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Convergence analysis for least-squares finite element approximations of second-order two-point boundary value problems

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ABSTRACT

In this paper, a C^0 least-squares finite element method for second-order two-point boundary value problems is considered. The problem is recast as a first-order system. Standard and improved optimal error estimates in maximum-norms are established. Superconvergence estimates at interelement, Lobatto, and Gauss points are developed. Numerical experiments are given to illustrate theoretical results.

1. Introduction

The least-squares finite element method (LSFEM) is a general methodology that has attracted increasing attention in the engineering and mathematical communities. The LSFEM usually recasts the original problem into a first-order system of differential equations, to whose residual an L^2 least-squares principle is then applied. Comparing with standard mixed finite element methods [1], the LSFEM possesses many desirable properties such as well-posed weak formulation (so that the inf–sup condition [2,3] can be circumvented), conforming stable discretization, symmetric positive-definite linear system of equations, and unified robust formulation for different differential equations. These and other advantages assure LSFEMs can be successfully applied to a large variety of problems arising in sciences and engineering. For detailed reviews and applications of the method, please refer to [4–6] and their extensive bibliographies.

The LSFEM has drawn wide attention in theoretical analysis besides applications. Optimal L^2 and $H(\mathrm{div})$ (or H^1 in one dimensional case) error estimates of LSFEMs for second-order elliptic problems have been established in, e.g., [7–14], which are analog to the error estimates of standard Galerkin finite element methods. Furthermore, there are also several papers devoted to superconvergence analysis for the LSFEM. Superconvergence technics have become standard practices in applications of classic Galerkin methods (cf., e.g., [15–19]). Regarding the LSFEM, superconvergence has been observed in numerical experiments of [20] for two-point boundary value problems, which are similar as those for Galerkin methods.

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In a later article [12], the authors studied error estimates of a least-squares mixed FEM for one-dimensional self-adjoint equations. Derivative superconvergence at Gauss points and function value superconvergence at interelement nodes have been proved. In [21,22], some *a priori* and superconvergence error estimates for multi-dimensional self-adjoint problems have been established in integral norms. In a recent paper [23], pointwise error estimates of first-order div LSFEM have been investigated for multi-dimensional self-adjoint problems. Function value superconvergence results are obtained, which are the same as those by standard Galerkin methods. A brief survey of superconvergence in LSFEMs is available in [24].

In the present note, error estimates in maximum-norms will be investigated. From the mechanism of LSFEMs, it is reasonable to anticipate that techniques for superconvergence and *a posteriori* error estimation for Galerkin methods can be extended and applied to least-squares methods. We first develop optimal maximum-norm error estimates of LSFEMs for second-order two-point boundary value problems. Natural superconvergence of LSFEMs at Lobatto points and Gauss points are then considered by using "superapproximation" (cf. [25,26]).

This paper is organized as follows. In Section 2, the prototype problem and a least-squares finite element formulation are proposed. In Section 3, we first review optimal error estimates of the method in L^2 and H^1 norms. Maximum-norm error estimates are developed. Superconvergence at Lobatto points and Gauss points is then investigated. In Section 4, numerical examples are given to illustrate the theoretical results. Concluding remarks are made in Section 5.

2. Problem formulation

Consider the second-order elliptic equation

$$\begin{cases} -au'' + bu' + cu = f \quad \text{in } \Omega = (0, 1), \\ u(0) = u(1) = 0, \end{cases} \tag{2.1}$$

where a, b, and c are sufficiently smooth on $\overline{\Omega}, f \in L^2(\Omega)$, and $a(x) \geq a_0 > 0$ for all $x \in \overline{\Omega}$. It is further assumed that the boundary value problem has a unique solution in $H^2(\Omega)$ for each $f \in L^2(\Omega)$. Here, and throughout this paper, standard notations for the Sobolev spaces and associated norms are used [27]. The problem (2.1) may be recast as a first-order equation system:

$$\begin{cases} p - u' = 0 \quad \text{in } \Omega, \\ -ap' + bp + cu = f \quad \text{in } \Omega, \\ u(0) = u(1) = 0. \end{cases} \tag{2.2}$$

For $\mathbf{u} = [p, u]^T \in H^1(\Omega) \times H_0^1(\Omega)$, define

$$\mathcal{A}\mathbf{u} = \begin{bmatrix} p - u' \\ -ap' + bp + cu \end{bmatrix} \quad \text{and} \quad \mathbf{f} = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

Eq. (2.2) thus reads

$$\mathcal{A}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega.$$

Define the least-squares functional $\mathcal{J} : H^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ as

$$\mathcal{J}(\mathbf{v}; f) = \frac{1}{2}\|\mathcal{A}\mathbf{v} - \mathbf{f}\|^2_{L^2(\Omega)} = \frac{1}{2}\left(\mathcal{A}\mathbf{v} - \mathbf{f}, \mathcal{A}\mathbf{v} - \mathbf{f}\right),$$

where $(\mathbf{u}, \mathbf{v}) = \int_0^1 \mathbf{u} \cdot \mathbf{v}\, dx$ is the standard inner product on $[L^2(\Omega)]^2$. A minimizer $\mathbf{u}$ of the functional $\mathcal{J}$ satisfies

$$\lim_{t\to 0} \frac{d}{dt}\mathcal{J}(\mathbf{u} + t\mathbf{v}; f) = (\mathcal{A}\mathbf{u} - \mathbf{f}, \mathcal{A}\mathbf{v}) = 0 \quad \forall \mathbf{v} \in H^1(\Omega) \times H_0^1(\Omega).$$

The least-squares variational formulation of (2.2) thus follows: Find $\mathbf{u} \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$B(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in H^1(\Omega) \times H_0^1(\Omega), \tag{2.3}$$

where the symmetric bilinear form B and the linear functional L are defined as

$$B(\mathbf{u}, \mathbf{v}) = (\mathcal{A}\mathbf{u}, \mathcal{A}\mathbf{v}) = \int_0^1 \left((p - u')(q - v') + (-ap' + bp + cu)(-aq' + bq + cv)\right)\, dx,$$

$$L(\mathbf{v}) = (\mathbf{f}, \mathcal{A}\mathbf{v}) = \int_0^1 f(-aq' + bq + cv)\, dx,$$

where $\mathbf{u} = [p, u]^T$ and $\mathbf{v} = [q, v]^T$ are arbitrary vectors in $H^1(\Omega) \times H_0^1(\Omega)$. The following coercivity and continuity results of the bilinear form can be obtained (cf. [8,10–14]).

