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Superconvergence of Discontinuous Galerkin Methods for Convection-Diffusion Problems

Zuozheng Zhang · Ziqing Xie · Zhimin Zhang

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Abstract Some discontinuous Galerkin methods for the linear convection-diffusion equation $-\epsilon u'' + bu' = f$ are studied. Based on superconvergence properties of numerical fluxes at element nodes established in some earlier works, e.g., Celiker and Cockburn in Math. Comput. 76(257), 67–96, 2007, we identify superconvergence points for the approximations of u or $q = u'$. Our results are twofold:

1) For the minimal dissipation LDG method (we call it md-LDG in this paper) using polynomials of degree p, we prove that the leading terms of the discretization errors for u and q are proportional to the right Radau and left Radau polynomials of degree $p+1$, respectively. Consequently, the zeros of the right-Radau and left-Radau polynomials of degree $p+1$ are the superconvergence points of order $p+2$ for the discretization errors of the potential and of the gradient, respectively.

2) For the consistent DG methods whose numerical fluxes at the mesh nodes converge at the rate of $O(h^{p+1})$, we prove that the leading term of the discretization error for q is proportional to the Legendre polynomial of degree p. Consequently, the approximation of the gradient superconverges at the zeros of the Legendre polynomial of degree p at the rate of $O(h^{p+1})$.

Numerical experiments are presented to illustrate the theoretical findings.

Keywords Discontinuous Galerkin method · Convection-diffusion problems · Superconvergence · Zeros of the Radau polynomial · Zeros of the Legendre polynomial

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Z. Zhang · Z. Xie (✉) · Z. Zhang
College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, People's Republic of China
e-mail: ziqingxie@hunnu.edu.cn

Z. Xie
e-mail: ziqingxie@yahoo.com.cn

1 Introduction

In this paper, we study superconvergence properties of some DG methods for convection-diffusion equation $-\epsilon u'' + bu' = f$. We first prove the existence and uniqueness of the discrete solution of the LDG method. We then establish asymptotic expansions of the discretization errors for the potential u or its derivative $q = u'$. For the md-LDG method [7] using polynomials of degree p, the leading terms of the discretization errors for u and q are proportional to the right Radau and left Radau polynomials of degree $p+1$ on each element, respectively, i.e.,

$$
\begin{aligned}
(u-U)(\xi) &= a_{p+1}R^+_{p+1}(\xi)h^{p+1} + O(h^{p+2}),\\
(q-Q)(\xi) &= b_{p+1}R^-_{p+1}(\xi)h^{p+1} + O(h^{p+2}),
\end{aligned}
\tag{1.1}
$$

where U and Q are the DG solutions, and R^+_{p+1} and R^-_{p+1} are the right Radau and left Radau polynomials of degree $p+1$, respectively. This fact implies that the zeros of the right-Radau and left-Radau polynomials of degree $p+1$ are the superconvergence points of order $p+2$ for U and Q, respectively. In addition, for the consistent DG methods whose numerical fluxes at the mesh nodes converge at the rate of $O(h^{p+1})$, the leading term for the error of Q is proportional to the Legendre polynomial of degree p on each element, i.e.,

$$(q-Q)(\xi) = b_pL_p(\xi)h^p + O(h^{p+1}),$$

where L_p is the Legendre polynomial of degree p. As a result, the zeros of the Legendre polynomial of degree p are the superconvergence points of order $p+1$ for $q = u'$. Note that in this case, the global convergence rate for q is suboptimal, i.e., $O(h^p)$.

Let us compare our results to some closely related works in the literature.

In [2], the authors presented a numerical study of the superconvergence properties for the LDG method for time-dependent convection-diffusion problems. For the equation $u_t + u_x - du_{xx} = f$ ($d = 10^{-4}$, a convection-dominated case) with the exact solution $u(x,t) = e^{-d\pi^2 t}\sin\pi(x-t)$, they observed that the error of the potential behaves like

$$u - U \approx \alpha(t)R^+_{p+1}(x). \tag{1.2}$$

Furthermore, for the purpose of obtaining *a posteriori* error estimates, they assumed that the error for the gradient behaves in the same way

$$q - Q \approx \beta(t)R^+_{p+1}(x),$$

even though their computational results indicated that the leading term in $q-Q$ is not proportional to $R^+_{p+1}(x)$. Our study here indicates that for the corresponding steady state problem, $q - Q \approx \beta R^-_{p+1}(x)$.

For the pure diffusion model $u_t = u_{xx}$ with the exact solution $u(x,t) = e^{-\pi^2 t}\sin\pi x$, they observed that U and Q do not exhibit any superconvergence, while U_x superconverges at $x_i = x(\xi_i), i = 1,\ldots,p$, with ξ_i the zeros of the derivative of the right-Radau polynomial of degree $p+1$ and Q_x at $y_i = x(\eta_i), i = 1,\ldots,p$, with η_i the zeros of the derivative of the left-Radau polynomial of degree $p+1$, where $x = x(\xi)$ is the function transforming the interval $[-1,1]$ into the element $[x_{j-1/2}, x_{j+1/2}]$. Based on this observation, they concluded that the errors behave like

$$u - U \approx \alpha(t)R^+_{p+1}(x) + c(t), \qquad q - Q \approx \beta(t)R^-_{p+1}(x) + z(t). \tag{1.3}$$

They made similar numerical observations for the Burgers' equation. Our study clearly indicates that U and Q do exhibit superconvergence for the corresponding steady state problem.

Similar superconvergence results at the zeros of the Radau polynomial were obtained earlier for a numerical scheme using a DG method to discretize in space a one-dimensional hyperbolic conservation law [4]. See also [3] and [1] for the extension to the two-dimensional hyperbolic problems on rectangular and triangular meshes, respectively.

Another closely related work is [6], where some DG methods are analyzed for the pure steady-state diffusion model $-u'' = f$. In this work, Castillo proved that a consistent and conservative DG method can capture the exact potential at all mesh nodes and the gradient error behaves like

$$q - Q = a_p L_p + O(h^{p+1}).$$

Consequently, an approximated gradient superconverges at a rate of $O(h^{p+1})$ at the zeros of the Legendre polynomial of degree p. Our results indicate that, although the DG solution is no longer exact at mesh nodes for the convection-diffusion problems, the superconvergence rate at the zeros of the Legendre polynomial of degree p is still the same provided that the DG method is consistent and its corresponding numerical fluxes at the mesh nodes converge at a rate of $O(h^{p+1})$. Obviously these assumptions are not restrictive as quite a large class of DG methods satisfy these conditions.

In [7], Celiker and Cockburn studied superconvergence properties of some DG methods for the same model problem as ours in this paper. They proved that both potential and its derivative superconverge at all nodes of the mesh provided that the fluxes are conservative. In particular, $O(h^{2p+1})$ superconvergence rate for the md-LDG method was established. Our study of identifying superconvergence points depends on the nodal superconvergence results obtained in [7].

An important issue for the convection-dominated problems ($\epsilon \ll 1$) is the uniform convergence, i.e., convergence independent of ϵ. In [10], Xie and Zhang implemented the LDG method for the one-dimensional convection-diffusion problem with mixed boundary conditions. In particular, they observed numerically the uniform convergence rate of $O(h^{2p+1})$ for the numerical fluxes under the Shishkin mesh and some graded meshes. For further results in this respect, see also [9, 11]. Theoretical verification of this fact is our on-going work.

The outline of this paper is as follows: We introduce the DG methods for our model problem in Sect. 2. The existence and uniqueness of the LDG solution is proved in Sect. 3. In Sect. 4, we establish asymptotic expansions of discretization errors for the md-LDG method, and hence prove the superconvergence rate of $O(h^{p+2})$ at the zeros of the right and left Radau polynomials of degree $p+1$ for U and Q, respectively. The asymptotic expansion for the discretization error of Q for the consistent DG methods whose numerical fluxes at the nodes converge at a rate of $O(h^{p+1})$, and its corresponding superconvergence rate of $O(h^{p+1})$ for Q at the zeros of the Legendre polynomial of degree p are obtained in Sect. 5. The numerical tests shown in Sect. 6 validate our theoretical results. Finally, in Sect. 7, we end with some concluding remarks.

2 The DG Method for Dirichlet Problem

Consider the following model problem

$$\begin{cases} -\varepsilon u'' + bu' = f & \text{in } \Omega = (0, 1), \\ u(0) = u_0, \qquad u(1) = u_1, \end{cases} \tag{2.4}$$

where $b > 0$ is a constant and ε is a small positive parameter. The choice of $b > 0$ guarantees that the location of the possible boundary layer is at the outflow boundary $x = 1$.

