SIAM J. NUMER. ANAL.
Vol. 50, No. 1, pp. 272–286




# FUNCTION VALUE RECOVERY AND ITS APPLICATION IN EIGENVALUE PROBLEMS*

AHMED NAGA† AND ZHIMIN ZHANG†

**Abstract.** Function value recovery techniques for linear finite elements are discussed. Using the recovered function and its gradient, we are able to enhance the eigenvalue approximation and increase its convergence rate to $h^{2\alpha}$, where $\alpha > 1$ is the superconvergence rate of the recovered gradient. This is true in both symmetric and nonsymmetric eigenvalue problems.

**Key words.** finite element method, recovery, superconvergence, eigenvalue

**AMS subject classifications.** 65N30, 65N15, 65N12, 65D10, 74S05, 41A10, 41A25

**DOI.** 10.1137/100797709

## 1. Introduction.

Finite element recovery techniques are postprocessing methods that are widely used in scientific and engineering applications [27]. The purpose is twofold: (1) to improve the solution or relevant values and (2) to provide a base for a posteriori error estimates. There have been intensive studies in the last decade; see, e.g., [1, 3, 4, 7, 13, 16, 17, 21, 24, 26, 28, 29] and references therein.

For second order elliptic problems, researchers have focused on gradient recovery, since the gradient is of primary interest in most practical problems and has the dominant error term compared to the solution itself. Meanwhile, there were some attempts to recover function values in the literature [14, 22, 19]. Different from gradient recovery, function value recovery does not result in superconvergence for the linear finite element. This fact is explained in [19] and we shall discuss it further in this paper.

It is well known that the convergence/superconvergence rate of $u - u_h$ for the $C^0$ finite elements cannot be better than $2k$ [20], where $k$ is the polynomial degree of the finite element space. Intuitively, for linear elements ($k = 1$), this limit is reached $\|u - u_h\|_{L_2(\Omega)} = O(h^2)$, and we would not expect better results except for a very extreme case: the equilateral triangle elements [5]. Nevertheless, we shall show in this paper that the gradient of the recovered function superconverges to the exact gradient on mildly structured meshes at a rate of $\alpha \in (1, 2]$. Using the recovered function and its superconvergent gradient, it is possible to improve eigenvalues in both symmetric and nonsymmetric eigenvalue problems. Furthermore, the improved, or recovered, eigenvalues converge to the exact eigenvalue at a rate of $2\alpha$. If $\alpha$ is close to 2, as it is in some cases, the convergence rate of the recovered eigenvalue is close to 4. Note that this is ultra-convergence. Also, note that the best achieved result in the nonsymmetric cases is 3; see [25, 12].

The motivation behind this paper came after publishing [18] in which we explored a technique to enhance the numerical eigenvalue $\lambda_h$ using $\|G_h u_h - \nabla u_h\|_{0,\Omega}$,

*Received by the editors June 14, 2010; accepted for publication (in revised form) November 14, 2011; published electronically February 9, 2012.
http://www.siam.org/journals/sinum/50-1/79770.html

†Department of Mathematics, Wayne State University, Detroit, MI 48202 (anaga@math.wayne.edu, zzhang@math.wayne.edu). This research was supported in part by the U.S. National Science Foundation through grants DMS-0612908 and DMS-1115530, the Ministry of Education of China through the Changjiang Scholars program, and Guangdong Provincial Government of China through the "Computational Science Innovative Research Team" program, and Guangdong Province Key Laboratory of Computational Science at the Sun Yat-sen University.

where $G_h$ is a gradient recovery operator (see section 4). The enhanced eigenvalue approximation, denoted by $\tilde{\lambda}_h$, was defined by

$$\tilde{\lambda}_h = \lambda_h - \|G_h u_h - \nabla u_h\|_{0,\Omega}^2. \tag{1.1}$$

Theoretically, it was shown that $\tilde{\lambda}_h$ converges at $h^{\alpha+1}$ ($\alpha = 1+\rho$, $\rho > 0$) when $G_h u_h$ superconverges, in the $L_2$ norm, to $\nabla u$ at a rate of $h^\alpha$ on mildly structured meshes. However, the numerical experiments in [18] indicated that $\tilde{\lambda}_h$ has a convergence rate of $h^{2\alpha}$. The gap between the numerical experiments and the theoretical results is mainly due to the theoretical difficulties imposed by (1.1). To bridge this gap, the authors looked into finding another method that achieves the convergence rate $h^{2\alpha}$ both theoretically and computationally. As we shall see, the recovered function and its gradient can be used to achieve this goal. For the Laplace operator, if $R_h u_h$ is the recovered function of the eigenfunction $u_h$, we define the recovered eigenvalue by

$$\hat{\lambda}_h = \frac{\|\nabla(R_h u_h)\|_{0,\Omega}^2}{\|R_h u_h\|_{0,\Omega}^2}. \tag{1.2}$$

As we shall see in section 5, $\hat{\lambda}_h$ converges to the exact eigenvalue at a rate of $h^{2\alpha}$ in symmetric eigenvalue problems. Moreover, (1.2) can be modified to achieve the same rate in nonsymmetric eigenvalue problems.

## 2. Preliminaries.

Consider the second order elliptic boundary value problem

$$\begin{cases} Lu = -\nabla\cdot(\mathbf{D}\nabla u) + \mathbf{b}\cdot\nabla u + cu = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain. The boundary $\partial\Omega = \Gamma$, $\mathbf{D}$ is a $2\times 2$ positive definite matrix on $\Omega$, and $\mathbf{b}$ is a vector function. For simplicity, all the coefficients and the boundary conditions in (2.1) are assumed to be real-valued functions. As usual, $H^m(\Omega)$ is the classical Sobolev space of order $m$ equipped with the norm $\|\cdot\|_{m,\Omega}$ and the seminorm $|\cdot|_{m,\Omega}$. The set of all polynomials defined on $\Omega' \subseteq \mathbb{R}^2$ of total degree $\le r$ will be denoted by $P_r(\Omega')$. We assume that $\mathbf{D}$, $\mathbf{b}$, $c$, and $f$ are sufficiently smooth such that $u \in H^3(\Omega)\cap W^2_\infty(\Omega)$.