Proposition 2.1. *There exists a constant $\alpha > 0$ and a constant $C > 0$ such that*

$$B(\mathbf{v},\mathbf{v}) \geq \alpha\left(\|v\|^2_{H^1(\Omega)} + \|q\|^2_{H^1(\Omega)}\right),$$

$$B(\mathbf{u},\mathbf{v}) \leq C\left(\|u\|^2_{H^1(\Omega)} + \|p\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}\left(\|v\|^2_{H^1(\Omega)} + \|q\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}},$$

where $\mathbf{u} = [p, u]^T$, $\mathbf{v} = [q, v]^T \in H^1(\Omega) \times H^1_0(\Omega)$.

It then follows from the Lax–Milgram lemma that problem (2.3) has a unique solution in $H^1(\Omega) \times H^1_0(\Omega)$.

Let $\mathcal{T}_h = \{e_i\}_{i=1}^N$ be a partition of Ω, where $e_i = [x_{i-1}, x_i]$ is the ith element, $i = 1, \ldots, N$, and $0 = x_0 < x_1 < \cdots < x_N = 1$. Set the mesh parameter $h = \max_i h_i$, where $h_i = x_i - x_{i-1}$, $i = 1, \ldots, N$. Define W_h and V_h as finite dimensional subspaces of $H^1(\Omega)$ and $H^1_0(\Omega)$ which consists of piecewise polynomials of degree r and k, respectively,

$$W_h = \left\{q_h \in C^0(\Omega) : q_h|_e \in P_r(e)\ \forall e \in \mathcal{T}_h\right\},$$

$$V_h = \left\{v_h \in C^0(\Omega) : v_h|_e \in P_k(e)\ \forall e \in \mathcal{T}_h, \quad v_h(0) = v_h(1) = 0\right\}.$$

Here, $P_k(e)$ is the space of functions whose restrictions to each element e are polynomials of degree not greater than k. The finite element approximation to problem (2.3) is posed as follows: find $\mathbf{u}_h \in W_h \times V_h$ such that

$$B(\mathbf{u}_h, \mathbf{v}_h) = L(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in W_h \times V_h. \tag{2.4}$$

By Proposition 2.1 and the Lax–Milgram lemma, problem (2.4) has a unique solution. Moreover, from (2.3) and (2.4), the following Galerkin orthogonality property holds:

$$B(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in W_h \times V_h. \tag{2.5}$$

Here, and throughout the remainder of the paper, we denote by $\mathbf{u} = [p, u]^T$ and $\mathbf{u}_h = [p_h, u_h]^T$ the solutions to (2.3) and (2.4), respectively.

3. Error estimates

In this section, we first review some error estimate results for LSFEMs in the literature in Section 3.1. Maximum-norm estimates, improved maximum-norm estimates, and superconvergence estimates at Lobatto and Gauss points will be developed in Sections 3.2–3.4, respectively. Without loss of generality, we may assume $a \equiv 1$ in this section. We use C to denote a generic positive constant that is independent of u, p, or h in the context of this section.

3.1. Error estimates in the literature

Let I_h be the standard polynomial interpolation operator mapping to W_h or V_h as appropriate. Then $I_h p$ and $I_h u$ are interpolants of p and u in W_h and V_h, respectively. By approximation theory (cf., e.g., [28,29]), it follows that

$$\|u - I_h u\|_{L^2(\Omega)} \leq Ch^{s+1}\|u\|_{H^{s+1}(\Omega)} \quad \text{for } 1 \leq s \leq k, \tag{3.1}$$

$$\|u - I_h u\|_{H^1(\Omega)} \leq Ch^{s}\|u\|_{H^{s+1}(\Omega)} \quad \text{for } 1 \leq s \leq k, \tag{3.2}$$

$$|u - I_h u|_{W^s_\infty(\Omega)} \leq Ch^{k+1-s}|u|_{W^{k+1}_\infty(\Omega)} \quad \text{for } 0 \leq s \leq k, \tag{3.3}$$

$$\|p - I_h p\|_{L^2(\Omega)} \leq Ch^{s+1}\|p\|_{H^{s+1}(\Omega)} \quad \text{for } 1 \leq s \leq r, \tag{3.4}$$

$$\|p - I_h p\|_{H^1(\Omega)} \leq Ch^{s}\|p\|_{H^{s+1}(\Omega)} \quad \text{for } 1 \leq s \leq r, \tag{3.5}$$

$$|p - I_h p|_{W^s_\infty(\Omega)} \leq Ch^{r+1-s}|p|_{W^{r+1}_\infty(\Omega)} \quad \text{for } 0 \leq s \leq r, \tag{3.6}$$

provided u and p have proper regularities.

The following error estimates in L^2 and H^1 norms can be obtained; cf. e.g. [8,10–14].

Proposition 3.1. *Let $s = \min(k, r)$ and assume that $u, p \in H^{s+1}(\Omega)$. Then*

$$\|u - u_h\|_{L^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq Ch^{s+1}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}).$$

Proposition 3.2. *Let $s = \min(k, r)$ and assume that $u, p \in H^{s+1}(\Omega)$. Then*

$$\|u - u_h\|_{H^1(\Omega)} + \|p - p_h\|_{H^1(\Omega)} \leq Ch^{s}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}).$$

Proof. Using Proposition 2.1 and orthogonality property (2.5), we have

$$\|u_h - I_h u\|^2_{H^1(\Omega)} + \|p_h - I_h p\|^2_{H^1(\Omega)} \le CB(\mathbf{u}_h - I_h\mathbf{u}, \mathbf{u}_h - I_h\mathbf{u}) \le CB(\mathbf{u} - I_h\mathbf{u}, \mathbf{u}_h - I_h\mathbf{u}).$$