By setting $q = u'$, (2.4) can be rewritten as

$$\begin{cases} -\varepsilon q' + bu' = f \quad \text{in } \Omega = (0, 1), \\ q - u' = 0 \quad \text{in } (0, 1), \\ u(0) = u_0, \qquad u(1) = u_1. \end{cases} \tag{2.5}$$

Denote the mesh by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ for $j = 1, 2, \dots, N$ with $x_{\frac{1}{2}} = 0$, $x_{N+\frac{1}{2}} = 1$. The center of the cell is $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$ and the length of the cell is $h_j = |I_j|$. Set $h = \max_{1\le j\le N} h_j$ and $\Omega_h = \bigcup_{j=1,\dots,N} I_j$. We denote by $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$ the values of u at $x_{j+\frac{1}{2}}$, from the right cell and the left cell of $x_{j+\frac{1}{2}}$, respectively. Multiplying the first two equations of (2.5) by test functions v and w, respectively, and integrating by parts in each cell I_j, we obtain

$$\begin{cases} \varepsilon \displaystyle\int_{I_j} qv'dx - b\int_{I_j} uv'dx - (\varepsilon q - bu)^-_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + (\varepsilon q - bu)^+_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = \int_{I_j} f v dx, \\ \displaystyle\int_{I_j} uw'dx + \int_{I_j} qwdx - u^-_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + u^+_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0, \end{cases} \tag{2.6}$$

which is the weak formulation we shall implement in the definition of the DG methods. Now denote the piecewise polynomial space V_h as the space of polynomials of degree $p \ge 1$ in each cell I_j, i.e.,

$$V_h = \{v : v \in P_p(I_j), j = 1, 2, \dots, N\}.$$

Moreover, define the space

$$V = H^k(\Omega_h) = \{v : v \in H^k(I_j), j = 1, 2, \dots, N\}$$

with $k \ge 0$. We will search for approximate solutions of (2.5) in terms of piecewise polynomial functions $U, Q \in V_h$ that satisfy (2.5) in a weak sense. Following Cockburn and Shu [8], we consider the following general formulation:

Find $U, Q \in V_h$ such that

$$\begin{cases} \varepsilon \displaystyle\int_{I_j} Qv'dx - b\int_{I_j} Uv'dx - (\varepsilon \hat{Q} - b\tilde{U})_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} + (\varepsilon \hat{Q} - b\tilde{U})_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = \int_{I_j} f v dx, \\ \displaystyle\int_{I_j} Uw'dx + \int_{I_j} Qwdx - \hat{U}_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \hat{U}_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0, \end{cases} \tag{2.7}$$

for any v and $w \in V_h$. To complete the specification of a DG method, one must define the numerical fluxes $\hat{Q}$, $\hat{U}$, and $\tilde{U}$ at all nodes. Towards this end, we adopt conventional notations of the average and jump of $v \in V$: $\{v\}_{i+\frac{1}{2}} = \frac{1}{2}(v(x^-_{i+\frac{1}{2}}) + v(x^+_{i+\frac{1}{2}}))$, $[v]_{i+\frac{1}{2}} = v(x^-_{i+\frac{1}{2}}) - v(x^+_{i+\frac{1}{2}})$, respectively.

The numerical flux associated with the convection is always the classical upwind one, namely,

$$\tilde{U}_{j+\frac{1}{2}} = \begin{cases} u_0, & j = 0, \\ U^-_{j+\frac{1}{2}}, & j = 1, 2, \dots, N-1, N. \end{cases} \tag{2.8}$$

It is the different choice of the numerical fluxes $\hat{U}$ and $\hat{Q}$ that specifies the corresponding DG method for the underlying problem. To fix the idea, two LDG methods, i.e., the md-LDG and LDG-I methods, will be specified. According to [7], the numerical fluxes of the md-LDG method are

$$\hat{U}_{j+\frac{1}{2}} = \begin{cases} u_0, & j=0, \\ \{U\}_{j+\frac{1}{2}} + \frac{1}{2}[U]_{j+\frac{1}{2}} = U^-_{j+\frac{1}{2}}, & j=1,2,\dots,N-1, \\ u_1, & j=N, \end{cases} \tag{2.9}$$

$$\hat{Q}_{j+\frac{1}{2}} = \begin{cases} Q^+_{\frac{1}{2}}, & j=0, \\ \{Q\}_{j+\frac{1}{2}} - \frac{1}{2}[Q]_{j+\frac{1}{2}} = Q^+_{j+\frac{1}{2}}, & j=1,2,\dots,N-1, \\ Q^-_{N+\frac{1}{2}} - \alpha\big(U^-_{N+\frac{1}{2}} - u_1\big), & j=N. \end{cases} \tag{2.10}$$

Another special case of the LDG methods is defined by taking its numerical fluxes as

$$\hat{U}_{j+\frac{1}{2}} = \begin{cases} u_0, & j=0, \\ \{U\}_{j+\frac{1}{2}} + \frac{1}{2}[U]_{j+\frac{1}{2}} = U^-_{j+\frac{1}{2}}, & j=1,2,\dots,N-1, \\ u_1, & j=N, \end{cases} \tag{2.11}$$

$$\hat{Q}_{j+\frac{1}{2}} = \begin{cases} Q^+_{\frac{1}{2}} - \alpha\big(u_0 - U^+_{\frac{1}{2}}\big), & j=0, \\ \{Q\}_{j+\frac{1}{2}} - \frac{1}{2}[Q]_{j+\frac{1}{2}} - \alpha[U]_{j+\frac{1}{2}} = Q^+_{j+\frac{1}{2}} - \alpha[U]_{j+\frac{1}{2}}, & j=1,2,\dots,N-1, \\ Q^-_{N+\frac{1}{2}} - \alpha\big(U^-_{N+\frac{1}{2}} - u_1\big), & j=N. \end{cases} \tag{2.12}$$

To distinguish it from the md-LDG method, it is called LDG-I method in this paper. The penalization parameter is typically $\alpha = p/h$.

When the numerical fluxes are consistent, the following error equations

$$\varepsilon \int_{I_n} e_q v' dx - b \int_{I_n} e_u v' dx - (\varepsilon \hat{e}_q - b\tilde{e}_u)_{n+\frac{1}{2}} v^-_{n+\frac{1}{2}} + (\varepsilon \hat{e}_q - b\tilde{e}_u)_{n-\frac{1}{2}} v^+_{n-\frac{1}{2}} = 0, \tag{2.13}$$

$$\int_{I_n} e_u w' dx + \int_{I_n} e_q w dx - (\hat{e}_u)_{n+\frac{1}{2}} w^-_{n+\frac{1}{2}} + (\hat{e}_u)_{n-\frac{1}{2}} w^+_{n-\frac{1}{2}} = 0 \tag{2.14}$$

hold, with $e_u = e_u^{(n)} = u - U^{(n)}$ and $e_q = e_q^{(n)} = q - Q^{(n)}$, where $U^{(n)}$ and $Q^{(n)}$ are U and Q in the n-th element. For simplicity, we omit the subscript n when a quantity is considered in the n-th element unless specified.

3 The Existence and Uniqueness of the LDG Solution

As we shall pay special attention to the study of the superconvergence points of the LDG methods, including the md-LDG and LDG-I methods, we investigate the existence and uniqueness of the numerical solution based on them. According to [8], the numerical fluxes are taken as

$$\widehat{U}_{j+\frac{1}{2}} = \begin{cases} u_0, & j=0, \\ \{U\}_{j+\frac{1}{2}} - \beta[U]_{j+\frac{1}{2}}, & j=1,2,\dots,N-1, \\ u_1, & j=N, \end{cases} \tag{3.1}$$

$$\widehat{Q}_{j+\frac{1}{2}} = \begin{cases} Q(0^+) - \alpha(u_0 - U(0^+)), & j=0, \\ \{Q\}_{j+\frac{1}{2}} + \beta[Q]_{j+\frac{1}{2}} - \alpha[U]_{j+1/2}, & j=1,2,\dots,N-1, \\ Q(1^-) - \alpha(U(1^-) - u_1), & j=N. \end{cases} \tag{3.2}$$

A direct computation leads to the following result, namely,

Lemma 3.1

$$\sum_{j=1}^{N}[(uv)^-_{j+\frac{1}{2}} - (uv)^+_{j-\frac{1}{2}}] = (uv)^-_{N+\frac{1}{2}} - (uv)^+_{\frac{1}{2}} - \sum_{j=1}^{N-1}(\{u\}_{j+\frac{1}{2}}[v]_{j+\frac{1}{2}} + [u]_{j+\frac{1}{2}}\{v\}_{j+\frac{1}{2}}). \tag{3.3}$$

Theorem 3.2 *With the numerical fluxes given by* (2.8), (3.1), *and* (3.2), *the LDG method defined in* (2.7) *determines a unique solution when* $\alpha \ge 0$ *in* (3.2).

Proof As (2.7) is a linear problem, we only need to verify that $U=0$, $Q=0$ if $u_0=u_1=0$ and $f=0$ in order to prove the existence and uniqueness of the numerical solution.