In the variational form of (2.1), we seek $u \in V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$ such that

$$a(u,v) = l(v) \quad \forall\, v \in V, \tag{2.2}$$

where

$$a(u,v) = \int_\Omega (\mathbf{D}\nabla u)\cdot\nabla v + (\mathbf{b}\cdot\nabla u)v + cuv \quad \text{and} \quad l(v) = \int_\Omega fv.$$

We further assume that $a(\cdot,\cdot)$ is continuous and V-elliptic, i.e.,

$$|a(u,v)| \lesssim \|u\|_{1,\Omega}\|v\|_{1,\Omega} \text{ and } a(u,u) \gtrsim \|u\|_{1,\Omega}^2 \quad \forall\, u,v \in V.$$

Note that for the V-elliptic property to hold, we must have $(c - \nabla\cdot\mathbf{b}/2) \ge 0$. Let $\mathcal{T}_h$ be a shape regular triangulation of $\Omega$ with mesh size parameter $h \in (0,1)$, let $\mathcal{N}_h$ denote the set of mesh nodes, let $\mathcal{M}_h$ denote the set of midpoints of the mesh edges, and set $\tilde{\mathcal{N}}_h = \mathcal{N}_h \cup \mathcal{M}_h$. If $z \in \mathcal{M}_h$, $z_1$ and $z_2$ will denote the vertices of the edge containing $z$. In linear finite element methods, the finite element space $V_h$ associated with $\mathcal{T}_h$ is defined by

$$V_h = \{v \in C(\overline{\Omega}) : v \in P_1(T) \text{ for every triangle } T \in \mathcal{T}_h\}.$$

The subspace $V_h \cap V$ will be denoted by $V_{h,0}$. The finite element approximation $u_h \in V_h$ of $u$ is computed by restricting $v$ in (2.2) to $V_{h,0}$ , i.e.,

$$a(u_h, v) = l(v) \quad \forall v \in V_{h,0}. \tag{2.3}$$

Let $I_h : C^0(\overline{\Omega}) \to V_h$ denote the usual Lagrange interpolation operator, let $\tilde{V}_h$ denote the piecewise quadratic $C^0$ finite element space, and let $\tilde{V}_{h,0} = \tilde{V}_h \cap V$. Note that $\tilde{V}_h = V_h \bigoplus \bar{V}_h$, where $\bar{V}_h$ is the space of quadratic "bump" functions, i.e., $I_h v = 0$ for all $v \in \bar{V}_h$. A triangular mesh $\mathcal{T}_h$ satisfies the condition $(\rho_1, \rho_2)$ for some $\rho_1, \rho_2 > 0$ if $\mathcal{T}_h$ admits a partition consisting of two sets that satisfy the following. In the first set, every pair of adjacent triangles forms an $O(h^{1+\rho_1})$ parallelogram for some $\rho_1 > 0$. (An $O(h^\delta)$ parallelogram, $\delta > 0$, is a quadrilateral in which the distance between the midpoints of its diagonals is $O(h^\delta)$.) In the second set, the triangles' total area is $O(h^{\rho_2})$ for some $\rho_2 > 0$. The ideal value for $\rho_1$ and $\rho_2$ is $\infty$. Note that $\rho_1 = \rho_2 = \infty$ if any edge in any triangle in $\mathcal{T}_h$ is parallel to one of three fixed directions. This ideal case was covered in [5], where the authors completely determined the expansion of $u - u_h$ at mesh vertices. The case $\rho_1 = 2$ and $\rho_2 = \infty$ is another interesting case that was studied in [15, 13], while the general case was treated in [24]. If a triangular mesh $\mathcal{T}_h$ satisfies the condition $(\rho_1, \rho_2)$ for some $\rho_1, \rho_2 > 0$ and if $V_h$ consists of piecewise linear polynomials, then $|I_h u - u_h|_{1,\Omega}$ is $O(h^{1+\rho})$, where $\rho = \frac{1}{2}\min\{1, 2\rho_1, \rho_2\}$; see [24] for the details. The following theorem [24] is crucial to our work.

THEOREM 2.1. *Let the solution of* (2.1) $u \in H^3(\Omega) \cap W^2_\infty(\Omega)$, *and let* $u_h \in V_h$ *be the solution of* (2.3). *If the triangulation* $\mathcal{T}_h$ *satisfies the condition* $(\rho_1, \rho_2)$ *for some* $\rho_1, \rho_2 > 0$, *then*

$$\|u_h - I_h u\|_{1,\Omega} \lesssim h^{1+\rho}(\|u\|_{3,\Omega} + \|u\|_{2,\infty,\Omega}), \tag{2.4}$$

*where* $\rho = \frac{1}{2}\min\{1, 2\rho_1, \rho_2\}$.

*Remark* 2.2. Theorem 2.1 provides sufficient conditions for the superconvergence result in (2.4). This requires the data for (2.1) to satisfy certain requirements. However, discussing these requirements is outside the scope of this work.

## 3. A class of function-recovery operators.

Let $\mathcal{R}$ denote the set of all linear recovery operators $R_h : V_h \to \tilde{V}_h$ such that

- R0: $\forall T \in \mathcal{T}_h$ and $v \in V_h$, $R_h v|_T$ is constructed using $v|_{K_T}$, where $K_T$ is the union of mesh elements in a bounded number of layers around $T$;
- R1: $R_h(I_h q) = q \quad \forall q \in P_2(\Omega)$;
- R2: $\forall v \in V_h, T \in \mathcal{T}_h$, $\|R_h v\|_{0,T} \le c_1 \|v\|_{0,K_T}$, where $c_1 > 0$ is a constant independent of $h$;
- R3: $\|R_h v\|_{0,\Omega} \ge c_2 \|v\|_{0,\Omega}$, where $c_2 > 0$ is a constant independent of $h$; and
- R4: $R_h v \in \tilde{V}_{h,0} \quad \forall v \in V_{h,0}$.

The property R0 ensures that $R_h$ is local in nature to minimize the computational cost. The property R1 says that $R_h$ preserves quadratic polynomials. This is important in establishing the approximation properties of $R_h$. The properties R2 and R3 ensure that $R_h$ is bounded and is well defined. The property R4 means that the recovery operator respects boundary conditions when applied to finite element solutions. This is achieved by setting $(R_h v)(z) = 0$ for all $z \in \tilde{\mathcal{N}}_h \cap \Gamma_D$.

The following lemma establishes the main approximation property for operators in $\mathcal{R}$, and its proof is a direct result of R1 and the Bramble–Hilbert lemma; see [9, pp. 116–126] for more details.

LEMMA 3.1. *If $R_h \in \mathcal{R}$, then*

$$|u - R_h(I_h u)|_{k,\Omega} \lesssim h^{3-k}|u|_{3,\Omega} \quad \forall\, u \in H^3(\Omega) \text{ and } k = 0, 1.$$

*Remark* 3.2. Using the Bramble–Hilbert lemma to prove Lemma 3.1 requires $R_h$ to be bounded. This follows from R2 and Lemma 3.5.

$R_h u_h$ is not expected to superconverge to $u$ because of the following lemma.

LEMMA 3.3. $\|u - R_h u_h\|_{0,\Omega} = o(h^2)$ *if and only if* $\|I_h u - u_h\|_{0,\Omega} = o(h^2)$.