We next proceed to estimate terms in $B(\mathbf{u}-I_h\mathbf{u}, \mathbf{u}_h-I_h\mathbf{u})$; see definition of the bilinear form $B(\cdot,\cdot)$. By (3.2) and (3.5), we get

$$\begin{aligned}
&\big(p - I_h p, (1+b^2)(p_h - I_h p) - b(p_h - I_h p)'\big) + \big((p - I_h p)', (p_h - I_h p)' - b(p_h - I_h p)\big)\\
&\quad \le C\|p - I_h p\|_{H^1(\Omega)}\|p_h - I_h p\|_{H^1(\Omega)} \le Ch^s\|p\|_{H^{s+1}(\Omega)}\|p_h - I_h p\|_{H^1(\Omega)},\\
&\big(p - I_h p, bc(u_h - I_h u) - (u_h - I_h u)'\big) - \big((p - I_h p)', c(u_h - I_h u)\big)\\
&\quad \le C\|p - I_h p\|_{H^1(\Omega)}\|u_h - I_h u\|_{H^1(\Omega)} \le Ch^s\|p\|_{H^{s+1}(\Omega)}\|u_h - I_h u\|_{H^1(\Omega)},\\
&\big(u - I_h u, bc(p_h - I_h p) - c(p_h - I_h p)'\big) - \big((u - I_h u)', (p_h - I_h p)\big)\\
&\quad \le C\|u - I_h u\|_{H^1(\Omega)}\|p_h - I_h p\|_{H^1(\Omega)} \le Ch^s\|u\|_{H^{s+1}(\Omega)}\|p_h - I_h p\|_{H^1(\Omega)},\\
&\big(u - I_h u, c^2(u_h - I_h u)\big) + \big((u - I_h u)', (u_h - I_h u)'\big)\\
&\quad \le C\|u - I_h u\|_{H^1(\Omega)}\|u_h - I_h u\|_{H^1(\Omega)} \le Ch^s\|u\|_{H^{s+1}(\Omega)}\|u_h - I_h u\|_{H^1(\Omega)}.
\end{aligned}$$

It thus follows that

$$\begin{aligned}
&\|u_h - I_h u\|^2_{H^1(\Omega)} + \|p_h - I_h p\|^2_{H^1(\Omega)}\\
&\quad \le Ch^s(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)})(\|u_h - I_h u\|_{H^1(\Omega)} + \|p_h - I_h p\|_{H^1(\Omega)})\\
&\quad \le Ch^s(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)})\left(\|u_h - I_h u\|^2_{H^1(\Omega)} + \|p_h - I_h p\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}},
\end{aligned}$$

which implies

$$\|u_h - I_h u\|_{H^1(\Omega)} + \|p_h - I_h p\|_{H^1(\Omega)} \le Ch^s(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}). \tag{3.7}$$

Using (3.2), (3.5) and (3.7), we obtain the desired result. □

The estimates in Propositions 3.1 and 3.2 are optimal for $k = r$, which are the same as the convergence rates obtained for standard Galerkin methods. The estimates in one component can be improved when $k \ne r$. In particular, the following estimates are given in [12,13].

Proposition 3.3. *Let $\kappa = \min(k, r+1)$ and $\rho = \min(k+1, r)$. Assume that $u \in H^{k+1}(\Omega)$ and $p \in H^{r+1}(\Omega)$. Then*

$$\begin{aligned}
&\|u - u_h\|_{L^2(\Omega)} \le Ch^{\kappa+1}(\|u\|_{H^{\kappa+1}(\Omega)} + \|p\|_{H^{\kappa}(\Omega)}) \quad \text{for } r > 1,\\
&\|p - p_h\|_{L^2(\Omega)} \le Ch^{\rho+1}(\|u\|_{H^{\rho}(\Omega)} + \|p\|_{H^{\rho+1}(\Omega)}) \quad \text{for } k > 1.
\end{aligned}$$

Proposition 3.4. *Let $\kappa = \min(k, r+1)$ and $\rho = \min(k+1, r)$. Assume that $u \in H^{k+1}(\Omega)$ and $p \in H^{r+1}(\Omega)$. Then*

$$\begin{aligned}
&\|u - u_h\|_{H^1(\Omega)} \le Ch^{\kappa}(\|u\|_{H^{\kappa+1}(\Omega)} + \|p\|_{H^{\kappa}(\Omega)}),\\
&\|p - p_h\|_{H^1(\Omega)} \le Ch^{\rho}(\|u\|_{H^{\rho}(\Omega)} + \|p\|_{H^{\rho+1}(\Omega)}).
\end{aligned}$$

When $|k - r| = 1$, the estimates of Propositions 3.3 and 3.4 are optimal, since the order of convergence matches that of Galerkin methods. When $|k-r| > 1$, the estimates are no longer optimal. See estimates in Section 3.3 and numerical results in Section 4 for details; cf. also [12,13].

Remark 3.1. Proofs of Propositions 3.1, 3.3 and 3.4 are not included in this paper for the concerned problem, but they can be obtained by analogue analyses in the above mentioned references. Notice that the results of these propositions are not used in any proof of this paper. On the other hand, stronger results in maximum norms are provided in Sections 3.2 and 3.3.

Superconvergence in standard Galerkin FEMs for two-point boundary value problems has been well known for a long time. In particular, for elements of degree k, the numerical solution converges with rate $\mathcal{O}(h^{2k})$ at interelement points and rate $\mathcal{O}(h^{k+2})$ at the other elemental Lobatto points, and the differentiated numerical solution converges as $\mathcal{O}(h^{k+1})$ at Gauss points.

For the LSFEM, the superconvergence phenomena at interelement nodes and elemental Gauss points have been observed in [20] and analyzed in [12] for self-adjoint problems. The results at interelement points can be extended for general elliptic problem (2.1) as follows (cf. also [30,26]). Other superconvergence results will be investigated in Section 3.4.

Proposition 3.5. *Let $s = \min(k, r)$ and x_i be an interelement point. Assume that $u, p \in H^{s+1}(\Omega)$. Then,*

$$|(u - u_h)(x_i)| + |(p - p_h)(x_i)| \le Ch^{2s}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}).$$

Proof. Let $\mathbf{g}(\cdot;\xi) = [\varpi(\cdot;\xi), \upsilon(\cdot;\xi)]^T$ be the Green's function so that

$$(u-u_h)(\xi) = B(\mathbf{g}(\cdot;\xi), \mathbf{u}-\mathbf{u}_h).$$

Let $\xi = x_i$ and denote $\mathbf{g}(\cdot;x_i) = [\varpi(\cdot;x_i), \upsilon(\cdot;x_i)]^T$ by $\mathbf{g}_i = [\varpi_i, \upsilon_i]^T$, it follows that

$$(u-u_h)(x_i) = B(\mathbf{g}_i, \mathbf{u}-\mathbf{u}_h).$$

Note that $\mathbf{g}_i$ is continuous and smooth on both sides of x_i, and

$$\|\varpi_i\|_{H^{r+1}([0,x_i])} + \|\varpi_i\|_{H^{r+1}([x_i,1])} + \|\upsilon_i\|_{H^{k+1}([0,x_i])} + \|\upsilon_i\|_{H^{k+1}([x_i,1])} \le C. \tag{3.8}$$