Integration by parts in the second expression of (2.7) leads to

$$-\int_{I_j} U'w dx + (U-\hat{U})_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} - (U-\hat{U})_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} + \int_{I_j} Qw dx = 0. \tag{3.4}$$

Summing up (3.4) over $j=1,2,\dots,N$, and implementing (3.3), we obtain

$$\int_{\Omega_h} Qw dx = \int_{\Omega_h} U'w dx + [(\hat{U}-U)w]^-_{N+\frac{1}{2}} - [(\hat{U}-U)w]^+_{\frac{1}{2}} - \sum_{j=1}^{N-1}(\{\hat{U}-U\}_{j+\frac{1}{2}}[w]_{j+\frac{1}{2}} + [\hat{U}-U]_{j+\frac{1}{2}}\{w\}_{j+\frac{1}{2}}). \tag{3.5}$$

By setting $w=Q$, (3.5) yields

$$\int_{\Omega_h} Q^2 dx = \int_{\Omega_h} U'Q dx + [(\hat{U}-U)Q]^-_{N+\frac{1}{2}} - [(\hat{U}-U)Q]^+_{\frac{1}{2}} - \sum_{j=1}^{N-1}(\{\hat{U}-U\}_{j+\frac{1}{2}}[Q]_{j+\frac{1}{2}} + [\hat{U}-U]_{j+\frac{1}{2}}\{Q\}_{j+\frac{1}{2}}). \tag{3.6}$$

In the first expression of (2.7) with $f=0$, taking $v=U$, summing up over $j=1,2,\dots,N$, and dividing by ϵ on both sides of it, we obtain

$$\int_{\Omega_h} U'Qdx = \frac{1}{\varepsilon}\Bigg[b\int_{\Omega_h} UU'dx - b\Bigg((\widetilde{U}U)^-_{N+\frac{1}{2}} - (\widetilde{U}U)^+_{\frac{1}{2}} - \sum_{j=1}^{N-1}(\{\widetilde{U}\}_{j+\frac{1}{2}}[U]_{j+\frac{1}{2}} + [\widetilde{U}]_{j+\frac{1}{2}}\{U\}_{j+\frac{1}{2}})\Bigg)\Bigg] + (\hat{Q}U)^-_{N+\frac{1}{2}} - (\hat{Q}U)^+_{\frac{1}{2}} - \sum_{j=1}^{N-1}(\{\hat{Q}\}_{j+\frac{1}{2}}[U]_{j+\frac{1}{2}} + [\hat{Q}]_{j+\frac{1}{2}}\{U\}_{j+\frac{1}{2}}). \tag{3.7}$$

Substituting (3.7) into (3.6), we have

$$\begin{aligned}\int_{\Omega_h} Q^2dx &= \frac{b}{\varepsilon}\Bigg[\int_{\Omega_h} UU'dx - \Bigg((\widetilde{U}U)^-_{N+\frac{1}{2}} - (\widetilde{U}U)^+_{\frac{1}{2}} - \sum_{j=1}^{N-1}(\{\widetilde{U}\}_{j+\frac{1}{2}}[U]_{j+\frac{1}{2}} + [\widetilde{U}]_{j+\frac{1}{2}}\{U\}_{j+\frac{1}{2}})\Bigg)\Bigg] \\ &\quad + (\hat{Q}U)^-_{N+\frac{1}{2}} - (\hat{Q}U)^+_{\frac{1}{2}} - \sum_{j=1}^{N-1}(\{\hat{Q}\}_{j+\frac{1}{2}}[U]_{j+\frac{1}{2}} + [\hat{Q}]_{j+\frac{1}{2}}\{U\}_{j+\frac{1}{2}}) \\ &\quad + [(\hat{U}-U)Q]^-_{N+\frac{1}{2}} - [(\hat{U}-U)Q]^+_{\frac{1}{2}} \\ &\quad - \sum_{j=1}^{N-1}(\{\hat{U}-U\}_{j+\frac{1}{2}}[Q]_{j+\frac{1}{2}} + [\hat{U}-U]_{j+\frac{1}{2}}\{Q\}_{j+\frac{1}{2}}) \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} I_1 &= \frac{b}{\varepsilon}\Bigg[\int_{\Omega_h} UU'dx - \Bigg((\widetilde{U}U)^-_{N+\frac{1}{2}} - (\widetilde{U}U)^+_{\frac{1}{2}} - \sum_{j=1}^{N-1}(\{\widetilde{U}\}_{j+\frac{1}{2}}[U]_{j+\frac{1}{2}} + [\widetilde{U}]_{j+\frac{1}{2}}\{U\}_{j+\frac{1}{2}})\Bigg)\Bigg] \\ &= -\frac{b}{2\varepsilon}(U^2_{N+\frac{1}{2}} - U^2_{\frac{1}{2}}) + \frac{b}{\varepsilon}\sum_{j=1}^{N-1}\{U\}_{j+\frac{1}{2}}[U]_{j+\frac{1}{2}} \\ &\quad - \frac{b}{\varepsilon}\Bigg[(\widetilde{U}_{N+\frac{1}{2}} - U_{N+\frac{1}{2}})U_{N+\frac{1}{2}} - (\widetilde{U}_{\frac{1}{2}} - U_{\frac{1}{2}})U_{\frac{1}{2}} \\ &\quad - \sum_{j=1}^{N-1}(\{\widetilde{U}-U\}_{j+\frac{1}{2}}[U]_{j+\frac{1}{2}} + [\widetilde{U}-U]_{j+\frac{1}{2}}\{U\}_{j+\frac{1}{2}})\Bigg], \end{aligned} \tag{3.9}$$

$$\begin{aligned} I_2 + I_3 &= (\hat{Q}U)^-_{N+\frac{1}{2}} - (\hat{Q}U)^+_{\frac{1}{2}} - \sum_{j=1}^{N-1}(\{\hat{Q}\}_{j+\frac{1}{2}}[U]_{j+\frac{1}{2}} + [\hat{Q}]_{j+\frac{1}{2}}\{U\}_{j+\frac{1}{2}}) \\ &\quad + [(\hat{U}-U)Q]^-_{N+\frac{1}{2}} - [(\hat{U}-U)Q]^+_{\frac{1}{2}} \\ &\quad - \sum_{j=1}^{N-1}(\{\hat{U}-U\}_{j+\frac{1}{2}}[Q]_{j+\frac{1}{2}} + [\hat{U}-U]_{j+\frac{1}{2}}\{Q\}_{j+\frac{1}{2}}). \end{aligned} \tag{3.10}$$

Herein, the integration by parts in the term $\int_{\Omega_h} UU'dx$ and Lemma 3.1 are implemented in the second equality of I_1. Inserting the numerical fluxes shown in (3.1), (3.2), and (2.8) into (3.9) and (3.10), respectively, and performing some algebraic manipulations yields

$$I_1 = -\frac{b}{2\varepsilon}U_{N+\frac{1}{2}}^2 - \frac{b}{2\varepsilon}U_{\frac{1}{2}}^2 - \sum_{j=1}^{N-1}\frac{b}{2\varepsilon}[U]_{j+\frac{1}{2}}^2, \tag{3.11}$$

$$I_2 + I_3 = -\alpha U_{N+\frac{1}{2}}^2 - \alpha U_{\frac{1}{2}}^2 - \sum_{j=1}^{N-1}\alpha[U]_{j+\frac{1}{2}}^2. \tag{3.12}$$

Substituting (3.11) and (3.12) into (3.8), and performing a simple algebraic manipulation, we obtain

$$\int_{\Omega_h} Q^2 dx + \Theta = 0 \tag{3.13}$$

with

$$\Theta = \left(\frac{b}{2\varepsilon}+\alpha\right)U_{N+\frac{1}{2}}^2 + \left(\frac{b}{2\varepsilon}+\alpha\right)U_{\frac{1}{2}}^2 + \frac{b}{2\varepsilon}\sum_{j=1}^{N-1}[U]_{j+\frac{1}{2}}^2 + \sum_{j=1}^{N-1}\alpha[U]_{j+\frac{1}{2}}^2, \tag{3.14}$$

which is always nonnegative. The combination of (3.14) and (3.13) yields

$$Q \equiv 0, \qquad U_{\frac{1}{2}} = U_{N+\frac{1}{2}} = 0, \qquad [U]_{j+\frac{1}{2}} = 0. \tag{3.15}$$

This fact, associated with (3.5), leads to

$$\int_{\Omega_h} U' w dx = 0 \quad \forall w \in V_h, \tag{3.16}$$

which implies that U is a piecewise constant. Once again the implementation of (3.15) reduces that $U = 0$. Consequently, the existence and uniqueness of the LDG solution are obtained. □

4 The Superconvergence of the md-LDG Method

In this section, we investigate the superconvergence points of the md-LDG method. To do this, we need the following nodal superconvergence results from [7].

Lemma 4.1 *For the md-LDG method with the numerical fluxes defined in* (2.8), (2.9), *and* (2.10), *the following error estimates*

$$\begin{aligned} |\hat{e}_u(x_{j+\frac{1}{2}})| &\le Ch^{2p+1}|u|_{p+2}, \\ |(\varepsilon\hat{e}_q - b\tilde{e}_u)(x_{j+\frac{1}{2}})| &\le Ch^{2p+1}|u|_{p+2} \end{aligned} \tag{4.1}$$

hold for all nodes with $C = C(\varepsilon, p)$, *where* $\hat{e}_u = u - \hat{U}$, $\hat{e}_q = q - \hat{Q}$ *and* $\tilde{e}_u = u - \tilde{U}$.

Based on Lemma 4.1, we summarize some facts in the following lemma. These facts will be used in the proof of our main results, Theorems 4.6 and 4.7.

Lemma 4.2

$$\begin{aligned}
&|\hat{e}_u(x_{j+\frac{1}{2}})| = |\tilde{e}_u(x_{j+\frac{1}{2}})| = |e_u^-(x_{j+\frac{1}{2}})| \le Ch^{2p+1}|u|_{p+2}, \quad j = 1, 2, \dots, N-1,\\
&|\hat{e}_q(x_{j+\frac{1}{2}})| = |e_q^+(x_{j+\frac{1}{2}})| \le Ch^{2p+1}|u|_{p+2}, \quad j = 0, 1, 2, \dots, N-1, \qquad (4.2)\\
&\hat{e}_u(x_{N+\frac{1}{2}}) = 0, \qquad \hat{e}_u(x_{\frac{1}{2}}) = \tilde{e}_u(x_{\frac{1}{2}}) = 0.
\end{aligned}$$

We need some more notations. Denote $L_j(\xi)$ the Legendre polynomial of degree j, $R^+_{p+1}(\xi) = L_{p+1}(\xi) - L_p(\xi)$ and $R^-_{p+1}(\xi) = L_{p+1}(\xi) + L_p(\xi)$ the right and left Radau polynomials of degree $p+1$, respectively. Moreover, the following three lemmas will be used repeatedly in the proof of Theorems 4.6 and 4.7.