*Proof.* By virtue of the property R2, Lemma 3.1, and the fact that

$$u - R_h u_h = (u - R_h(I_h u)) + R_h(I_h u - u_h),$$

$\|u - R_h u_h\|_{0,\Omega} = o(h^2)$ if $\|u_h - I_h u\|_{0,\Omega} = o(h^2)$. On the other hand, R3 leads to

$$\|I_h u - u_h\|_{0,\Omega} \lesssim \|R_h(I_h u - u_h)\|_{0,\Omega} \le \|u - R_h(I_h u)\|_{0,\Omega} + \|u - R_h u_h\|_{0,\Omega}.$$

Again, Lemma 3.1 implies that $\|I_h u - u_h\|_{0,\Omega} = o(h^2)$ if $\|u - R_h u_h\|_{0,\Omega} = o(h^2)$. ∎

The previous lemma shows that $\|u - R_h u_h\|_{0,\Omega}$ has superconvergence if and only if $\|I_h u - u_h\|_{0,\Omega}$ has superconvergence. Unfortunately, the later quantity has no superconvergence even under very ideal conditions; see [19] for an example that verifies this statement. So, we will not think of the function recovery as a tool for enhancing the function itself, at least for linear elements.

*Remark* 3.4. Since recovered functions will not have superconvergence, we can change Lemma 3.1 so that

$$|u - R_h(I_h u)|_{k,\Omega} \lesssim h^{2-k}|u|_{2,\Omega} \quad \forall\, u \in H^2(\Omega) \text{ and } k = 0, 1.$$

Moreover, condition R1 can be relaxed so that

$$R_h(I_h q) = q \quad \forall\, q \in P_1(\Omega).$$

Although the operators in $\mathcal{R}$ do not improve the convergence rate of the recovered function, the gradient of the recovered function may have a better convergence rate than the gradient of the original function. This is the main ingredient in studying the eigenvalue enhancement. The following lemma is crucial for the rest of this work.

LEMMA 3.5. *If $R_h \in \mathcal{R}$ and $v \in V_h$, then*

$$\|\nabla(R_h v)\|_{0,\Omega} \lesssim \|\nabla v\|_{0,\Omega}.$$

*Proof.* Let $T \in \mathcal{T}_h$. Since $R_h$ is linear and satisfies the condition R1, we have

$$\|\nabla(R_h v)\|_{0,T} = \|\nabla(R_h(v - \overline{v}_T))\|_{0,T},$$

where $\overline{v}_T = \frac{1}{|K_T|}\int_{K_T} v$. Using the inverse inequalities [1, 6] and the boundedness condition in R2, we get

$$\|\nabla(R_h(v - \overline{v}_T))\|_{0,T} \lesssim h^{-1}\|R_h(v - \overline{v}_T)\|_{0,T} \lesssim h^{-1}\|v - \overline{v}_T\|_{0,K_T}.$$

By virtue of Poincaré's inequality [6, pp. 134–135], the last inequality leads to

$$\|\nabla(R_h v)\|_{0,T} \lesssim \|\nabla v\|_{0,K_T}.$$

This inequality together with the definition of $\|\nabla(R_h v)\|_{0,\Omega}$ completes the proof. ∎

We are now ready to prove the first key result in this paper.

THEOREM 3.6. *Let the triangulation $\mathcal{T}_h$ satisfy the condition $(\rho_1, \rho_2)$ for some $\rho_1, \rho_2 > 0$. If $R_h \in \mathcal{R}$, $u \in H^3(\Omega) \cap W^2_\infty(\Omega)$, then*

$$\|\nabla(u - R_h u_h)\|_{0,\Omega} \lesssim h^\alpha,$$

*where $\alpha = 1 + \frac{1}{2}\min\{1, 2\rho_1, \rho_2\}$.*

*Proof.* Note that

$$\|\nabla(u - R_h u_h)\|_{0,\Omega} \le \|\nabla(u - R_h(I_h u))\|_{0,\Omega} + \|\nabla(R_h(I_h u - u_h))\|_{0,\Omega}.$$

From Lemma 3.5,

$$\|\nabla(R_h(I_h u - u_h))\|_{0,\Omega} \lesssim \|\nabla(I_h u - u_h)\|_{0,\Omega}$$

and the last term is $O(h^\alpha)$ by Theorem 2.1. Also, $\|\nabla(u - R_h(I_h u))\|_{0,\Omega}$ is $O(h^2)$ by Lemma 3.1, and this concludes the proof. ∎

*Remark* 3.7. According to Theorem 3.6, $\nabla R_h u_h$ superconverges to $\nabla u$. Consequently, $\nabla R_h u_h$ can be used in constructing the asymptotically exact a posteriori error estimator $\eta_h = \|\nabla(R_h u_h) - \nabla u_h\|_{0,\Omega}$; see [16] for a standard argument to prove this fact.

*Remark* 3.8. We would like to point out that the result in Theorem 3.6 can be localized as in Theorem 4.3 in [24]. Since eigenvalue approximation needs global recovery, we will not go over local versions of Theorem 3.6.

*Remark* 3.9. Although Theorem 3.6 is stated in the context of mildly structured meshes, its result is true as long as $\|\nabla(I_h u - u_h)\|_{0,\Omega}$ is $O(h^\alpha)$ for some $\alpha > 1$. In fact, the recovered gradient can be superconvergent even for adaptive meshes [23].

**4. Examples for operators in the class $\mathcal{R}$.** In this section we will go over two examples for operators in $\mathcal{R}$.

**4.1. Example 1.** Our first example is about operators that are based on gradient recovery operators. Let $\mathcal{G}$ denote the set of all the linear gradient recovery operators $G_h : V_h \to V_h \times V_h$ such that

- G0: $\forall T \in \mathcal{T}_h$ and $v \in V_h$, $G_h v|_T$ is constructed using $v|_{K_T}$ or $\nabla v|_{K_T}$, where $K_T$ is the union of mesh elements in a bounded number of layers around $T$;
- G1: $G_h(I_h q) = \nabla q \;\; \forall q \in P_2(\Omega)$; and
- G2: $\forall T \in \mathcal{T}_h$ and $v \in V_h$, $\|G_h v\|_{0,T} \le c_1 \|\nabla v\|_{0,K_T}$, and $c_1$ is a finite positive constant that is independent of $h$.

Note that $G_h v$, called the recovered gradient of $v$, is a continuous approximation of $\nabla v$. An example for operators in $\mathcal{G}$ is the polynomial-preserving gradient recovery operator (PPR) that was introduced in [26] and theoretically studied in [16, 17].