Using orthogonality property (2.5), we have

$$(u-u_h)(x_i) = B(\mathbf{g}_i - I_h\mathbf{g}_i, \mathbf{u}-\mathbf{u}_h),$$

where $I_h\mathbf{g}_i = [I_h\varpi_i, I_h\upsilon_i]^T$ is the interpolant of $\mathbf{g}_i$ in $W_h \times V_h$. We next investigate terms in $B(\mathbf{g}_i - I_h\mathbf{g}_i, \mathbf{u}-\mathbf{u}_h)$ (cf. definition of the bilinear form). By interpolation estimates (3.1)–(3.6) and (3.8), we get

$$\begin{aligned}
&\big(p-p_h, (1+b^2)(\varpi_i - I_h\varpi_i) - b(\varpi_i - I_h\varpi_i)'\big) + \big((p-p_h)', (\varpi_i - I_h\varpi_i)' - b(\varpi_i - I_h\varpi_i)\big)\\
&\quad \le C\|\varpi_i - I_h\varpi_i\|_{H^1(\Omega)}\|p-p_h\|_{H^1(\Omega)} \le Ch^r\|p-p_h\|_{H^1(\Omega)},\\
&\big(p-p_h, bc(\upsilon_i - I_h\upsilon_i) - (\upsilon_i - I_h\upsilon_i)'\big) - \big((p-p_h)', c(\upsilon_i - I_h\upsilon_i)\big)\\
&\quad \le C\|\upsilon_i - I_h\upsilon_i\|_{H^1(\Omega)}\|p-p_h\|_{H^1(\Omega)} \le Ch^k\|p-p_h\|_{H^1(\Omega)},\\
&\big(u-u_h, bc(\varpi_i - I_h\varpi_i) - c(\varpi_i - I_h\varpi_i)'\big) - \big((u-u_h)', (\varpi_i - I_h\varpi_i)\big)\\
&\quad \le C\|\varpi_i - I_h\varpi_i\|_{H^1(\Omega)}\|u-u_h\|_{H^1(\Omega)} \le Ch^r\|u-u_h\|_{H^1(\Omega)},\\
&\big(u-u_h, c^2(\upsilon_i - I_h\upsilon_i)\big) + \big((u-u_h)', (\upsilon_i - I_h\upsilon_i)'\big)\\
&\quad \le C\|\upsilon_i - I_h\upsilon_i\|_{H^1(\Omega)}\|u-u_h\|_{H^1(\Omega)} \le Ch^k\|u-u_h\|_{H^1(\Omega)}.
\end{aligned}$$

It follows from Proposition 3.2 that

$$|(u-u_h)(x_i)| \le Ch^{2s}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}).$$

Similarly, we get the estimate

$$|(p-p_h)(x_i)| \le Ch^{2s}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}).$$

The desired result thus follows. □

Proposition 3.5 cannot be improved as in the cases of Propositions 3.3 and 3.4. See numerical results in Section 4.

3.2. Maximum-norm error estimates

We are now in a position to prove a maximum-norm error estimate for the LSFEM (cf. e.g., [25,31]).

Theorem 3.1. *Let $s = \min(k, r)$ and assume that $u, p \in H^{s+1}(\Omega)$. Then*

$$\|u-u_h\|_{L^\infty(\Omega)} + \|p-p_h\|_{L^\infty(\Omega)} \le Ch^{s+1}\left(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}\right).$$

Proof. Define projections $S_h : H_0^1(\Omega) \to V_h$ such that

$$\int_0^1 \big(c^2(S_hu-u)v + (S_hu-u)'v'\big)\,dx = 0 \quad \forall v \in V_h, \tag{3.9}$$

and $R_h : H^1(\Omega) \to W_h$ such that

$$\begin{cases} \displaystyle\int_0^1 \big((R_hp-p)q + (-(R_hp-p)' + b(R_hp-p))(-q'+bq)\big)\,dx = 0 \quad \forall q \in \mathring{W}_h,\\ (R_hp-p_h)(0) = (R_hp-p_h)(1) = 0, \end{cases} \tag{3.10}$$

where $\overset{\circ}{W}_h = W_h \cap H_0^1(\Omega)$. Note that $R_hp - p_h \in \overset{\circ}{W}_h$, by the coercivity in Proposition 2.1, (2.5), (3.9), (3.10), and integration by parts, we have

$$\begin{aligned}
&\|S_hu-u_h\|^2_{H^1(\Omega)}+\|R_hp-p_h\|^2_{H^1(\Omega)}\\
&\quad\le CB([R_hp-p_h,S_hu-u_h]^T,[R_hp-p_h,S_hu-u_h]^T)\\
&\quad= CB([R_hp-p,S_hu-u]^T,[R_hp-p_h,S_hu-u_h]^T)\\
&\quad= C\int_0^1\Big((1-c)(S_hu-u)(R_hp-p_h)'+bc(S_hu-u)(R_hp-p_h)\\
&\qquad+(R_hp-p)((c-1)(S_hu-u_h))'+bc(R_hp-p)(S_hu-u_h)\Big)\,dx\\
&\quad\le C\left(\|S_hu-u\|^2_{L^2(\Omega)}+\|R_hp-p\|^2_{L^2(\Omega)}\right)^{\frac12}\left(\|S_hu-u_h\|^2_{H^1(\Omega)}+\|R_hp-p_h\|^2_{H^1(\Omega)}\right)^{\frac12},
\end{aligned}$$

which leads to

$$\|S_hu-u_h\|_{H^1(\Omega)}+\|R_hp-p_h\|_{H^1(\Omega)}\le C\left(\|S_hu-u\|_{L^2(\Omega)}+\|R_hp-p\|_{L^2(\Omega)}\right). \tag{3.11}$$