Lemma 4.3 *If*

$$\int_{-1}^{1} U_k(\xi) w'(\xi) d\xi = 0, \quad \forall w \in P_p([-1, 1]), \tag{4.3}$$

where $U_k(\xi)$ is a polynomial of degree k for $k = 0, 1, \dots, p-1$. Then

$$U_k(\xi) = 0, \quad k = 0, 1, \dots, p-1.$$

Moreover, if (4.3) holds for $k = p$, then $U_p(\xi) = a_p L_p(\xi)$ with a_p a constant independent of ξ.

Proof When $k \le p-1$, (4.3) implies $U_k(\xi) = 0$. For $k = p$, (4.3) leads to $U_k(\xi) = U_p(\xi) = a_p L_p(\xi)$ by the property of the Legendre polynomial. □

Lemma 4.4 *Assume that*

$$\int_{-1}^{1} U_k(\xi) w'(\xi) d\xi - U_k(1^-) w(1^-) = 0, \quad \forall w \in P_p([-1, 1]), \tag{4.4}$$

where $U_k(\xi)$ is a polynomial of degree k , $k = 0, 1, \dots, p$. Then

$$U_k(\xi) = 0, \quad k = 0, 1, \dots, p.$$

Further, if (4.4) holds for $k = p+1$, then

$$U_{p+1}(\xi) = a_{p+1}(L_{p+1}(\xi) - L_p(\xi)) = a_{p+1} R^+_{p+1}(\xi), \qquad U_{p+1}(1^-) = 0.$$

Proof Taking $w = 1$ in (4.4) results in

$$U_k(1^-) = 0, \quad k = 0, 1, \dots, p, \tag{4.5}$$

which, together with Lemma 4.3 and (4.4), yields

$$U_k(\xi) = 0, \quad k = 0, 1, \dots, p-1, \qquad U_p(\xi) = a_p L_p(\xi).$$

By $U_p(1^-) = 0$, we obtain $U_p(1^-) = a_p L_p(1^-) = 0$, which leads to $a_p = 0$. As a result,

$$U_p(\xi) = 0.$$

When (4.4) holds for $k = p + 1$, we have $U_{p+1}(1^-) = 0$ and

$$\int_{-1}^{1} U_{p+1}(\xi)w'(\xi)d\xi = 0, \quad \forall w \in P_p([-1, 1]),$$

which yields

$$U_{p+1}(\xi) = a_{p+1}L_{p+1}(\xi) + a'_{p+1}L_p(\xi).$$

This, associated with $U_{p+1}(1^-) = 0$, gives rise to

$$U_{p+1}(1^-) = a_{p+1}L_{p+1}(1^-) + a'_{p+1}L_p(1^-) = a_{p+1} + a'_{p+1} = 0,$$

which implies

$$U_{p+1}(\xi) = a_{p+1}(L_{p+1}(\xi) - L_p(\xi)) = a_{p+1}R^+_{p+1}(\xi). \qquad \square$$

Lemma 4.5 *Assume*

$$\int_{-1}^{1} Q_k(\xi)v'(\xi)d\xi + Q_k(-1^+)v(-1^+) = 0, \quad \forall v \in P_p([-1, 1]), \tag{4.6}$$

where $Q_k(\xi)$ is a polynomial of degree k, $k = 0, 1, \ldots, p$. Then

$$Q_k(\xi) = 0, \quad k = 0, 1, \ldots, p.$$

Further, if (4.6) holds for $k = p + 1$, then

$$Q_{p+1}(\xi) = b_{p+1}(L_{p+1}(\xi) + L_p(\xi)) = b_{p+1}R^-_{p+1}(\xi), \qquad Q_{p+1}(-1^+) = 0.$$

Proof The proof is similar to that of Lemma 4.4 and therefore, is omitted. $\square$

We are now ready to prove the main results of this section.

Theorem 4.6 *For the errors of the md-LDG solutions in the n-th element, $n = 1, 2, \ldots, N - 1$, we have the following expressions*

$$e_u^{(n)} = \sum_{k=p+1}^{\infty} U_k^{(n)}(\xi)h_n^k, \qquad e_q^{(n)} = \sum_{k=p+1}^{\infty} Q_k^{(n)}(\xi)h_n^k, \tag{4.7}$$

where $U_k^{(n)}(\xi)$ and $Q_k^{(n)}(\xi)$ are polynomials of degree k on $[-1, 1]$ with the leading terms

$$\begin{aligned} U_{p+1}^{(n)}(\xi) &= a_{p+1}^{(n)}R^+_{p+1}(\xi) = a_{p+1}^{(n)}(L_{p+1}(\xi) - L_p(\xi)), \\ Q_{p+1}^{(n)}(\xi) &= b_{p+1}^{(n)}R^-_{p+1}(\xi) = b_{p+1}^{(n)}(L_{p+1}(\xi) + L_p(\xi)), \end{aligned} \tag{4.8}$$

and the rest terms

$$\begin{aligned} U_k^{(n)}(\xi) &= \sum_{j=2(p+1)-(k+1)}^{k} b_{jk}^{(n)}L_j(\xi), \qquad U_k^{(n)}(1^-) = 0, \quad k = p + 1, \ldots, 2p, \\ Q_k^{(n)}(\xi) &= \sum_{j=2(p+1)-(k+1)}^{k} c_{jk}^{(n)}L_j(\xi), \qquad Q_k^{(n)}(-1^+) = 0, \quad k = p + 1, \ldots, 2p. \end{aligned} \tag{4.9}$$

Proof As the DG scheme in (2.7) with the md-LDG fluxes is consistent, the error equations (2.13) and (2.14) hold. The transformation

$$x(\xi) = \frac{x_{n-\frac{1}{2}} + x_{n+\frac{1}{2}}}{2} + \frac{h_n}{2}\xi \tag{4.10}$$

maps the canonical element $[-1, 1]$ to $[x_{n-\frac{1}{2}}, x_{n+\frac{1}{2}}]$. As $U^{(n)}$, $Q^{(n)}$, u and q are all smooth, the standard Taylor expansion leads to the following series expansions in terms of h_n,

$$\begin{aligned} e_u^{(n)} &= \sum_{k=0}^{\infty} U_k^{(n)}(\xi) h_n^k, \\ e_q^{(n)} &= \sum_{k=0}^{\infty} Q_k^{(n)}(\xi) h_n^k. \end{aligned} \tag{4.11}$$

Applying Lemma 4.2, (2.13), (2.14), and the transformation defined in (4.10) yields

$$\begin{aligned} &\varepsilon \int_{-1}^{1} e_q v' d\xi - b \int_{-1}^{1} e_u v' d\xi + b e_u(1^-) v(1^-) + \varepsilon e_q(-1^+) v(-1^+) \\ &\quad = O(h_n^{2p+1}) v(1^-) + O(h_n^{2p+1}) v(-1^+), \end{aligned} \tag{4.12}$$

$$\int_{-1}^{1} e_u w' d\xi + \frac{h_n}{2} \int_{-1}^{1} e_q w d\xi - e_u(1^-) w(1^-) = O(h_n^{2p+1}) w(-1^+). \tag{4.13}$$

Substituting (4.11) into (4.12) and (4.13), respectively, and grouping terms with common powers of h_n, gives

$$\begin{aligned} &\sum_{k=0}^{\infty} \left[\int_{-1}^{1} [\varepsilon Q_k(\xi) - b U_k(\xi)] v'(\xi) d\xi + b U_k(1^-) v(1^-) + \varepsilon Q_k(-1^+) v(-1^+) \right] h_n^k \\ &\quad = O(h_n^{2p+1}) v(1^-) + O(h_n^{2p+1}) v(-1^+), \end{aligned} \tag{4.14}$$

$$\begin{aligned} &\int_{-1}^{1} U_0(\xi) w'(\xi) d\xi - U_0(1^-) w(1^-) + \sum_{k=1}^{\infty} \left[\int_{-1}^{1} U_k(\xi) w'(\xi) d\xi \right. \\ &\qquad \left. + \frac{1}{2} \int_{-1}^{1} Q_{k-1}(\xi) w(\xi) d\xi - U_k(1^-) w(1^-) \right] h_n^k \\ &\quad = O(h_n^{2p+1}) w(-1^+). \end{aligned} \tag{4.15}$$

Consequently, we obtain

$$\int_{-1}^{1} U_0(\xi) w'(\xi) d\xi - U_0(1^-) w(1^-) = 0, \tag{4.16}$$

$$\begin{aligned} &\int_{-1}^{1} U_k(\xi) w'(\xi) d\xi + \frac{1}{2} \int_{-1}^{1} Q_{k-1}(\xi) w(\xi) d\xi - U_k(1^-) w(1^-) = 0, \\ &k = 1, 2, \ldots, 2p, \end{aligned} \tag{4.17}$$

$$\int_{-1}^{1}[\varepsilon Q_k(\xi)-bU_k(\xi)]v'(\xi)d\xi+bU_k(1^-)v(1^-)+\varepsilon Q_k(-1^+)v(-1^+)=0,$$
$$k=0,1,\dots,2p, \tag{4.18}$$

for any $w, v \in P^p([-1,1])$. Based on (4.16), (4.17), and (4.18), we will use an induction argument to prove the following conclusion, i.e.,

$$U_k(\xi)=Q_k(\xi)=0, \quad k=0,1,2,\dots,p. \tag{4.19}$$

As a matter of fact, by Lemma 4.4 and (4.16), we have

$$U_0(\xi)=0. \tag{4.20}$$

Setting $k=0$ in (4.18) and using (4.20), we obtain

$$\varepsilon\int_{-1}^{1}Q_0(\xi)v'(\xi)d\xi+\varepsilon Q_0(-1^+)v(-1^+)=0, \quad \forall v\in P^p([-1,1]). \tag{4.21}$$

The application of Lemma 4.5 leads to

$$Q_0(\xi)=0. \tag{4.22}$$

The combination of (4.20) and (4.22) implies (4.19) with $k=0$.