Now, let $v \in V_h$, $G_h \in \mathcal{G}$, and let $R_h : V_h \to \tilde{V}_h$. We will define $R_h$ using $G_h$. To define $R_h v \in \tilde{V}_h$, it suffices to define it on $\tilde{\mathcal{N}}_h$. Let $z \in \tilde{\mathcal{N}}_h$. We define $(R_h v)(z)$ as follows:

1. If $z \in \mathcal{N}_h$, set $(R_h v)(z) = v(z)$.
2. If $z \in \mathcal{M}_h$ is the midpoint of an element edge $e$ with ends $z_1$ and $z_2$, we first compute $p_e \in P_3(e)$ such that

$$p_e(z_i) = v(z_i) \text{ and } p_e'(z_i) = G_h v(z_i) \cdot \frac{\vec{\ell}}{|\vec{\ell}|} \qquad \text{for } i = 1, 2, \tag{4.1}$$

where $\vec{\ell} = z_2 - z_1$ is the vector from $z_1$ to $z_2$. Then

$$(R_h v)(z) = p_e(z) = \frac{v(z_1) + v(z_2)}{2} + \frac{\vec{\ell} \cdot [(G_h v)(z_1) - (G_h v)(z_2)]}{8}. \tag{4.2}$$

Since $v$ is linear, (4.2) is equivalent to

$$(R_h v)(z) = v(z) + \frac{\vec{\ell} \cdot [(G_h v)(z_1) - (G_h v)(z_2)]}{8}.$$

From the definitions of $R_h v$ and $\tilde{V}_h$, one can write

$$R_h v = v + v_b, \tag{4.3}$$

where $v_b \in \bar{V}_h$ and

$$v_b(z) = \begin{cases} 0 & \text{if } z \in \mathcal{N}_h, \\ \frac{1}{8}\vec{\ell} \cdot [(G_h v)(z_1) - (G_h v)(z_2)] & \text{if } z \in \mathcal{M}_h. \end{cases} \tag{4.4}$$

Lemma 4.1. *Let $v \in V_h$, $G_h \in \mathcal{G}$, and let $v_b \in \bar{V}_h$ be defined as in* (4.4). *Then,*

$$\|v_b\|_{0,T} \lesssim h\|G_h v - \nabla v\|_{0,T} \quad \forall T \in \mathcal{T}_h.$$

*Proof.* We first note that

$$\|v_b\|_{0,T} \lesssim h\|v_b\|_{\infty,T} \quad \text{and} \quad \|v_b\|_{\infty,T} = |v_b(z)| \text{ for some } z \in T \cap \mathcal{M}_h.$$

By the definition of $v_b$ and noting that $\nabla v|_T$ is constant,

$$\begin{aligned} |v_b(z)| &\lesssim h\,|(G_h v)(z_1) - (G_h v)(z_2)| \\ &= h\,|(G_h v - \nabla v)(z_1) - (G_h v - \nabla v)(z_2)| \\ &\lesssim h\|G_h v - \nabla v\|_{\infty,T}. \end{aligned}$$

Applying the inverse inequality to the right-hand side of the last term, we obtain

$$|v_b(z)| \lesssim \|G_h v - \nabla v\|_{0,T},$$

which concludes the proof of our lemma. ∎

Theorem 4.2. *Let $v \in V_h$, $G_h \in \mathcal{G}$, and let $v_b \in \bar{V}_h$ be defined as in* (4.4). *Let $R_h : V_h \to \tilde{V}_h$ be defined as in* (4.3). *Then, $R_h \in \mathcal{R}$.*

*Proof.* Since $G_h$ satisfies G0, $R_h$ satisfies R0. To prove that $R_h$ satisfies R1, let $q \in P_2(\Omega)$. From the definition of $R_h$, $R_h(I_h q) = q$ if and only if $(R_h(I_h q))(z) = q(z)$ for all $z \in \mathcal{M}_h$. So, let $z \in \mathcal{M}_h$ be the midpoint of an element edge $e$ with ends $z_1$ and $z_2$. Using the definition of $R_h$ and G1, $(R_h(I_h q))(z_i) = q(z_i)$ and $(G_h(I_h q))(z_i) = \nabla q(z_i)$ for $i = 1, 2$. If $p_e \in P_3(e)$ satisfies the conditions in (4.1), when $v = I_h q$, then

$p_e = q|_e$. This is because we are fitting the data of the quadratic polynomial $q$ along the edge $e$ by the cubic polynomial $p_e$. Hence, $(R_h(I_h q))(z) = p_e(z) = q(z)$.

For the rest of the proof, we need to show that

$$\|v_b\|_{0,T} \lesssim \|v - \bar{v}_T\|_{0,K_T},$$

where $\bar{v}_T = \frac{1}{|K_T|}\int_{K_T} v$ for all $T \in \mathcal{T}_h$. Using Lemma 4.1, we have

$$\begin{aligned}
\|v_b\|_{0,T} &\lesssim h\|G_h v - \nabla v\|_{0,T} \\
&= h\|G_h(v - \bar{v}_T) - \nabla(v - \bar{v}_T)\|_{0,T} \\
&\lesssim h(\|G_h(v - \bar{v}_T)\|_{0,T} + \|\nabla(v - \bar{v}_T)\|_{0,T}) \\
&\lesssim h(\|\nabla(v - \bar{v}_T)\|_{0,K_T} + \|\nabla(v - \bar{v}_T)\|_{0,T}) && (4.5) \\
&\lesssim h\|\nabla(v - \bar{v}_T)\|_{0,K_T} && (4.6) \\
&\lesssim \|v - \bar{v}_T\|_{0,K_T}. && (4.7)
\end{aligned}$$

Equation (4.5) is due to the fact that $G_h$ satisfies G2 and (4.7) follows by applying the inverse inequality to (4.6). To prove that $R_h$ satisfies the inequality in R2, we have

$$\begin{aligned}
\|R_h v\|_{0,T} &\lesssim \|v\|_{0,T} + \|v_b\|_{0,T} \\
&\lesssim \|v\|_{0,K_T} + \|v - \bar{v}_T\|_{0,K_T}.
\end{aligned}$$

Note that $\|v\|^2_{0,K_T} = \|\bar{v}_T\|^2_{0,K_T} + \|v - \bar{v}_T\|^2_{0,K_T}$. Hence, $\|v - \bar{v}_T\|_{0,K_T} \leq \|v\|_{0,K_T}$. Using equivalence of norms on finite dimensional spaces and a scaling argument, we verify that $\|R_h v\|_{0,T} \geq C\|v\|_{0,K_T}$, where $C$ is independent of $h$ and depends only on the mesh geometry; see [16, 17] for a similar argument. Hence, $R_h$ satisfies R3, and the proof is complete. ◻

*Remark* 4.3. The restriction in G1 is because of the restriction in R1. However, if we relax R1 as we mentioned in Remark 3.4, we can relax G1 to the following form:

$$G_h(I_h p) = p \quad \forall\, p \in P_1(\Omega) \quad \text{and} \quad G_h(I_h p) = p + O(h^2) \quad \forall\, p \in P_2(\Omega)\backslash P_1(\Omega).$$

This allows for including the famous ZZ superconvergence patch recovery [29] into $\mathcal{G}$.