For $x\in\Omega$, since $(R_hp-p_h)(0)=0$ and $(S_hu-u_h)(0)=0$, it follows

$$(S_hu-u_h)(x)=\int_0^x(S_hu-u_h)'(t)\,dt, \tag{3.12}$$

$$(R_hp-p_h)(x)=\int_0^x(R_hp-p_h)'(t)\,dt. \tag{3.13}$$

Therefore, by (3.12), (3.13), (3.11), (3.1) and (3.4), we have

$$\begin{aligned}
\|S_hu-u_h\|_{L^\infty(\Omega)}+\|R_hp-p_h\|_{L^\infty(\Omega)} &\le \|S_hu-u_h\|_{H^1(\Omega)}+\|R_hp-p_h\|_{H^1(\Omega)}\\
&\le C\left(\|S_hu-u\|_{L^2(\Omega)}+\|R_hp-p\|_{L^2(\Omega)}\right)\\
&\le Ch^{s+1}(\|u\|_{H^{s+1}(\Omega)}+\|p\|_{H^{s+1}(\Omega)}).
\end{aligned} \tag{3.14}$$

We next bound $\|S_hu-u\|_{L^\infty(\Omega)}$ and $\|R_hp-p\|_{L^\infty(\Omega)}$. As in [32], for each interelement point x_i, define

$$G_i(x)=\begin{cases}x(1-x_i) & 0\le x\le x_i,\\ (1-x)x_i & x_i\le x\le 1.\end{cases}$$

Then, on the one hand, by (3.9) and (3.1), we have

$$\begin{aligned}
|(S_hu-u)(x_i)| &= |((S_hu-u)',G_i')|=|(c(S_hu-u),cG_i)|\\
&\le C\|S_hu-u\|_{L^2(\Omega)}\le Ch^{s+1}\|u\|_{H^{s+1}(\Omega)}.
\end{aligned} \tag{3.15}$$

On the other hand, since $G_i\in\overset{\circ}{W}_h$, using (3.10), Proposition 3.5, and (3.4),

$$\begin{aligned}
|(R_hp-p)(x_i)| &= |((R_hp-p)',G_i')+(1-x_i)(R_hp-p)(0)+x_i(R_hp-p)(1)|\\
&\le |(R_hp-p,(1+b'+b^2)G_i)|+|(p_h-p)(0)|+|(p_h-p)(1)|\\
&\le C\|R_hp-p\|_{L^2(\Omega)}+Ch^{2s}(\|u\|_{H^{s+1}(\Omega)}+\|p\|_{H^{s+1}(\Omega)})\\
&\le Ch^{s+1}\|p\|_{H^{s+1}(\Omega)}+Ch^{2s}(\|u\|_{H^{s+1}(\Omega)}+\|p\|_{H^{s+1}(\Omega)})\\
&\le Ch^{s+1}(\|u\|_{H^{s+1}(\Omega)}+\|p\|_{H^{s+1}(\Omega)}).
\end{aligned} \tag{3.16}$$

Hence, for $x\in[x_{i-1},x_i]$, by (3.15), (3.16), (3.2) and (3.5),

$$\begin{aligned}
(S_hu-u)(x) &= (S_hu-u)(x_{i-1})+\int_{x_{i-1}}^x(S_hu-u)'(t)\,dt\\
&\le Ch^{s+1}\|u\|_{H^{s+1}(\Omega)}+h\|(S_hu-u)'\|_{L^2(\Omega)}\\
&\le Ch^{s+1}\|u\|_{H^{s+1}(\Omega)},\\
(R_hp-p)(x) &= (R_hp-p)(x_{i-1})+\int_{x_{i-1}}^x(R_hp-p)'(t)\,dt\\
&\le Ch^{s+1}(\|u\|_{H^{s+1}(\Omega)}+\|p\|_{H^{s+1}(\Omega)})+h\|(R_hp-p)'\|_{L^2(\Omega)}\\
&\le Ch^{s+1}(\|u\|_{H^{s+1}(\Omega)}+\|p\|_{H^{s+1}(\Omega)}).
\end{aligned}$$

It follows immediately from the above inequalities that

$$\|S_h u - u\|_{L^\infty(\Omega)} + \|R_h p - p\|_{L^\infty(\Omega)} \le Ch^{s+1}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}). \tag{3.17}$$

We finally complete the proof by combining (3.14) and (3.17). □

Remark 3.2. By Sobolev embedding theorem, it holds that $W_\infty^{s+1}(\Omega) \subset H^{s+1}(\Omega)$ (cf. e.g. [27,28]). Therefore, if $u, p \in W_\infty^{s+1}(\Omega)$, then the error estimate in Theorem 3.1 can be bounded by the corresponding $W_\infty^{s+1}(\Omega)$ norms of u and p. This same argument applies to results elsewhere in this paper as appropriate.

Next, we establish a superapproximation estimate. Let $\mathbf{N}_h\mathbf{u} = [M_h p, N_h u]^T$ be the projection of $\mathbf{u}$ into $W_h \times V_h$ so that

$$((M_h p - p)', q') = 0 \quad \forall q \in W_h, \tag{3.18}$$

$$((N_h u - u)', v') = 0 \quad \forall v \in V_h. \tag{3.19}$$

We have the following superapproximation property.

Theorem 3.2. *Let $s = \min(k, r)$ and assume that $u, p \in H^{s+1}(\Omega)$. Then*

$$\left\|(N_h u - u_h)'\right\|_{L^\infty(\Omega)} + \left\|(M_h p - p_h)'\right\|_{L^\infty(\Omega)} \le Ch^{s+1}\left(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}\right).$$

Proof. It follows from definitions (3.18)–(3.19) and orthogonal property (2.5) that

$$\begin{aligned}
\left((\mathbf{N}_h\mathbf{u} - \mathbf{u}_h)', \mathbf{v}'\right) &= \left((\mathbf{N}_h\mathbf{u} - \mathbf{u})', \mathbf{v}'\right) + \left((\mathbf{u} - \mathbf{u}_h)', \mathbf{v}'\right) \\
&= \left((\mathbf{u} - \mathbf{u}_h)', \mathbf{v}'\right) = \left((u - u_h)', v'\right) + \left((p - p_h)', q'\right) \\
&= \left(u_h - u,\ (1-c)q' + bcq + c^2 v\right) + \left(p_h - p,\ (b^2 + b' + 1)q + (c-1)v' + (bc + c')v\right) \\
&\quad - b(1)(p_h - p)(1)q(1) + b(0)(p_h - p)(0)q(0),
\end{aligned}$$

for all $\mathbf{v} = [q, v]^T \in W_h \times V_h$. Using Theorem 3.1, we have

$$\begin{aligned}
\left|\left((\mathbf{N}_h\mathbf{u} - \mathbf{u}_h)', \mathbf{v}'\right)\right| &\le C\left(\|u - u_h\|_{L^\infty(\Omega)} + \|p - p_h\|_{L^\infty(\Omega)}\right)\left(\|v\|_{W_1^1(\Omega)} + \|q\|_{W_1^1(\Omega)}\right) \\
&\le Ch^{s+1}\left(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}\right)\left(\|v\|_{W_1^1(\Omega)} + \|q\|_{W_1^1(\Omega)}\right).
\end{aligned}$$