It is observed that the induction hypothesis is (4.19) with k replaced by $k-1$, i.e.,

$$U_{k-1}(\xi)=Q_{k-1}(\xi)=0.$$

The substitution of $Q_{k-1}(\xi)=0$ into (4.17) yields

$$\int_{-1}^{1}U_k(\xi)w'(\xi)d\xi-U_k(1^-)w(1^-)=0, \quad \forall w\in P_p([-1,1]). \tag{4.23}$$

A straightforward application of Lemma 4.4 in (4.23) leads to

$$U_k(\xi)=0, \quad k=0,1,2,\dots,p. \tag{4.24}$$

Substituting (4.24) into (4.18) results in

$$\varepsilon\int_{-1}^{1}Q_k(\xi)v'(\xi)d\xi+\varepsilon Q_k(-1^+)v(-1^+)=0, \quad \forall v\in P_p([-1,1]). \tag{4.25}$$

The implementation of Lemma 4.5 in (4.25) yields

$$Q_k(\xi)=0, \quad k=0,1,2,\dots,p. \tag{4.26}$$

The verification of (4.19) is done, and (4.7) is a straightforward conclusion of (4.19).

Now we turn to the proof of (4.8). Actually, we note that $Q_p(\xi)=0$ in (4.19). By taking $k=p+1$, (4.17) can be simplified as

$$\int_{-1}^{1}U_{p+1}(\xi)w'(\xi)d\xi-U_{p+1}(1^-)w(1^-)=0, \quad \forall w\in P_p([-1,1]), \tag{4.27}$$

which implies, by Lemma 4.4,

$$U_{p+1}(\xi) = a_{p+1}(L_{p+1}(\xi) - L_p(\xi)) = a_{p+1} R^+_{p+1}(\xi), \qquad U_{p+1}(1^-) = 0. \tag{4.28}$$

On the other hand, setting $k = p + 1$ and making use of (4.28), (4.18) can be written as

$$\varepsilon \int_{-1}^{1} Q_{p+1}(\xi) v'(\xi) d\xi + \varepsilon Q_{p+1}(-1^+) v(-1^+) = 0, \quad \forall v \in P_p([-1, 1]), \tag{4.29}$$

which yields, by Lemma 4.5,

$$Q_{p+1}(\xi) = b_{p+1}(L_{p+1}(\xi) + L_p(\xi)) = b_{p+1} R^-_{p+1}(\xi), \qquad Q_{p+1}(-1^+) = 0. \tag{4.30}$$

Now we once again adopt an induction argument to prove (4.9). Having established (4.28) and (4.30), we see that (4.9) is satisfied with $k = p + 1$. Noting that the induction hypothesis is (4.9) with k replaced by $k - 1$, i.e.

$$U_{k-1}(\xi) = \sum_{j=2(p+1)-k}^{k-1} b_{j,k-1} L_j(\xi), \qquad Q_{k-1}(\xi) = \sum_{j=2(p+1)-k}^{k-1} c_{j,k-1} L_j(\xi).$$

Based on this hypothesis, (4.17) can be written as

$$\int_{-1}^{1} U_k(\xi) w'(\xi) d\xi - U_k(1^-) w(1^-) + \frac{1}{2} \int_{-1}^{1} \sum_{j=2(p+1)-k}^{k-1} c_{j,k-1} L_j(\xi) w(\xi) d\xi = 0,$$
$$\forall w(\xi) \in P_p([-1, 1]).$$

Taking $w(\xi) = \xi^j$, $j = 0, 1, \ldots, 2(p + 1) - (k + 1)$, the expression above can be simplified as

$$\int_{-1}^{1} U_k(\xi) w'(\xi) d\xi - U_k(1^-) w(1^-) = 0, \tag{4.31}$$

which leads to

$$\int_{-1}^{1} U_k(\xi) w'(\xi) d\xi = 0 \quad \text{for } w = \xi^j,\ j = 0, 1, \ldots, 2(p + 1) - (k + 1), \qquad U_k(1^-) = 0. \tag{4.32}$$

This implies

$$U_k(\xi) = \sum_{j=2(p+1)-(k+1)}^{k} a_{jk} L_j(\xi). \tag{4.33}$$

Based on $U_k(1^-) = 0$, (4.18) can be expressed as

$$\int_{-1}^{1} [\varepsilon Q_k(\xi) - b U_k(\xi)] v'(\xi) d\xi + \varepsilon Q_k(-1^+) v(-1^+) = 0, \quad \forall v \in P^p([-1, 1]),$$

which results in

$$Q_k(-1^+) = 0,$$

$$\varepsilon Q_k(\xi) - b U_k(\xi) = \sum_{j=2(p+1)-(k+1)}^{k} c'_{jk} L_j(\xi).$$

This, combing with (4.33), yields

$$Q_k(\xi) = \sum_{j=2(p+1)-(k+1)}^{k} c_{jk} L_j(\xi).$$

□

Because of the different definition of the numerical fluxes in the last element I_N, the corresponding result (of Theorem 4.6) and its proof are slightly different. For the sake of completeness, we show the result in the following theorem.

Theorem 4.7 *For the errors of the md-LDG solutions in the N-th element, i.e., I_N, we have the following expressions*

$$e_u^{(N)} = \sum_{k=p}^{\infty} U_k^{(N)}(\xi) h_N^k, \qquad e_q^{(N)} = \sum_{k=p+1}^{\infty} Q_k^{(N)}(\xi) h_N^k, \tag{4.34}$$

with the leading terms

$$\begin{aligned} &U_p^{(N)}(\xi) = a_p^{(N)} L_p(\xi), \\ &Q_{p+1}^{(N)}(\xi) = b_{p+1}^{(N)} R_{p+1}^{-}(\xi) = b_{p+1}^{(N)} \big(L_{p+1}(\xi) + L_p(\xi)\big). \end{aligned} \tag{4.35}$$

Further

$$\begin{aligned} &U_k^{(N)}(\xi) = \sum_{j=2(p+1)-(k+1)}^{k} b_{jk}^{(N)} L_j(\xi), \quad k = p+1, \ldots, 2p, \\ &Q_k^{(N)}(\xi) = \sum_{j=2(p+1)-(k+1)}^{k} c_{jk}^{(N)} L_j(\xi), \qquad Q_k^{(N)}(-1) = 0, \quad k = p+1, \ldots, 2p. \end{aligned} \tag{4.36}$$

Proof The error equations of (2.13) and (2.14), combining with Lemma 4.2, yield

$$\begin{cases} \varepsilon \int_{-1}^{1} e_q v' d\xi - b \int_{-1}^{1} e_u v'(\xi) d\xi + \varepsilon e_q(-1^+) v(-1^+) \\ \quad = O(h_N^{2p+1}) v(1^-) + O(h_N^{2p+1}) v(-1^+), \\ \int_{-1}^{1} e_u w' d\xi + \frac{h_N}{2} \int_{-1}^{1} e_q(\xi) w(\xi) d\xi = O(h_N^{2p+1}) w(-1^+). \end{cases} \tag{4.37}$$

Substituting (4.11) into (4.37), then grouping terms having common powers of h_N leads to

$$\begin{aligned} &\sum_{k=0}^{\infty} \Big[\int_{-1}^{1} (\varepsilon Q_k(\xi) - b U_k(\xi)) v'(\xi) d\xi + \varepsilon Q_k(-1^+) v(-1^+) \Big] h_N^k \\ &\quad = O(h_N^{2p+1}) v(1^-) + O(h_N^{2p+1}) v(-1^+), \\ &\int_{-1}^{1} U_0(\xi) w'(\xi) d\xi + \sum_{k=1}^{\infty} \Big[\int_{-1}^{1} U_k(\xi) w'(\xi) d\xi + \frac{1}{2} \int_{-1}^{1} Q_{k-1}(\xi) w(\xi) d\xi \Big] h_N^k \\ &\quad = O(h_N^{2p+1}) w(-1^+). \end{aligned}$$