**4.2. Example 2.** Our second example for operators in $\mathcal{R}$ relies on an algorithm similar to the one used in the PPR gradient recovery operator [26, 16, 17]. As we mentioned in Example 1, to completely define $R_h v$ for $R_h \in \mathcal{R}$ and $v \in V_h$, we need to define $R_h v$ on $\tilde{\mathcal{N}}_h$. For $z \in \mathcal{N}_h$, let $K_z$ denote a patch of mesh elements around $z$. We will define $K_z$ in a moment. Let $p_z \in P_2(K_z)$ be the polynomial that best fits $v$ at the mesh nodes in $K_z$ in the discrete least-squares sense, i.e.,

$$\sum_{\tilde{z} \in \mathcal{N}_h \cap K_z} |(v - p_z)(\tilde{z})|^2 = \min_{p \in P_2(K_z)} \sum_{\tilde{z} \in \mathcal{N}_h \cap K_z} |(v - p)(\tilde{z})|^2. \tag{4.8}$$

Then,

$$(R_h v)(z) = p_z(z).$$

For $z \in \mathcal{M}_h$, let $z$ be the midpoint of an edge with ends $z_1, z_2 \in \mathcal{N}_h$. Then,

$$(R_h v)(z) = \frac{p_{z_1}(z) + p_{z_2}(z)}{2},$$

where $p_{z_i} \in P_2(K_{z_i})$ is the polynomial used in defining $(R_h v)(z_i)$ for $i = 1, 2$. To complete the definition of the $R_h$, we need to define the patches $\{K_z : z \in \mathcal{N}_h\}$. Note that $K_z$ must have at least six nodes distributed around $z$ such that $p_z$ is unique.

For $z \in \mathcal{N}_h \cap \Omega$, we first define $K_z$ to be the union of the mesh triangles that meet at $z$. If $K_z$ has less than six nodes or if $p_z$ is not unique, then we extend $K_z$ out by adding mesh triangles that have common edges with $K_z$. This procedure may be repeated till we get a unique $p_z$.

For $z \in \mathcal{N}_h \cap \partial\Omega$, let $K_{z,0}$ be the union of the triangles in the first $n_0$ layers of triangles around $z$, where $n_0$ is the smallest positive integer such that $K_{z,0}$ has at least one internal mesh vertex. Then,

$$K_z = K_{z,0} \cup \{K_{\tilde{z}} : \tilde{z} \in K_{z,0} \text{ and } \tilde{z} \in \mathcal{N}_h \cap \Omega\}.$$

Of course $K_z$ has to satisfy certain conditions to get unique $p_z$. For more details about these conditions, and to prove that $R_h \in \mathcal{R}$, see [16, 17].

*Remark* 4.4. One may replace the discrete least-squares in (4.8) by the continuous $L_2$ least-squares, i.e.,

$$\|v - p_z\|^2_{L_2(K_z)} = \min_{p \in P_2(K_z)} \|v - p\|^2_{L_2(K_z)}.$$

**5. Eigenvalue recovery.** This section is devoted to studying the eigenvalue recovery using the operators in $\mathcal{R}$. As is well known, the eigenfunctions are not unique. Indeed, each eigenvalue has a corresponding eigenspace whose dimension is at most the multiplicity of this eigenvalue. This is true for both exact and approximate eigenvalues. So, in studying the convergence between the exact and the approximate eigenvalues/eigenvectors, one has to study the convergence between the exact and the approximate eigenspaces. This kind of study is involved and needs some deep analysis; see, e.g., [11]. In this paper, the focus is on using the function recovery operators in enhancing eigenvalues. For this reason, we will consider problems with simple eigenvalues, i.e., the multiplicity is 1. Although this assumption is a limitation, it greatly simplifies our analysis. Also, as we will see in our numerical example, the convergence in this simple case applies to the multiple eigenvalue case. Normalizing the eigenfunctions, we can say that the eigenfunctions are unique up to a sign. Even the sign can be fixed through a suitable sign condition, e.g., the average (or the maximum) is positive. Hence, the simplicity, the normalization, and the sign conditions allow us to speak about the convergence of the approximate eigenfunctions to their corresponding exact eigenfunctions without using the eigenspaces.

Before going through the main results in this section, we need to cover some basic facts about eigenvalue problems. Consider the classic eigenvalue problem in which we want to find the eigenvalue $\lambda \in \mathbb{C}$ and a corresponding eigenfunction $0 \neq u$ such that

$$\begin{cases} Lu = \lambda m u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{5.1}$$

where $L, \Omega$, and $\Gamma$ as defined in section 2. The weak formulation of (5.1) reads as follows: find an eigenvalue $\lambda \in \mathbb{C}$ and a corresponding eigenfunction $0 \neq u \in V$ such that

$$a(u, v) = \lambda b(u, v) \quad \forall\, v \in V, \tag{5.2}$$

where $a(\cdot,\cdot)$ is continuous and V-elliptic, and $b(u, v) = \int_\Omega muv$. Assuming that $m \in L_\infty(\Omega)$ such that $m(x, y) \geq \beta > 0$ for all $(x, y) \in \Omega$ for some constant $\beta$, then

$$\beta\|u\|^2_{0,\Omega} \leq b(u, u) \text{ and } |b(u, v)| \leq \|m\|_{\infty,\Omega}\|u\|_{0,\Omega}\|v\|_{0,\Omega} \quad \forall\, u, v \in V. \tag{5.3}$$

Under these assumptions, (5.2) has a countable set of eigenvalues as detailed in [2, 8]. The linear finite element discretization of (5.2) is as follows: find an eigenvalue $\lambda_h \in \mathbb{C}$ and its corresponding eigenfunction $0 \neq u_h \in V_{h,0}$ such that

$$a(u_h, v) = \lambda_h b(u_h, v) \quad \forall\, v \in V_{h,0}. \tag{5.4}$$

Let $M_\lambda$ be the space of all eigenfunctions corresponding to $\lambda$ obtained by solving (5.2), and let $M_{\lambda,h}$ be the space of all eigenfunctions corresponding to $\lambda_h$ obtained by solving (5.4). Given two closed subspaces $V_1 \subset V$ and $V_2 \subset V$, we define the gap between them by

$$\widehat{\delta}(V_1, V_2) = \max(\delta(V_1, V_2), \delta(V_2, V_1)), \text{ where } \delta(V_1, V_2) = \sup_{\substack{v_1 \in V_1 \\ \|v_1\|_{1,\Omega}=1}} \operatorname{dist}(v_1, V2).$$

We have the following theorem (see [2]).