Hence

$$\left|\left((M_h p - p_h)', q'\right)\right| \le Ch^{s+1}\left(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}\right)\|q\|_{W_1^1(\Omega)}, \tag{3.20}$$

$$\left|\left((N_h u - u_h)', v'\right)\right| \le Ch^{s+1}\left(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}\right)\|v\|_{W_1^1(\Omega)}, \tag{3.21}$$

for all $q \in W_h$ and $v \in V_h$.

For all $\psi \in L^1(\Omega)$, let $P\psi$ be the L^2-projection of ψ into $W_h' = \{q' : q \in W_h\}$, such that

$$(P\psi - \psi, q') = 0 \quad \forall q \in W_h. \tag{3.22}$$

Setting

$$q_\psi(x) = \int_0^x P\psi(t)\, dt - x\int_0^1 P\psi(t)\, dt,$$

we conclude that

$$q_\psi'(x) = P\psi(x) - \int_0^1 P\psi(t)\, dt \tag{3.23}$$

and $q_\psi \in \overset{\circ}{W}_h \subset H_0^1(\Omega)$. By Poincaré's inequality [27, p. 183], we obtain

$$\|q_\psi\|_{L^1(\Omega)} \le C|q_\psi|_{W_1^1(\Omega)},$$

which leads to

$$\|q_\psi\|_{W_1^1(\Omega)} \le C\|q_\psi'\|_{L^1(\Omega)}. \tag{3.24}$$

Note that

$$|q_\psi'(x)| - \int_0^1 |P\psi(t)|dt \le \left|q_\psi'(x) + \int_0^1 P\psi(t)\, dt\right| = |P\psi(x)|,$$

which implies

$$\|q'_\psi\|_{L^1(\Omega)} \leq 2\|P\psi\|_{L^1(\Omega)} \leq C\|\psi\|_{L^1(\Omega)}. \tag{3.25}$$

The second inequality in (3.25) is due to boundedness of the L^2-projection P. By using (3.22), (3.23), (3.18), (3.20), (3.24) and (3.25), we get

$$\begin{aligned}\left|\left((M_hp-p_h)',\psi\right)\right| &= \left|\left((M_hp-p_h)',P\psi\right)\right| = \left|\left((M_hp-p_h)',q'_\psi\right)\right| \\ &\leq Ch^{s+1}\left(\|u\|_{H^{s+1}(\Omega)}+\|p\|_{H^{s+1}(\Omega)}\right)\|q_\psi\|_{W_1^1(\Omega)} \\ &\leq Ch^{s+1}\left(\|u\|_{H^{s+1}(\Omega)}+\|p\|_{H^{s+1}(\Omega)}\right)\|\psi\|_{L^1(\Omega)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|(M_hp-p_h)'\right\|_{L^\infty(\Omega)} &= \sup_{\psi\neq 0}\frac{\left|\left((M_hp-p_h)',\psi\right)\right|}{\|\psi\|_{L^1(\Omega)}} \\ &\leq Ch^{s+1}\left(\|u\|_{H^{s+1}(\Omega)}+\|p\|_{H^{s+1}(\Omega)}\right).\end{aligned}$$

Similarly, we obtain

$$\left\|(N_hu-u_h)'\right\|_{L^\infty(\Omega)} \leq Ch^{s+1}\left(\|u\|_{H^{s+1}(\Omega)}+\|p\|_{H^{s+1}(\Omega)}\right).$$

The superapproximation estimate thus follows. □

Remark 3.3. Unlike the projection N_h, projection M_h is not unique but unique up to a constant. Nevertheless, since only the derivative of M_h is of concern, the constant difference is ignored.

We then have another estimate in maximum-norm.

Theorem 3.3. *Let $s=\min(k,r)$ and assume that $p\in W_\infty^{r+1}(\Omega)$ and $u\in W_\infty^{k+1}(\Omega)$. Then*

$$\|u-u_h\|_{W_\infty^1(\Omega)}+\|p-p_h\|_{W_\infty^1(\Omega)} \leq Ch^s\left(\|u\|_{W_\infty^{s+1}(\Omega)}+\|p\|_{W_\infty^{s+1}(\Omega)}\right).$$

Proof. By Theorems 3.1 and 3.2, it suffices to estimate $\left\|(M_hp-p)'\right\|_{L^\infty(\Omega)}$ and $\left\|(N_hu-u)'\right\|_{L^\infty(\Omega)}$. Using (3.3) and (3.6), we have

$$\left\|(M_hp-p)'\right\|_{L^\infty(\Omega)} \leq C\min_{\varpi\in W_h}\left\|(\varpi-p)'\right\|_{L^\infty(\Omega)} \leq Ch^r|p|_{W_\infty^{r+1}(\Omega)},$$

$$\left\|(N_hu-u)'\right\|_{L^\infty(\Omega)} \leq C\min_{\upsilon\in V_h}\left\|(\upsilon-u)'\right\|_{L^\infty(\Omega)} \leq Ch^k|u|_{W_\infty^{k+1}(\Omega)}.$$

We obtain the theorem. □

When $k=r$, the estimates in Theorems 3.1 and 3.3 are optimal. When $k\neq r$, the error estimates of p_h or u_h can be improved, due to higher degree in W_h or V_h, which will be developed in next section.

3.3. Improved maximum-norm error estimates

The results of Theorems 3.1 and 3.3 can be improved in one component if $k\neq r$, cf. Propositions 3.3 and 3.4.