Therefore,

$$\int_{-1}^{1} U_0(\xi)w'(\xi)d\xi = 0, \quad \forall w \in P_p[(-1,1)], \tag{4.38}$$

$$\int_{-1}^{1} U_k(\xi)w'(\xi)d\xi + \frac{1}{2}\int_{-1}^{1} Q_{k-1}(\xi)w(\xi)d\xi = 0,$$
$$\forall w \in P_p[(-1,1)],\ k = 1, 2, \ldots, 2p, \tag{4.39}$$

$$\int_{-1}^{1} (\varepsilon Q_k(\xi) - bU_k(\xi))v'(\xi)d\xi + \varepsilon Q_k(-1^+)v(-1^+) = 0,$$
$$\forall v \in P_p[(-1,1)],\ k = 0, 1, 2, \ldots, 2p. \tag{4.40}$$

The induction argument (which is omitted here) based on (4.38), (4.39), and (4.40), together with Lemmas 4.3 and 4.5, yields

$$U_k(\xi) = Q_k(\xi) = 0, \quad k = 0, 1, 2, \ldots, p-1. \tag{4.41}$$

By (4.39) with $k = p$, we have

$$\int_{-1}^{1} U_p(\xi)w'(\xi)d\xi = 0, \quad \forall w \in P_p[(-1,1)],$$

which implies, by Lemma 4.3,

$$U_p(\xi) = a_p L_p(\xi). \tag{4.42}$$

Setting $k = p$ in (4.40) and taking (4.42) into account, leads to

$$\varepsilon \int_{-1}^{1} Q_p(\xi)v'(\xi)d\xi + \varepsilon Q_p(-1^+)v(-1^+) = 0, \quad \forall v \in P_p[(-1,1)],$$

which yields, by Lemma 4.5,

$$Q_p(\xi) = 0. \tag{4.43}$$

In (4.39), setting $k = p+1$, we have

$$\int_{-1}^{1} U_{p+1}(\xi)w'(\xi)d\xi = 0, \quad \forall w \in P_p[(-1,1)].$$

Consequently,

$$U_{p+1}(\xi) = a_{p+1}L_{p+1}(\xi) + a_p L_p(\xi). \tag{4.44}$$

Based on (4.44), setting $k = p+1$, (4.40) can be written as

$$\varepsilon \int_{-1}^{1} Q_{p+1}(\xi)v'(\xi)d\xi + \varepsilon Q_{p+1}(-1^+)v(-1^+) = 0, \quad \forall v \in P_p[(-1,1)].$$

This, in combination with Lemma 4.5, implies

$$Q_{p+1}(\xi) = b_{p+1}R^-_{p+1}(\xi), \qquad Q_{p+1}(-1^+) = 0. \tag{4.45}$$

The combination of (4.41), (4.42), (4.43), and (4.45) results in (4.34) and (4.35).

Based on (4.44) and (4.45), (4.36) can be derived by the induction argument with the same strategy as that in the n-th element, $n = 1, \dots, N-1$. □

Corollary 4.8 *For the md-LDG method, let $U, Q \in V_h$ be the approximate solutions of the model problem (2.5) in I_n. Then we have*

$$|u(\zeta_l^n) - U(\zeta_l^n)| = O(h_n^{p+2}), \quad l = 1, 2, \dots, p, \ n = 1, \dots, N-1,$$
$$|q(\eta_l^n) - Q(\eta_l^n)| = O(h_n^{p+2}), \quad l = 1, 2, \dots, p, \ n = 1, \dots, N,$$

where $\zeta_l^n = x(\xi_l)$, $l = 1, \dots, p$, with ξ_l the zeros of the right-Radau polynomial of degree $p+1$ except for 1, and $\eta_l^n = x(\tau_l)$, $l = 1, \dots, p$, with τ_l the zeros of the left-Radau polynomial of degree $p+1$ except for -1, and $x(\xi)$ defined in (4.10).

Remark 4.1 Although we have not verified that the zeros of the right Radau polynomial of degree $p+1$ are the superconvergence points of order $p+2$ for U in the last element I_N, our numerical experiment does show this result.

5 The Superconvergence of Some Consistent DG Methods

Now we shall investigate the superconvergence points of the consistent DG methods whose numerical fluxes at the mesh nodes converge at the rate of $O(h^{p+1})$. The main conclusion of this section is

Theorem 5.1 *Consider the DG methods, which are consistent. Assume that the numerical fluxes satisfy the following estimates, i.e.,*

$$|\hat{e}_u(x_{j+\frac{1}{2}})| \le Ch^{p+1}|u|_{p+2}, \quad j = 0, 1, 2, \dots, N,$$
$$|(\epsilon\hat{e}_q - b\tilde{e}_u)(x_{j+\frac{1}{2}})| \le Ch^{p+1}|u|_{p+2}, \quad j = 0, 1, 2, \dots, N.$$

Then we have the following error expansion, namely,

$$e_q^{(n)} = \sum_{k=p}^{\infty} Q_k^{(n)}(\xi) h_n^k \tag{5.1}$$

with the leading term

$$Q_p^{(n)}(\xi) = b_p^{(n)} L_p(\xi). \tag{5.2}$$

Proof For the consistent DG methods, the error equations (2.13) and (2.14) hold. Implementing the transformation (4.10) in $I_n = [x_{n-1/2}, x_{n+1/2}]$ and the assumptions above, (2.13) and (2.14) lead to

$$\int_{-1}^{1} e_u w' d\xi + \frac{h_n}{2}\int_{-1}^{1} e_q(\xi) w(\xi) d\xi = O(h_n^{p+1}) w(1^-) + O(h_n^{p+1}) w(-1^+),$$
$$\forall w \in P_p([-1, 1]), \tag{5.3}$$

$$\varepsilon \int_{-1}^{1} e_q v' d\xi - b\int_{-1}^{1} e_u(\xi) v'(\xi) d\xi = O(h_n^{p+1}) v(1^-) + O(h_n^{p+1}) v(-1^+),$$
$$\forall v \in P_p([-1, 1]). \tag{5.4}$$

Again, the error estimates can be expressed in the form (4.11). Similar to the proof of Theorem 4.6, substituting (4.11) into (5.3) and (5.4) and grouping the terms with common powers of h_n, we obtain

$$\begin{aligned}
&\int_{-1}^{1} U_0(\xi)w'(\xi)d\xi + \sum_{k=1}^{\infty}\left[\int_{-1}^{1} U_k(\xi)w'(\xi)d\xi + \frac{1}{2}\int_{-1}^{1} Q_{k-1}(\xi)w(\xi)d\xi\right]h_n^k \\
&\quad = O(h_n^{p+1})w(1^-) + O(h_n^{p+1})w(-1^+), \quad \forall w \in P_p([-1,1]), \\
&\sum_{k=0}^{\infty}\left[\int_{-1}^{1} \varepsilon Q_k(\xi)v'(\xi)d\xi - b\int_{-1}^{1} U_k(\xi)v'(\xi)d\xi\right]h_n^k \\
&\quad = O(h_n^{p+1})v(1^-) + O(h_n^{p+1})w(-1^+), \quad \forall v \in P_p([-1,1]).
\end{aligned} \tag{5.5}$$

These two expressions yield

$$\int_{-1}^{1} U_0(\xi)w'(\xi)d\xi = 0, \quad \forall w(\xi) \in P_p([-1,1]), \tag{5.6}$$

$$\int_{-1}^{1} U_k(\xi)w'(\xi)d\xi + \frac{1}{2}\int_{-1}^{1} Q_{k-1}(\xi)w(\xi)d\xi = 0, \quad \forall w(\xi) \in P_p([-1,1]),\ k = 1, 2, \ldots, p, \tag{5.7}$$

$$\int_{-1}^{1} [\varepsilon Q_k(\xi) - bU_k(\xi)]v'(\xi)d\xi = 0, \quad \forall v(\xi) \in P_p([-1,1]), k = 0, 1, 2, \ldots, p. \tag{5.8}$$

We use an induction argument to prove that

$$U_k(\xi) = Q_k(\xi) = 0, \quad k = 0, 1, 2, \ldots, p-1. \tag{5.9}$$

Actually, when $p \geq 1$, the implementation of Lemma 4.3 in (5.6) and (5.8) implies

$$U_0(\xi) = Q_0(\xi) = 0, \tag{5.10}$$

which shows that (5.9) is satisfied when $k = 0$. Suppose that the induction hypothesis holds, i.e., (5.9) holds with k replaced by $k-1$,