Theorem 5.1. *There exist constants $C > 0$ and $h_0 > 0$ independent of $h$ such that*

$$\widehat{\delta}(M_\lambda, M_{\lambda,h}) \leq \sup_{\substack{u \in M_\lambda \\ \|u\|_{1,\Omega}=1}} \inf_{v \in V_h} \|u - v\|_{1,\Omega}$$

*for all $0 < h < h_0$.*

*Remark* 5.2. Using a duality argument and Theorem 5.1, it is straightforward to show that if $u$ and $u_h$ are solutions of (5.2) and (5.4), respectively, and if $u \in H^2(\Omega)$, then $\|u - u_h\|_{0,\Omega} \lesssim h^2$. Note that as mentioned earlier, $u$ and $u_h$ are normalized and their global maximums, or their averages, have the same sign.

*Remark* 5.3. If $u$ and $u_h$ are solutions of (5.2) and (5.4), respectively, then Theorem 2.1 holds true. Of course, it is assumed that $u$ and $u_h$ are appropriately normalized as we discussed before.

The problem in (5.2) is said to be symmetric, or self-adjoint, if $a(u, v) = a(v, u)$ for all $u, v \in V$; otherwise, it is nonsymmetric. For a nonsymmetric eigenvalue problem, we will need to consider its adjoint problem: find an eigenvalue $\lambda^* \in \mathbb{C}$ and a corresponding eigenfunction $0 \neq u^*$ such that

$$\begin{cases} L^*u^* = -\nabla \cdot (\mathbf{D}^*\nabla u^*) - \mathbf{b} \cdot \nabla u^* + (c - \nabla \cdot \mathbf{b})u = \lambda^* m u^* & \text{in } \Omega, \\ u^* = 0 & \text{on } \Gamma, \end{cases} \tag{5.5}$$

where $\mathbf{D}^*$ is the transpose of $\mathbf{D}$. The weak formulation of (5.5) reads as follows: find an eigenvalue $\lambda^* \in \mathbb{C}$ and its corresponding eigenfunction $0 \neq u^* \in V$ such that

$$a^*(u^*, v) = \int_\Omega (\mathbf{D}^*\nabla u^*) \cdot \nabla v - (\mathbf{b} \cdot \nabla u^*) v + (c - \nabla \cdot \mathbf{b}) u^* v = \lambda^* b(u^*, v) \quad \forall\, v \in V. \tag{5.6}$$

Finally, the linear finite element discretization of (5.6) is as follows: find an eigenvalue $\lambda_h^* \in \mathbb{C}$ and its corresponding eigenfunction $0 \neq u_h^* \in V_{h,0}$ such that

$$a^*(u_h^*, v) = \lambda_h^* b(u_h^*, v) \;\; \forall\, v \in V_{h,0}. \tag{5.7}$$

*Remark* 5.4. Note that $a^*(\cdot, \cdot)$ is still continuous and V-elliptic. Moreover, if $M_\lambda^*$ is the space of all eigenfunctions corresponding to $\lambda^*$ in (5.6), and $M_{\lambda,h}^*$ is the space of all eigenfunctions corresponding to $\lambda_h^*$ in (5.7), then Theorem 5.1 is still valid after replacing $M_\lambda$ and $M_{\lambda_h}$ with $M_\lambda^*$ and $M_{\lambda_h}^*$, respectively. This means that $\|u^* - u_h^*\|_{0,\Omega} \lesssim h^2$ when $u^* \in H^2(\Omega)$.

Let us now see how to use the recovery operators in $\mathcal{R}$ to enhance the accuracy of eigenvalues computed with linear finite element methods.

**5.1. Symmetric eigenvalue problems.** Let us assume that $a(\cdot,\cdot)$ in (5.2) is symmetric. Then, (5.2) has a countable sequence of simple real eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots \nearrow +\infty$ and their corresponding eigenfunctions are $u_1, u_2, \ldots$. On the other hand, (5.4) has a finite sequence of eigenvalues $0 < \lambda_{1,h} < \lambda_{2,h} < \cdots < \lambda_{n_h,h}$ and the corresponding eigenfunctions are $u_{1,h}, u_{2,h}, \ldots, u_{n_h,h}$, where $n_h = \dim V_{h,0}$. The discrete eigenvalues of (5.4) do not have to be simple even under the assumption that (5.2) has simple ones. However, we assumed that (5.4) has simple eigenvalues to simplify the analysis. In addition, $\lambda_{i,h} \geq \lambda_i$ for $1 \leq i \leq n_h$ and the following identity holds true (see [2]).

LEMMA 5.5. *Let $(u, \lambda)$ be a solution of* (5.2) *and let $a(\cdot,\cdot)$ be symmetric. Then, for any $w \in H_0^1(\Omega) \setminus \{0\}$, there holds*

$$\frac{a(w,w)}{b(w,w)} - \lambda = \frac{a(w-u,w-u)}{b(w,w)} - \lambda\frac{b(w-u,w-u)}{b(w,w)}. \tag{5.8}$$

*Remark* 5.6. Using (5.4), one can note that $\lambda_h = a(u_h,u_h)/b(u_h,u_h)$. Using $w = u_h$ in Lemma 5.5, it is straightforward to prove that maximum convergence rate for $|\lambda - \lambda_h|$ is $O(h^2)$; see [2] for more details.

Now, let $R_h \in \mathcal{R}$ and define the recovered eigenvalue

$$\hat{\lambda}_h = \frac{a(R_h u_h, R_h u_h)}{b(R_h u_h, R_h u_h)}. \tag{5.9}$$

Our second major result is in the following theorem.

THEOREM 5.7. *Let $(u, \lambda)$ be a solution of* (5.2) *when $a(\cdot,\cdot)$ is symmetric, let $(u_h, \lambda_h)$ be the corresponding finite element solution of* (5.4)*, and let $u$ and $u_h$ be appropriately normalized. Assume that $u \in H^3(\Omega) \cap W^2_\infty(\Omega)$. If $\|\nabla(I_h u - u_h)\|_{0,\Omega} \lesssim h^\alpha$ for some $\alpha > 0$; then*

$$|\hat{\lambda}_h - \lambda| \lesssim h^{2\alpha}.$$

*Proof.* Setting $w = R_h u_h$ in Lemma 5.5, we have

$$\hat{\lambda}_h - \lambda = \frac{a(u - R_h u_h, u - R_h u_h)}{b(R_h u_h, R_h u_h)} - \lambda\frac{b(u - R_h u_h, u - R_h u_h)}{b(R_h u_h, R_h u_h)}. \tag{5.10}$$