Theorem 3.4. *Assume that $u,p\in H^{s+1}(\Omega)$. Assume further that $u,p\in W_\infty^{s+2}(\Omega)$ as appropriate. If $k\geq r+1$, then*

$$\|u-u_h\|_{W_\infty^1(\Omega)} \leq Ch^{r+1}\left(\|u\|_{W_\infty^{r+2}(\Omega)}+\|p\|_{H^{r+1}(\Omega)}\right), \tag{3.26}$$

$$\|u-u_h\|_{L^\infty(\Omega)} \leq Ch^{r+2}\left(\|u\|_{W_\infty^{r+2}(\Omega)}+\|p\|_{H^{r+1}(\Omega)}\right) \quad \text{for } r>1. \tag{3.27}$$

If $r\geq k+1$, then

$$\|p-p_h\|_{W_\infty^1(\Omega)} \leq Ch^{k+1}\left(\|u\|_{H^{k+1}(\Omega)}+\|p\|_{W_\infty^{k+2}(\Omega)}\right), \tag{3.28}$$

$$\|p-p_h\|_{L^\infty(\Omega)} \leq Ch^{k+2}\left(\|u\|_{H^{k+1}(\Omega)}+\|p\|_{W_\infty^{k+2}(\Omega)}\right) \quad \text{for } k>1. \tag{3.29}$$

Proof. When $k \geq r+1$, $s = \min(k,r) = r$. From Theorem 3.2, we have

$$\left\|(N_h u - u_h)'\right\|_{L^\infty(\Omega)} \leq Ch^{r+1}\left(\|u\|_{H^{r+1}(\Omega)} + \|p\|_{H^{r+1}(\Omega)}\right).$$

From interpolation error estimate (3.3), we have

$$\|N_h u - u\|_{W^1_\infty(\Omega)} \leq Ch^{r+1}|u|_{W^{r+2}_\infty(\Omega)}.$$

Estimate (3.26) thus follows from combining the above inequalities. Analogously, (3.27)–(3.29) are obtained. □

3.4. Superconvergence estimates at Lobatto and Gauss points

We will now establish a superconvergence error estimate at Lobatto points. Let F_i be the affine mapping from $[-1,1]$ to e_i. Denote L_k be the Legendre polynomial of degree k in $[-1,1]$. Let $l_{j,k}$ be the jth interior Lobatto point of order k in $[-1,1]$ (i.e. $L_k'(l_{j,k}) = 0$), $1 \leq j \leq k-1$. Then $F_i(l_{j,k})$ is the jth interior Lobatto point of order k in e_i. Note that there are two other Lobatto points of e_i, which are interelement points x_{i-1} and x_i discussed in Proposition 3.5 and are not considered in the following theorem.

Theorem 3.5. *Let $s = \min(k,r)$ and assume that $u, p \in H^{s+1}(\Omega)$. Then for $s > 1$, $1 \leq i \leq N$, $1 \leq j \leq k-1$, and $1 \leq \rho \leq r-1$,*

$$|(u-u_h)(F_i(l_{j,k}))| + |(p-p_h)(F_i(l_{\rho,r}))| \leq Ch^{s+2}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}).$$

Proof. We first consider $\mathbf{u} - \mathbf{N}_h\mathbf{u}$. For each element e_i, $i = 1, \dots, N$, expanding $(u - N_h u)'$ at the midpoint $(x_{i-1}+x_i)/2$ and writing the expansion in terms of Legendre polynomials, we have a decomposition

$$(u - N_h u)'(x) = c_0 L_0(F_i^{-1}(x)) + c_1 L_1(F_i^{-1}(x)) + \cdots + c_k L_k(F_i^{-1}(x)) + \mathcal{O}(h^{k+1}).$$

From definition (3.19), we conclude that $(u - N_h u)'$ is orthogonal to any polynomial of degree less than k in e_i. Therefore,

$$(u - N_h u)'(x) = c_k L_k(F_i^{-1}(x)) + \mathcal{O}(h^{k+1}) \quad \forall x \in e_i. \tag{3.30}$$

Similarly, we have

$$(p - M_h p)'(x) = c_r L_r(F_i^{-1}(x)) + \mathcal{O}(h^{r+1}) \quad \forall x \in e_i. \tag{3.31}$$

Now, for any $x \in e_i$, we have

$$(u-u_h)(x) = (u-u_h)(x_{i-1}) + \int_{x_{i-1}}^{x} \left((u - N_h u)'(t) + (N_h u - u_h)'(t)\right) dt. \tag{3.32}$$

Notice that $k(k+1)L_k(x) = -\left((1-x^2)L_k'(x)\right)'$, it follows from (3.30) that

$$\begin{aligned}\int_{x_{i-1}}^{x} (u - N_h u)'(t)\, dt &= c_k \int_{x_{i-1}}^{x} L_k(F_i^{-1}(t))\, dt + \mathcal{O}(h^{k+2}) \\ &= -\frac{c_k}{k(k+1)}(1 - (F_i^{-1}(x))^2)L_k'(F_i^{-1}(x)) + \mathcal{O}(h^{k+2}).\end{aligned} \tag{3.33}$$

Using Proposition 3.5, Theorem 3.2, (3.32) and (3.33), we reach to

$$\begin{aligned}\left|(u-u_h)(F_i(l_{j,k}))\right| &\leq C(h^{2s} + h^{s+2})\left(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}\right) \\ &\leq Ch^{s+2}\left(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}\right)\end{aligned}$$

for $1 \leq j \leq k-1$, provided $s > 1$. Similarly, for $s > 1$ and $1 \leq \rho \leq r-1$, we have

$$\left|(p-p_h)(F_i(l_{\rho,r}))\right| \leq Ch^{s+2}\left(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}\right).$$

The superconvergence result follows. □

Finally, we give another version of superconvergence at Gauss points, cf. [12]. Let $g_{j,k}$ be the jth Gauss point of order k in $[-1,1]$ (i.e. $L_k(g_{j,k}) = 0$), $1 \leq j \leq k$. We have the following result.

Theorem 3.6. *Let $s = \min(k,r)$ and assume that $u, p \in H^{s+1}(\Omega)$. Then for $1 \leq i \leq N$, $1 \leq j \leq k$, and $1 \leq \rho \leq r$,*

$$|(u-u_h)'(F_i(g_{j,k}))| + |(p-p_h)'(F_i(g_{\rho,r}))| \leq Ch^{s+1}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}).$$

Table 1
Convergence rates in different norms.