$$U_{k-1}(\xi) = Q_{k-1}(\xi) = 0.$$

Equation (5.7) leads to

$$\int_{-1}^{1} U_k(\xi)w'(\xi)d\xi = 0, \quad \forall w \in P_p([-1,1]),$$

which results in, by Lemma 4.3,

$$U_k(\xi) = 0, \quad k = 0, 1, \ldots, p-1.$$

This, combing with (5.8), gives rise to

$$\int_{-1}^{1} Q_k(\xi)v'(\xi)d\xi = 0, \quad \forall v \in P_p([-1,1]),$$

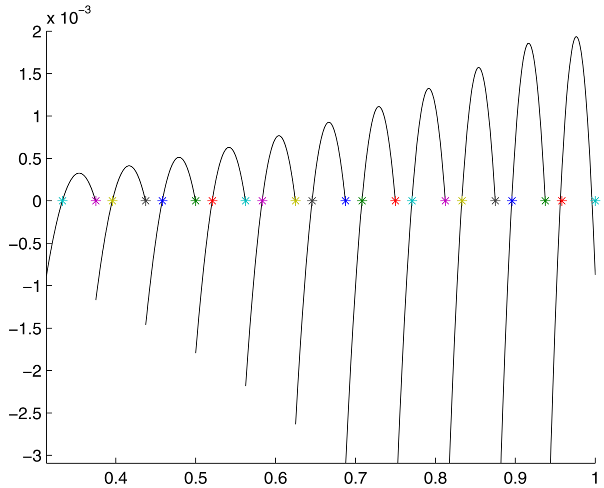

Fig. 1 Error curve of $u - U$ for the md-LDG, $p = 1$, $\varepsilon = 0.5$

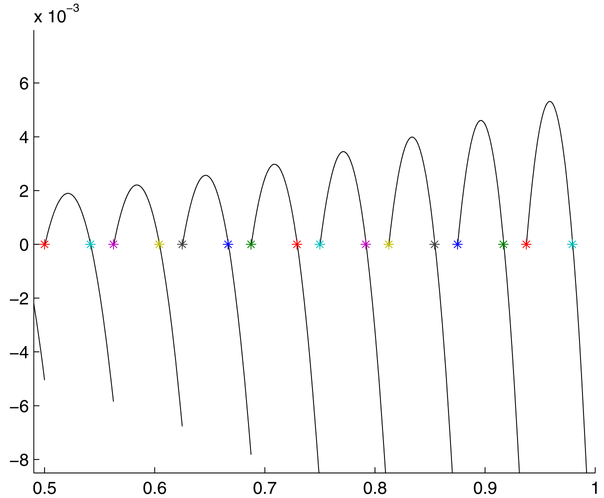

Fig. 2 Error curve of $q - Q$ for the md-LDG, $p = 1$, $\varepsilon = 0.5$

which implies

$$Q_k(\xi) = 0, \quad k = 0, 1, \dots, p - 1.$$

Taking $k = p$ in (5.7), we have

$$\int_{-1}^{1} U_p(\xi) w'(\xi) d\xi = 0, \quad \forall w(\xi) \in P_p([-1, 1]),$$

which yields, by Lemma 4.3,

$$U_p(\xi) = a_p L_p(\xi). \tag{5.11}$$

Based on (5.11), setting $k = p$ in (5.8), we have

$$Q_p(\xi) = b_p L_p(\xi).$$

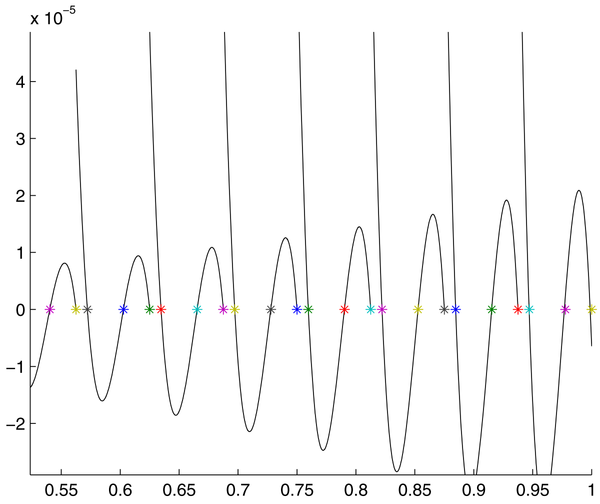

Fig. 3 Error curve of $u - U$ for the md-LDG, $p = 2$, $\varepsilon = 0.5$

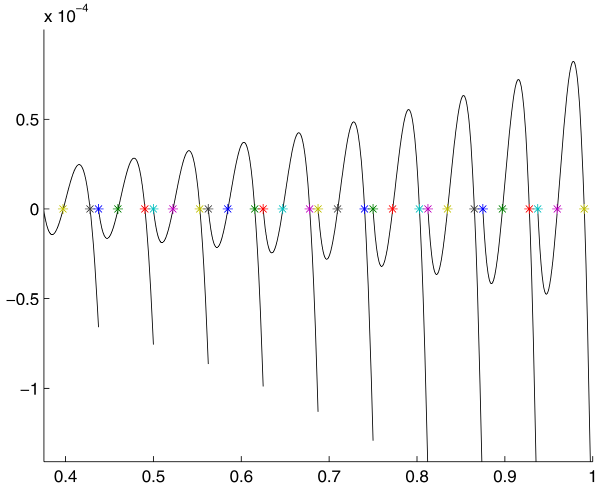

Fig. 4 Error curve of $q - Q$ for the md-LDG, $p = 2$, $\varepsilon = 0.5$

Then (5.2) follows. □

A straightforward application of this theorem is

Corollary 5.2 *Let $U, Q \in V_h$ be the approximate solutions of the model problem* (2.5) *in $I_n, n = 1, 2, \ldots, N$. Under the assumptions of Theorem* 5.1, *we have*

$$|q(\omega_l^n) - Q(\omega_l^n)| = O(h_n^{p+1}), \quad l = 1, 2, \ldots, p,$$

where $\omega_l^n = x(\sigma_l)$, $l = 1, \ldots, p$, with σ_l the zeros of the Legendre polynomial of degree p and $x = x(\xi)$ defined in (4.10).

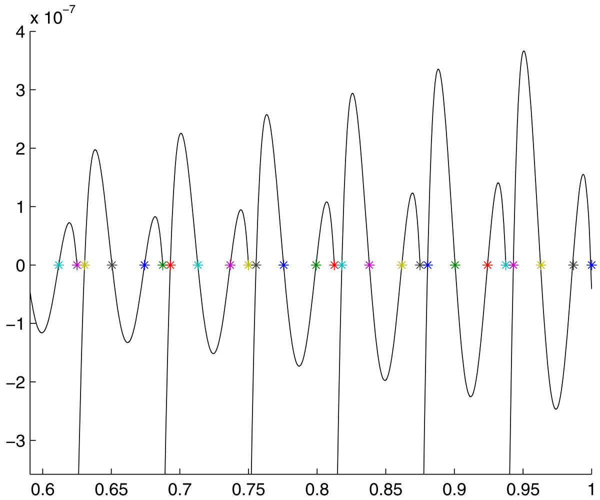

Fig. 5 Error curve of $u - U$ for the md-LDG, $p = 3$, $\varepsilon = 0.5$

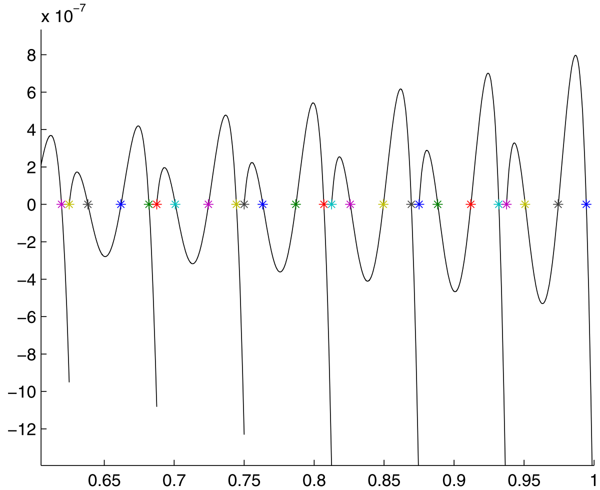

Fig. 6 Error curve of $q - Q$ for the md-LDG, $p = 3$, $\varepsilon = 0.5$

Remark 5.1 According to [7], for a group of consistent and conservative DG methods including the so-called IP DG method [5], their numerical fluxes satisfy

$$|\hat{e}_u(x_{j+\frac{1}{2}})| \le Ch^{2p}|u|_{p+2},$$
$$|(\epsilon\hat{e}_q - b\tilde{e}_u)(x_{j+\frac{1}{2}})| \le Ch^{2p}|u|_{p+2}.$$

As a result, Theorem 5.1 hold for all the DG methods shown in Table 4 of [7].

6 Numerical Results

In this section, we present numerical results to confirm our theoretical findings in Sects. 4 and 5. For simplicity, we use the uniform mesh.

Table 1 Convergence rate at the zeros of the Radau polynomial of degree $p+1$ for the md-LDG method, $\varepsilon = 0.5$

N	$p=1$		$p=2$		$p=3$	
	$\|u-U\|_{\infty.*}$	order	$\|u-U\|_{\infty.*}$	order	$\|u-U\|_{\infty.*}$	order
3	5.76e-03	2.49	9.02e-05	3.63	1.18e-06	4.69
4	8.72e-04	2.72	6.44e-06	3.81	4.11e-08	4.84
5	1.21e-04	2.85	4.31e-07	3.90	1.36e-09	4.92
6	1.59e-05	2.92	2.79e-08	3.95	4.37e-11	4.96
7	2.04e-06	2.96	1.77e-09	3.97	1.39e-12	4.98
N	$\|q-Q\|_{\infty.*}$	order	$\|q-Q\|_{\infty.*}$	order	$\|q-Q\|_{\infty.*}$	order
3	1.74e-03	2.65	2.75e-05	3.76	3.13e-07	4.80
4	2.46e-04	2.82	1.87e-06	3.88	1.05e-08	4.90
5	3.27e-05	2.91	1.22e-07	3.94	3.40e-10	4.95
6	4.22e-06	2.96	7.78e-09	3.97	1.07e-11	4.99
7	5.35e-07	2.98	4.88e-10	4.00	1.14e-12	–