By virtue of R3 and (5.3), $b(R_h u_h, R_h u_h) \gtrsim \|u_h\|^2_{0,\Omega}$. For the first term in the right-hand side of (5.10), the continuity of $a(\cdot,\cdot)$, and Theorem 3.6, we get

$$\begin{aligned} a(u - R_h u_h, u - R_h u_h) &\lesssim \|u - R_h u_h\|^2_{1,\Omega} \\ &\lesssim \|\nabla(u - R_h u_h)\|^2_{0,\Omega} \\ &\lesssim h^{2\alpha}. \end{aligned}$$

For the second term, (5.3), Lemma 3.1, and Theorem 5.1 lead to

$$b(u - R_h u_h, u - R_h u_h) \lesssim \|u - R_h u_h\|^2_{0,\Omega} \lesssim h^4,$$

and this completes the proof. ∎

Theorem 5.7 indicates that in mildly structured meshes, where $\alpha$ is close to 2, $\hat{\lambda}_h$ converges to $\lambda$ at a rate close to 4, which is ultra-convergence. Based on the numerical results in [18], we can claim that the estimate in Theorem 5.7 is sharp. Note that [18] used the formula (1.1) for the recovered eigenvalue. However, the numerical experiments based on (5.9) produces results close to the ones reported in [18]. For this reason, we will not include any numerical examples for symmetric eigenvalue problems in this paper.

**5.2. Nonsymmetric eigenvalue problems.** Now, we consider nonsymmetric eigenvalue problems. First, we should mention that Lemma 5.7 is not valid when $a(\cdot,\cdot)$ is nonsymmetric. However, it can be extended to cover the nonsymmetric case as follows (see [25]).

LEMMA 5.8. *Let $(u, \lambda)$ be a solution of (5.2) and let $a(\cdot,\cdot)$ be nonsymmetric. Then for any $w \in H_0^1(\Omega) \setminus \{0\}$, there holds*

$$\frac{a(w,w)}{b(w,w)} - \lambda = \frac{a(w-u,w-u)}{b(w,w)} - \lambda\frac{b(w-u,w-u)}{b(w,w)} + \frac{a(w-u,u)-a(u,w-u)}{b(w,w)}.$$

Proceeding as in Theorem 5.7, it is easy to prove that the last term in Lemma 5.8 is $O(h^\alpha)$. Hence, if we use the formula in (5.9), the convergence rate of $|\lambda - \hat{\lambda}_h|$ will be around $\alpha$. Therefore, $\hat{\lambda}_h$ does not really improve $\lambda_h$. That is because we used only $u_h$. However, if we use both $u_h$ and $u_h^*$, we can get a result similar to Theorem 5.7. For the nonsymmetric case, we change the definition in (5.9) to the following:

$$\hat{\lambda}_h = \frac{a(R_h u_h, R_h u_h^*)}{b(R_h u_h, R_h u_h^*)}. \tag{5.11}$$

Note that (5.11) reduces to (5.9) when $a(\cdot,\cdot)$ is symmetric. Proving a result similar to Theorem 5.7 needs the following lemma from [12].

LEMMA 5.9. *Let $(u, \lambda)$ be a solution of (5.2), let $a(\cdot,\cdot)$ be nonsymmetric, and let $(u^*, \lambda^* = \overline{\lambda})$ be the solution of (5.6). Then for any $w$ and $w^* \in H_0^1(\Omega) \setminus \{0\}$, there holds*

$$\frac{a(w,w^*)}{b(w,w^*)} - \lambda = \frac{a(w-u,w^*-u^*)}{b(w,w^*)} - \lambda\frac{b(w-u,w^*-u^*)}{b(w,w^*)}. \tag{5.12}$$

Now, we are ready to prove the ultra-convergence of $\hat{\lambda}_h$ for nonsymmetric eigenvalue problems.

THEOREM 5.10. *Let $(u, \lambda)$ be a solution of (5.2), let $a(\cdot,\cdot)$ be nonsymmetric, let $(u_h, \lambda_h)$ be the corresponding finite element solution of (5.4), and let $u$ and $u_h$ be appropriately normalized. Let $u^*$ and $u_h^*$ be the corresponding solutions of (5.6) and (5.7), respectively. Assume that $u$ and $u^*$ are both in $H^3(\Omega) \cap W_\infty^2(\Omega)$. If $\|\nabla(I_h u - u_h)\|_{0,\Omega} \lesssim h^\alpha$ and $\|\nabla(I_h u^* - u_h^*)\|_{0,\Omega} \lesssim h^\alpha$ for some $\alpha > 0$, then*

$$|\hat{\lambda}_h - \lambda| \lesssim h^{2\alpha}.$$

*Proof.* Setting $w = R_h u_h$ and $w^* = R_h u_h^*$ in Lemma 5.9, we get

$$\hat{\lambda}_h - \lambda = \frac{a(u-R_h u_h, u^*-R_h u_h^*)}{b(R_h u_h, R_h u_h^*)} - \lambda\frac{b(u-R_h u_h, u^*-R_h u_h^*)}{b(R_h u_h, R_h u_h^*)}. \tag{5.13}$$

The rest of the proof follows the steps used in Theorem 5.7. ∎

As in the case of the symmetric eigenvalue problems, Theorem 5.10 says that $\hat{\lambda}_h$ enjoys ultra-convergence in the nonsymmetric eigenvalue problems.

**5.3. A note about the computational cost.** At first glance, it seems that (5.11) has a very expensive computational cost compared to (5.9). We need to solve (5.7) for $u_h^*$ and to construct $R_h u_h^*$ in addition to constructing $R_h u_h$. However, we really do not need to solve for $u_h^*$ separately. Note that the nodal values of $u_h$ compose an eigenvector of the generalized eigenvalue problem $Ax = \lambda_h Mx$, where $A$ and

$M$ are the assembled matrices of the system in (5.4). Note that the mass matrix $M$ is symmetric and positive definite. Using the sparse Cholesky decomposition, we can find a sparse matrix $Q$ such that $M = QQ^*$. Now, the eigenvalue problem can be written as $Q^{-1}AQ^{-*}Q^*x = Q^*x$. Setting $B = Q^{-1}AQ^{-*}$ and $y = Q^*x$, the generalized eigenvalue problem is now reduced to $By = \lambda_h y$. Since $B$ is nonsymmetric, it has both left and right eigenvectors. When we solve for $u_h$, we use the right eigenvectors of the eigenvalue problem $By = \lambda_h y$, whereas $u_h^*$ comes from the left eigenvectors of the same system. There are eigenvalue solvers that can compute both the left and right eigenvectors of $B$ simultaneously. For small systems, LAPACK and BLAS packages have routines for calculating both left and right eigenvectors. For large systems, the focus is usually on the first few smallest eigenvalues. For that, there are algorithms that can compute both the left and the right eigenvectors. For example, [10] introduced a two-sided Arnoldi method that can do this job. Also, Zhaojun Bai and David Day wrote the MATLAB package ABLEPACK for solving generalized non-Hermitian eigenvalue problems. This package, which can be downloaded at www.cs.ucdavis.edu/ bai/ABLEpack, finds the eigenvalues, or eigentriplets, using an adaptive block Lanczos method. By an eigentriplet, we mean the eigenvalue, a corresponding right eigenvector, and a corresponding left eigenvector.