$\lvert k-r\rvert$	k	r	s	$\lVert e_h\rVert_\infty$	$\lVert \epsilon_h\rVert_\infty$	$\lVert e_h\rVert_\infty+\lVert e_h'\rVert_\infty$	$\lVert \epsilon_h\rVert_\infty+\lVert \epsilon_h'\rVert_\infty$	$\lVert e_h\rVert_M$	$\lVert \epsilon_h\rVert_M$	$\lVert e_h\rVert_L$	$\lVert \epsilon_h\rVert_L$	$\lVert e_h'\rVert_G$	$\lVert \epsilon_h'\rVert_G$
0	1	1	1	1.99	2.03	0.99	0.99	2.00	2.00	–	–	1.90	1.95
	2	2	2	3.04	2.95	1.96	2.00	4.04	3.97	3.80	3.87	2.81	2.90
	3	3	3	3.99	3.99	2.99	2.95	6.10	5.75	4.88	4.91	3.87	3.90
	4	4	4	4.95	4.96	4.60	4.34	7.90	8.01	5.88	5.73	4.88	4.79
1	2	1	1	1.98	2.06	1.96	0.97	1.98	1.97	1.98	–	1.97	1.98
	1	2	1	2.00	2.25	1.00	2.00	1.99	2.01	–	1.98	1.97	2.39
	3	2	2	4.00	2.96	2.98	2.00	3.87	3.91	4.00	3.94	2.99	3.02
	2	3	2	2.95	4.01	1.97	2.96	3.99	4.13	3.99	4.30	2.99	3.45
	4	3	3	4.89	4.00	3.94	2.95	6.19	5.84	4.88	4.91	3.88	3.91
	3	4	3	3.99	4.94	3.00	4.36	6.02	5.93	4.90	4.94	3.90	4.10
2	4	2	2	3.99	2.93	2.98	2.01	4.08	3.96	3.99	3.94	2.99	3.03
	2	4	2	2.94	4.15	1.96	2.90	3.98	4.01	3.99	3.92	2.99	2.91
	5	3	3	4.93	4.00	3.94	2.95	5.99	5.99	4.95	5.07	3.94	3.93
	3	5	3	3.99	4.29	3.00	4.61	5.89	5.90	4.97	4.99	3.95	3.95

Proof. From (3.30), we have that, in each e_i,

$$(u-u_h)'(x) = (N_h u - u_h)'(x) + c_k L_k(F_i^{-1}(x)) + \mathcal{O}(h^{k+1}).$$

It follows immediately from Theorem 3.2 that

$$|(u-u_h)'(F_i(g_{j,k}))| \le Ch^{s+1}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}).$$

Similarly, using (3.31) and Theorem 3.2, we have

$$|(p-p_h)'(F_i(g_{\rho,r}))| \le Ch^{s+1}(\|u\|_{H^{s+1}(\Omega)} + \|p\|_{H^{s+1}(\Omega)}).$$

We obtain the theorem. □

The superconvergence results in Theorems 3.5 and 3.6 cannot be improved when $k \neq r$. See numerical examples in Section 4.

Remark 3.4. Compared to the superconvergence estimate at Gauss points in [12], Theorem 3.6 requires less regularity of the weak solution $\mathbf{u}$.

4. Numerical results

In this section we consider the following test problem

$$\begin{cases} -(x+1)u''(x) + \dfrac{x^2+1}{2}\,u'(x) + 2u(x) = f(x) \quad \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $f(x)$ is determined from the exact solution

$$u(x) = 2(x^9 - \sin 2\pi x + e^x)(x - x^2).$$

The discrete problem is set up as described in the preceding sections using C^0 polynomial elements for V_h and W_h. The stiffness matrices and load vectors are calculated by symbolic algebra software (e.g. Maple™), so that no competitive numerical errors may arise in numerical integrations. A set of equidistance meshes of decreasing size are used for all numerical tests.

Let $e_h = u - u_h$ and $\epsilon_h = p - p_h$. Let $\|\cdot\|_\infty$, $\|\cdot\|_M$, $\|\cdot\|_L$, and $\|\cdot\|_G$ be discrete maximum-norms evaluated at randomly selected points, the interelement points (including boundary points 0 and 1), the interior Lobatto points, and the Gauss points, respectively, in all elements. Convergence rates of e_h and ϵ_h in different norms are summarized in Table 1.

In the case $k = r$, we computed least-squares results for linear, quadratic, cubic, and quartic elements. We observe from Table 1 that $\|e_h\|_\infty$ and $\|\epsilon_h\|_\infty$ converge with rate $\mathcal{O}(h^{s+1})$ and $\|e_h\|_\infty + \|e_h'\|_\infty$ and $\|\epsilon_h\|_\infty + \|\epsilon_h'\|_\infty$ converge in $\mathcal{O}(h^s)$, which coincide with the estimates in Theorems 3.1 and 3.3 respectively. On the other hand, the data verify that the errors of u_h and p_h converge with rates $\mathcal{O}(h^{2s})$ at interelement points, as indicated in Proposition 3.5. The convergence rates of e_h' and ϵ_h' at Gauss points are $\mathcal{O}(h^{s+1})$, which confirm Theorem 3.6. When $s > 1$, the errors of u_h and p_h at interior Lobatto points (not including the interelement points) are $\mathcal{O}(h^{s+2})$ as predicted in Theorem 3.5. See also [20,12].

When $k \neq r$, we considered the cases $|k-r| = 1$ and $|k-r| = 2$. We are concerned with the improved maximum-norm error estimates in Theorem 3.4. In particular, when $|k-r| = 1$, the estimates are optimal, since the least-squares convergence rates corresponding to the orders of the finite element spaces are the same as those of the standard Galerkin method. When $|k-r| = 2$, the improved estimates cannot however reach the optimal Galerkin rates of the finite element space of higher degree. Moreover, the "standard" superconvergence rate specified in Proposition 3.5, Theorems 3.5 and 3.6 cannot be improved further. Therefore, in order to achieve higher convergence rates, it is not sufficient or efficient to keep increasing the degree for only one of spaces V_h and W_h, and cases of $|k-r| \geq 2$ are not recommended.

5. Conclusions

In this paper, we considered the convergence and superconvergence properties of LSFEM for general second-order two-point boundary value problems. Optimal maximum-norm error estimates are obtained for the case $k = r$, which can be improved for the component of higher degree when $|k-r| > 0$. Superconvergence for solution values at Lobatto points as well as for the derivatives at Gauss points are investigated, which nonetheless cannot be improved when $|k-r| > 0$. The convergence and superconvergence results of LSFEM coincide with those by using standard Galerkin methods. All numerical results are consistent with our theoretical results, which indicate that $|k-r| \geq 2$ is not an efficient choice for finite element spaces V_h and W_h.

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