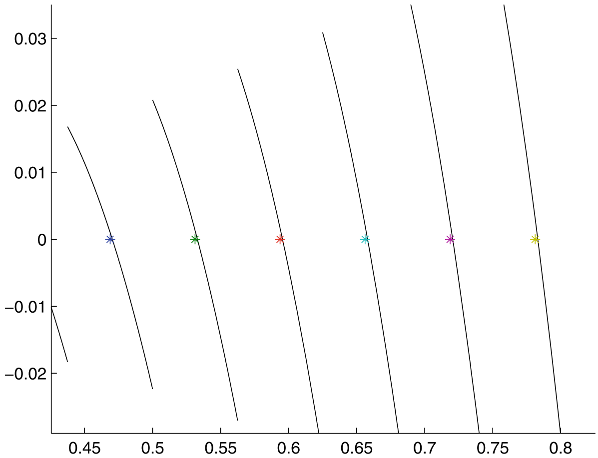

Fig. 7 Error curve of $q-Q$ for the LDG I method, $p=1$, $\varepsilon = 0.5$

In all numerical experiments, we choose in (2.4) $b = 1$ and $f = e^x$ with the Dirichlet boundary conditions $u_0 = u_1 = 0$. Therefore, the exact solution is given by

$$u = \begin{cases} \frac{e^x(1-e^{-\frac{1}{\varepsilon}})+e^{1-\frac{1}{\varepsilon}}-1+(1-e)e^{\frac{x-1}{\varepsilon}}}{(1-\varepsilon)(1-e^{-\frac{1}{\varepsilon}})}, & \varepsilon \neq 1, \\ \frac{e}{e-1}(e^x-1) - xe^x, & \varepsilon = 1. \end{cases} \tag{6.1}$$

First, the md-LDG method is implemented. The errors are depicted in Figs. 1–6 with $\varepsilon = 0.5$ and $p = 1, 2, 3$ for U and Q, respectively. For clarity, we enlarge the original error figures, only parts of which are shown. The stars on each element represent the zeros of the right-Radau polynomials in Figs. 1, 3, and 5. Meanwhile, the stars in Figs. 2, 4, and 6 represent the zeros of the left-Radau polynomials. We observe that the zeros of the right-Radau and left-Radau polynomials are close to the zeros of the errors $u-U$ and $q-Q$, respectively.

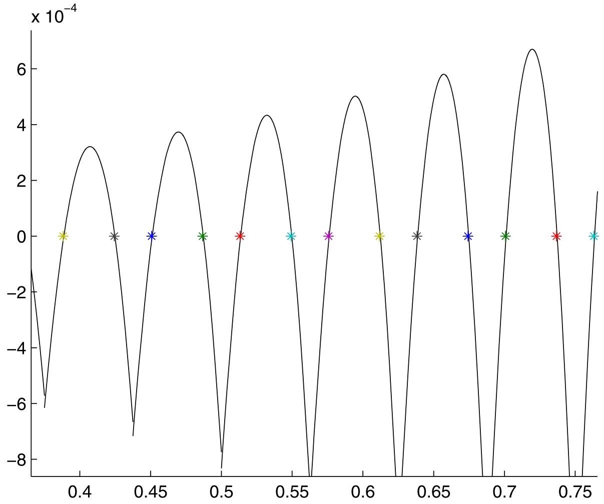

Fig. 8 Error curve of $q-Q$ for the LDG I method, $p=2$, $\varepsilon=0.5$

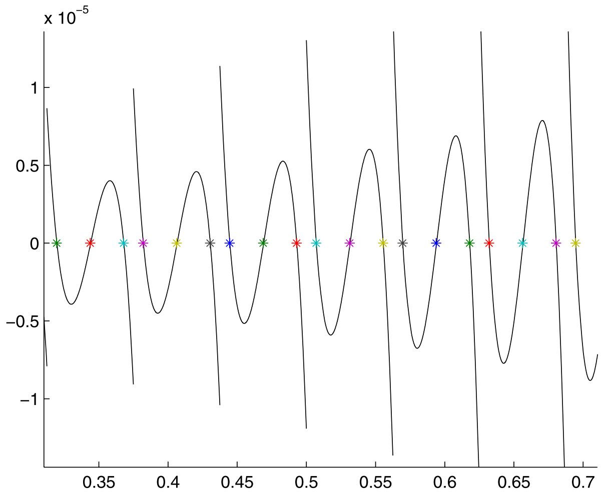

Fig. 9 Error curve of $q-Q$ for the LDG I method, $p=3$, $\varepsilon=0.5$

Listed in Table 1 are the errors and their corresponding convergence rates in the discrete L_∞ norm at the zeros of the right Radau polynomials for U and the zeros of the left Radau polynomials for Q for the md-LDG method with p varying from 1 to 3 and $\varepsilon=0.5$. Herein, we define

$$\|u-U\|_{\infty.*}=\max_{1\le j\le N}\max_{1\le i\le p}|(u-U)(x_{j,i})|, \tag{6.2}$$

where $x_{j,i}=x(\xi_i)$, $i=1,2,\ldots,p$, $j=1,\ldots,N$, with ξ_i the zeros of the right-Radau polynomial of degree $p+1$ except for 1, and $x=x(\xi)$ defined in (4.10). Then $\|q-Q\|_{\infty.*}$ is defined in a similar way with zeros of the right-Radau polynomial of degree $p+1$ replaced by the zeros of the left-Radau polynomial of degree $p+1$. The superconvergence of order $p+2$ which was predicted by Theorem 4.6 or Theorem 4.7 is observed for both U and Q.

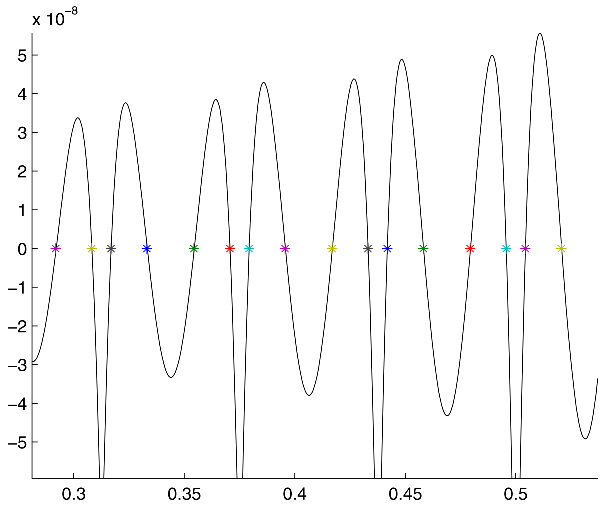

Fig. 10 Error curve of $q - Q$ for the LDG I method, $p = 4$, $\varepsilon = 0.5$

Table 2 Convergence rate of Q at the zeros of the Legendre polynomial of degree p for the LDG-I and IP DG methods with $\varepsilon = 0.5$

Methods	N	$p = 1$		$p = 2$		$p = 3$	
		$\|q - Q\|_{\infty,\star}$	order	$\|q - Q\|_{\infty,\star}$	order	$\|q - Q\|_{\infty,\star}$	order
	3	1.70e-02	1.79	3.99e-04	2.84	7.19e-06	3.82
	4	4.57e-03	1.89	5.28e-05	2.92	4.78e-07	3.92
LDG I	5	1.25e-03	1.87	7.18e-06	2.88	3.26e-08	3.87
	6	3.33e-04	1.91	9.42e-07	2.93	2.14e-09	3.93
	7	8.58e-05	1.96	1.21e-07	2.97	1.36e-10	3.97
	3	1.98e-01	1.01	3.05e-03	2.90	2.42e-05	3.86
	4	7.75e-02	1.35	3.96e-04	2.95	1.58e-06	3.93
IP	5	2.48e-02	1.64	5.04e-05	2.97	1.01e-07	3.97
	6	7.07e-03	1.81	6.36e-06	2.99	6.40e-09	3.98
	7	1.89e-03	1.91	7.98e-07	2.99	4.03e-10	3.99

The error curves of Q are plotted in Figs. 7–10 for the LDG I method with $\varepsilon = 0.5$ and p varying from 1 to 4. Again we enlarge the original error figures. The stars plotted in Figs. 7–10 represent the zeros of the Legendre polynomial of degree p. Clearly, they are close to the zeros of error $q - Q$. Comparing with Figs. 2, 4, and 6, we note that the md-LDG and LDG I methods have different types of superconvergence points. For other consistent DG methods whose numerical fluxes at the mesh nodes converge at a rate of $O(h^{p+1})$, e.g., the IP DG method, our numerical results have also predicted the same superconvergence points as those obtained by the LDG I method. In Table 2, the convergence rate of the error for Q at the zeros of the Legendre polynomial of degree p in a discrete L_∞ norm, which is defined in a similar way as that in (6.2) with zeros of the right-Radau polynomial of degree $p + 1$ replaced by the zeros of the Legendre polynomial of degree p, is displayed for the LDG I and IP DG methods. Obviously, the superconvergence of order $p + 1$ at the zeros of the Legendre polynomial of degree p is observed. This observation is in accordance with Theorem 1 and Corollary 5.2.

7 Conclusions

In this paper, we investigate superconvergence properties of several DG methods for the one-dimensional steady state convection-diffusion equation. The asymptotic expansions of the discretization errors of the md-LDG method and their superconvergence of order $p+2$ at the zeros of the right-Radau and left-Radau polynomials of degree $p+1$ for U and Q are established, respectively. On the other hand, for the consistent DG methods whose numerical fluxes at the mesh nodes converge at the rate of $O(h^{p+1})$, the superconvergence of order $p+1$ for Q at the zeros of the Legendre polynomial of degree p is also shown. The corresponding numerical results validate the theoretical analysis.

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