From this discussion, we can see that the computational cost for enhancing $\lambda_h$ is mainly in the function recovery. Since $R_h$ is linear, $R_h u_h$ and $R_h u_h^*$ can be computed by applying $R_h$ to both $u_h$ and $u_h^*$ simultaneously.

Going over the existing techniques for enhancing eigenvalues in nonsymmetric eigenvalue problems, we believe that this cost is practically acceptable. For example, one of the existing techniques uses the two-grids discretization approach as in [25]. In this approach, the eigenvalue recovery is done in two steps on two grids where the second grid is a finer version of the first one. Briefly speaking, the algorithm in this approach uses the coarse mesh to compute $(\lambda_H, u_H)$, then solves the finite element problem $a(u_h, v) = \lambda_H b(u_H, v)$ for all $v \in V_h$ for $u_h$ on the finer mesh. Finally, $u_h$ is used in a Rayleigh quotient, as in (5.9), to calculate the improved eigenvalue. As explained earlier, this formula does not give good results in nonsymmetric eigenvalue problems. The numerical experiments reported in [12] support our claim. Another approach [12] for enhancing the eigenvalue approximation combines the two-grids approach with solving the adjoint problem.

**5.4. A numerical example.** Consider the convection-diffusion equation

$$\begin{cases} -\nu\Delta u + \mathbf{b}\cdot\nabla u = \lambda u & \text{in } \Omega = (0,1)\times(0,1), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.14}$$

where $\mathbf{b} = (b_x, b_y)$. The eigenvalues of (5.14) are

$$\lambda_{k,l} = \frac{b_x^2 + b_y^2}{4\nu} + \nu\pi^2(k^2 + l^2)$$

for $k, l \in \mathbb{N}_+$, and the corresponding eigenfunctions are

$$u(x,y) = \exp\left(\frac{b_x x + b_y y}{2\nu}\right)\sin(k\pi x)\sin(l\pi y).$$

The adjoint eigenvalue problem has eigenvalues $\lambda^*_{k,l} = \lambda_{k,l}$ and the corresponding eigenfunctions are

$$u^*(x,y) = \exp\left(-\frac{b_x x + b_y y}{2\nu}\right)\sin(k\pi x)\sin(l\pi y).$$

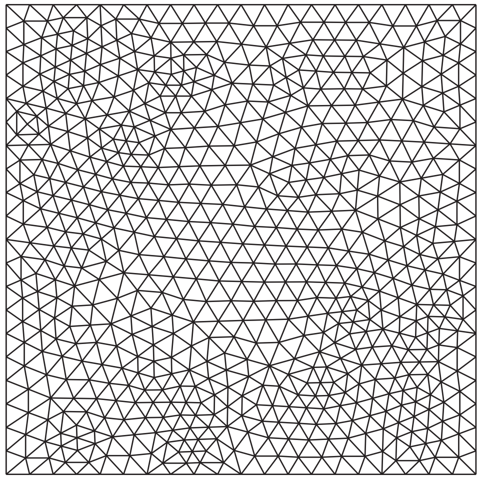

FIG. 5.1. *The initial Delaunay mesh used in solving* (5.14).

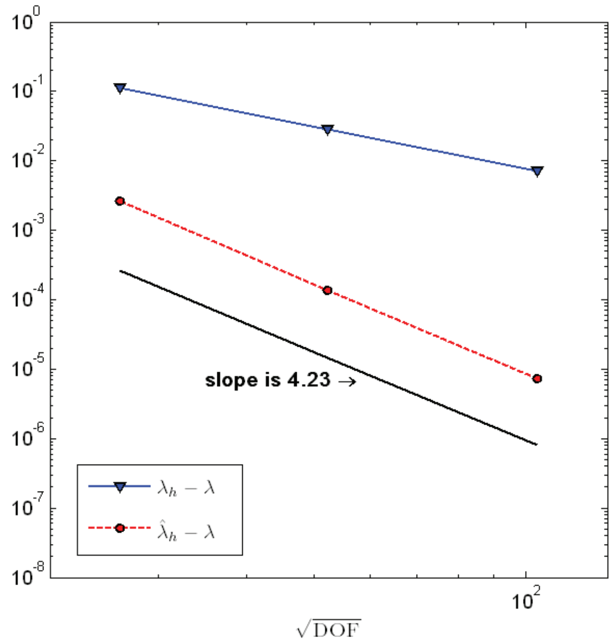


FIG. 5.2. *The accuracy of the recovered eigenvalues using* (5.11).

In this example, we use $\nu = 1, b_x = 10$, and $b_y = 1$. The computations in this example are conducted with FEMLAB version 2.3 under MATLAB version 7.0. The eigenvalues are computed using the MATLAB algorithms which are based on LAPACK routines. Let us consider the smallest eigenvalue $\lambda = \lambda_{1,1}$. For computing the finite element solution of (5.14), we use a sequence of meshes $\{\mathcal{T}_i : i = 1, 2, \dots\}$, where $\mathcal{T}_1$ is the Delaunay mesh shown in Figure 5.1 and $\mathcal{T}_{i+1}$ is obtained from $\mathcal{T}_i$ using regular refinement. As we can see in Figure 5.2, the convergence rate is close to 4, which confirms the estimate in Theorem 5.10. Repeating the same steps for

$\lambda = \lambda_{4,4}$, the convergence rate is 4.85. Indeed, $|\lambda_h - \lambda|$ dropped from 8.647877 in the first iteration to 0.526391 in the last iteration, while $|\lambda_h^* - \lambda|$ dropped from 2.497003 to 0.004124. In this example, some of the eigenvalues have multiplicities more than one. For example, the multiplicity of the eigenvalue $\lambda_{3,4}$ is 2. Although this is not a simple eigenvalue, it enjoys the same convergence properties as the simple ones. Using the same sequence of meshes, $|\hat{\lambda}_h - \lambda|$ goes from 1.004846 in the first iteration to 0.001247 in the last iteration at a rate of 4.92.

**Acknowledgments.** The authors would like to thank an anonymous referee for the valuable and constructive comments on our submitted manuscript. We also extend our thanks to Zhaojun Bai of University of California at Davis for the fruitful discussions on solving nonsymmetric eigenvalue problems.

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