dx}{(x^2 + 1)(x^2 + 9)} = \frac{27e^{-3} - 3e^{-1}}{48} \pi.$$

**7(P.277:6).** Show that

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

*Answer:* We approach this by using the complex function  $\frac{1}{z^{1/2}(z^2+1)}$ . Besides the singularity due to a branch cut associated with  $z^{1/2}$ , there are two isolated singularities at  $\pm i$ , respectively. We could use two methods to solve this. One is based on an indented upper semicircle as the integral contour. (See figure 97 in the book.) We could use the negative  $y$ -axis as the branch cut, which only touch our semicircle at the origin, which is excluded by using a small semicircle  $C_\epsilon$ . And  $z^{1/2} = \sqrt{|z|}e^{i\frac{1}{2}\arg z}$ , with  $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$ . Let the closed indented semicircle be  $C = C_\epsilon + (\epsilon, R) + C_R + (-R, -\epsilon)$ , oriented counterclockwise. Then

$$\int_C \frac{1}{z^{1/2}(z^2+1)} dz = 2\pi i \operatorname{Res}_{z=i} \frac{1}{z^{1/2}(z^2+1)} = 2\pi i \frac{1}{e^{i\pi/4} 2i} = \frac{\sqrt{2}\pi(1-i)}{2}.$$

We also have

$$\int_{-R}^{-\epsilon} \frac{1}{z^{1/2}(z^2+1)} dz = \int_{-R}^{-\epsilon} \frac{1}{\sqrt{|x|}e^{i\pi/2}(x^2+1)} dx = -i \int_\epsilon^R \frac{1}{\sqrt{x}(x^2+1)} dx.$$

We then let  $\epsilon \rightarrow 0$  and  $R \rightarrow \infty$ , argue that both the  $C_\epsilon$  and  $C_R$  integrals converge to zero, and get

$$(1-i) \int_0^{\infty} \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\sqrt{2}\pi(1-i)}{2}.$$

Thus  $\int_0^{\infty} \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\sqrt{2}\pi}{2}$ .

One could also choose the closed contour as depicted in Figure 99.

**8(P.280:5).** Find the integral

$$\int_0^\pi \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2} \quad (-1 < a < 1).$$

*Answer:* This is equal to one half of the integral on the interval  $(-\pi, \pi)$ . By changing variable  $e^{i\theta} = z$ , using the fact that  $\cos\theta = (z+z^{-1})/2$ ,  $\cos 2\theta = (z^2+z^{-2})/2$ , and  $d\theta = dz/iz$ , we can write the latter integral as a closed contour integral on the unit circle  $C$  in the  $z$ -plane. Thus

$$\begin{aligned} \int_0^\pi \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2} &= \frac{1}{2} \int_{-\pi}^\pi \frac{\cos 2\theta d\theta}{1-2a\cos\theta+a^2} \\ &= -\frac{1}{4ai} \int_C \frac{z^4+1}{z^2(z-a)(z-\frac{1}{a})} dz = -\frac{1}{4ai} 2\pi i \operatorname{Res}_{z=0, z=a} \frac{z^4+1}{z^2(z-a)(z-\frac{1}{a})} = \frac{a^2\pi}{1-a^2}. \end{aligned}$$

Note that the residue at 0 is equal to the derivative of  $\frac{z^4+1}{(z-a)(z-\frac{1}{a})}$  at 0, and the residue at  $a$  is just the value of  $\frac{z^4+1}{z^2(z-\frac{1}{a})}$  at  $z = a$ .

**9(P.286:8).** Determine the number of roots, counting multiplicity, of the equation  $2z^5 - 6z^2 + z + 1 = 0$  in the annulus  $1 \leq z < 2$ .

*Answer:* We determine the number of zeros in  $|z| < 2$  and in  $|z| < 1$ , and to get the answer by minus the latter from the former. We use Rouché's theorem. To resolve the issue on  $|z| < 2$ , we consider taking  $f(z) = 2z^5$  and  $g(z) = -6z^2 + z + 1$ . Since  $|g(z)| < |f(z)|$  for  $|z| = 2$ , the number of zeros in  $|z| < 2$  of  $f + g$  is the same as that of  $f$ , which is 5.

To resolve the problem on  $|z| < 1$ , we let  $f(z) = -6z^2$  and  $g(z) = 2z^5 + z + 1$  to verify the condition of Rouché's theorem. Thus there are 2 zeros. The number of zeros in the annulus is then 3.

**10(P.286:9).** Show that if  $c$  is a complex number such that  $|c| > e$ , then the equation  $cz^n = e^z$  has  $n$  roots, counting multiplicity, inside the circle  $|z| = 1$ .

*Answer:* A root of the equation  $cz^n = e^z$  is a zero of the function  $cz^n - e^z$ . Let  $f(z) = cz^n$  and  $g(z) = -e^z$ . Since  $|c| > e$ , we have  $|g(z)| < |f(z)|$  for  $|z| = 1$ . Thus, according to Rouché, in  $|z| < 1$ , the number of zeros of  $f + g$  is the same as that of  $f$ , which is  $n$ .